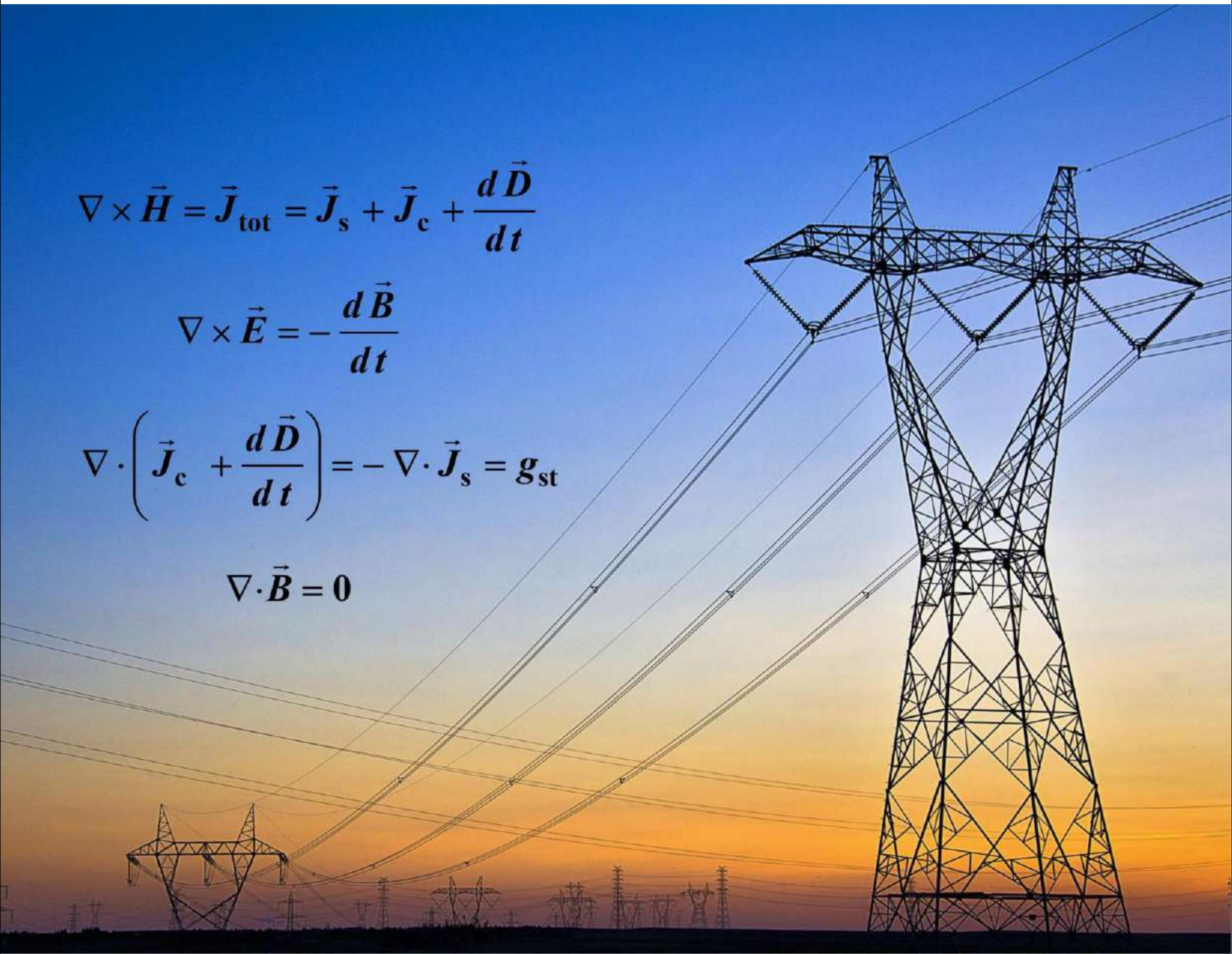
UNIVERSITY OF SPLIT

FACULTY OF ELECTRICAL ENGINEERING, MECHANICAL ENGINEERING AND NAVAL ARCHITECTURE



Slavko Vujević

ELECTROMAGNETICS

UNIVERSITY OF SPLIT
FACULTY OF ELECTRICAL ENGINEERING, MECHANICAL ENGINEERING AND NAVAL ARCHITECTURE

Slavko Vujević

ELECTROMAGNETICS

Split, April 2026

Publisher

FACULTY OF ELECTRICAL ENGINEERING, MECHANICAL ENGINEERING AND NAVAL ARCHITECTURE

Editor

Associate Professor Ivo Marinić-Kragić

Author

Professor Emeritus Slavko Vujević

Reviewer

Professor Dragan Poljak

Proofreader

Ana Mršić Zdilar, Senior Lecturer

Graphic Design and Illustrations

Professor Emeritus Slavko Vujević

ISBN 978-953-290-150-4 (PDF)

ISBN 978-953-290-146-7 (Print)

To my family: Eda,
Andrija, Anđela, and
Petar

I would like to thank
Professor Dragan Poljak for
his review, and Professor
Mateo Bašić for his careful
reading of the manuscript
prior to the review

PREFACE

This is the English-language edition of a textbook originally published in Croatian in 2024, listed as the final reference in the bibliography of this textbook. This university-level textbook is the result of my forty years of work in electromagnetic theory and numerical modeling of electromagnetic phenomena. It is primarily intended for electrical engineering students specializing in electric power engineering and fully covers the content of the current course *Theoretical Electrical Engineering*. Today, a more suitable name for this course might be *Engineering Electromagnetics*.

The structure of the course and textbook is based on the lectures of Professor Mate Kurtović, Ph.D., a highly respected supervisor whose approach served as the foundation for the material. Significant influence also came from renowned university textbooks authored by Professors Zijad Haznadar, Željko Štih, Tomo Bosanac, Jovan Surutka and Dragutin M. Veličković. Among problem collections, the one compiled by Professor Sead Berberović, Ph.D., was particularly valuable.

This textbook stands out from other works on electromagnetism in several important respects. Maxwell's equations are originally formulated for conducting media that contain sources of the electromagnetic field – an approach not satisfactorily addressed in existing literature. In addition, electromagnetic fields are systematically classified based on the rate of their temporal variation.

Another key distinction lies in the definition of phasor magnitude, which is taken here as the root-mean-square (RMS) value of the corresponding sinusoidal quantity. This approach aligns with definitions used in foundational courses such as electrical engineering fundamentals and electric power systems.

Split, April 2026

Author

CONTENTS

1. FUNDAMENTAL TERMS OF VECTOR ANALYSIS

Vector algebra is a branch of mathematics that enables the algebraic representation of vectors in three-dimensional (3D) space and supports algebraic operations on vectors, whereas *vector analysis*, also known as vector calculus, deals with the differentiation and integration of vector and scalar fields. Linear algebra is a branch of mathematics concerned with the study of vectors, vector spaces, matrices, linear transformations, linear operators and systems of linear equations.

A *scalar* is a quantity defined solely by its magnitude. A *scalar function* of one or more variables is a function that assigns numbers to ordered n-tuples of numbers. In a narrower sense, scalars are real numbers, but complex numbers are also referred to as scalars, or more precisely, *complex scalars*. Accordingly, a *complex scalar function* of one or more variables is a function that assigns complex numbers to ordered n-tuples of numbers.

A *vector* is a quantity defined by both magnitude and direction. For practical purposes, a vector is often expressed as the sum of the products of its scalar components and the corresponding unit vectors of a chosen coordinate system. The scalar components of a vector may be real or complex numbers. A vector function is a function that assigns a vector to each real number (or, more generally, to each point in its domain).

A *scalar field* is a scalar function that assigns a real or complex number to each point in space. A *vector field* is a vector function that assigns a vector to each point in space, with components that may be real or complex numbers.

1.1. Coordinate Systems

The fundamental coordinate systems are three-dimensional (3D) coordinate systems, whereas two-dimensional (2D) and one-dimensional (1D) systems are considered subsets of the corresponding 3D systems. Therefore, it is sufficient to provide a brief description of the 3D coordinate systems used in this textbook.

Three-dimensional coordinate systems can be categorized as follows:

- Orthogonal rectilinear coordinate systems,
- Orthogonal curvilinear coordinate systems,
- Non-orthogonal rectilinear coordinate systems,
- Non-orthogonal curvilinear coordinate systems.

3D orthogonal coordinate systems are coordinate systems in which the unit vectors at every point in space are orthogonal, i.e., mutually perpendicular.

This textbook employs only the following three 3D orthogonal coordinate systems:

- Cartesian (also known as Descartes or rectangular) coordinate system: (x, y, z),
- Cylindrical (or circular cylindrical) coordinate system: (r, φ, z) or (r, ϕ, z),
- Spherical coordinate system (r, ϑ, φ) or (r, ϑ, ϕ).

In a cylindrical coordinate system, r represents the shortest distance from the field point to the z-axis, whereas in a spherical coordinate system, r denotes the distance from the field point to the origin of the coordinate system. The polar coordinate system (r, φ) or (r, ϕ) is a 2D coordinate system and can be viewed as a subset of the cylindrical coordinate system (r, φ, z) or (r, ϕ, z).

The Cartesian coordinate system is an orthogonal rectilinear coordinate system, whereas the cylindrical and spherical coordinate systems are orthogonal curvilinear coordinate systems. All three are classified as right-handed orthogonal coordinate systems because the orientation of their unit vectors follows the right-hand rule:

$$\vec{e}_1 = \vec{e}_2 \times \vec{e}_3 \tag{1.1}$$

where $\vec{e}_1$, $\vec{e}_2$ and $\vec{e}_3$ are the unit vectors of the corresponding coordinate system.

1.2. Differential Vector Element Along an Oriented 3D Curve

The differential vector element along an oriented (directed) curve in an orthogonal 3D coordinate system (u_1, u_2, u_3), also known as the infinitesimal displacement vector, is given by the following expression:

$$d\vec{\ell} = h_1 \cdot du_1 \cdot \vec{e}_1 + h_2 \cdot du_2 \cdot \vec{e}_2 + h_3 \cdot du_3 \cdot \vec{e}_3 \tag{1.2}$$

For a Cartesian coordinate system, the following expression holds:

$$(u_1, u_2, u_3) \to (x, y, z) \quad ; \quad h_1 = h_2 = h_3 = 1 \tag{1.3}$$

whereas for a cylindrical coordinate system, the following expression holds:

$$(u_1, u_2, u_3) \to (r, \varphi, z) \quad ; \quad h_1 = 1 \quad ; \quad h_2 = r \quad ; \quad h_3 = 1 \tag{1.4}$$

and for the spherical coordinate system, the following expression holds:

$$(u_1, u_2, u_3) \to (r, \vartheta, \varphi) \quad ; \quad h_1 = 1 \quad ; \quad h_2 = r \quad ; \quad h_3 = r \cdot \sin\vartheta \tag{1.5}$$

From expressions (1.2) - (1.5), it follows directly that the differential vector element along an oriented 3D curve in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$d\vec{\ell} = \vec{\ell}_0 \cdot d\ell = dx \cdot \vec{i} + dy \cdot \vec{j} + dz \cdot \vec{k} \tag{1.6}$$

whereas the differential vector element along the oriented 3D curve in the cylindrical coordinate system (r, φ, z) is given by the expression:

$$d\vec{\ell} = \vec{\ell}_0 \cdot d\ell = dr \cdot \vec{e}_r + r \cdot d\varphi \cdot \vec{e}_\varphi + dz \cdot \vec{e}_z \tag{1.7}$$

and the differential vector element along an oriented 3D curve in the spherical coordinate system (r, ϑ, φ) is given by the expression:

$$d\vec{\ell} = \vec{\ell}_0 \cdot d\ell = dr \cdot \vec{e}_r + r \cdot d\vartheta \cdot \vec{e}_\vartheta + r \cdot \sin\vartheta \cdot d\varphi \cdot \vec{e}_\varphi \tag{1.8}$$

where $\vec{\ell}_0$ is the unit vector tangent to the oriented 3D curve at the point of interest.

Let the parametric representation of the 3D curve in the Cartesian coordinate system (x, y, z) be given by:

$$x = x(t) \quad ; \quad y = y(t) \; ; \quad z = z(t) \tag{1.9}$$

and let the parametric representation of the 3D curve in the cylindrical coordinate system (r, φ, z) be given by:

$$r = r(t) \quad ; \quad \varphi = \varphi(t) \; ; \quad z = z(t) \tag{1.10}$$

and let the parametric representation of the 3D curve in the spherical coordinate system read (r, ϑ, φ) be given by:

$$r = r(t) \quad ; \quad \vartheta = \vartheta(t) \; ; \quad \varphi = \varphi(t) \tag{1.11}$$

If the curve is oriented in the direction of increasing parameter t, it is said to be positively oriented. Otherwise, the curve is negatively oriented.

From expressions (1.6) and (1.9), the differential arc length $d\ell$ in the Cartesian coordinate system (x, y, z) is readily obtained as:

$$d\ell = \sqrt{\left(\frac{\partial x}{\partial t}\right)^2 + \left(\frac{\partial y}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2} \cdot dt \tag{1.12}$$

whereas, based on the expressions (1.7) and (1.10), the differential arc length $d\ell$ in the cylindrical coordinate system (r, φ, z) is given by:

$$d\ell = \sqrt{\left(\frac{\partial r}{\partial t}\right)^2 + \left(r\cdot\frac{\partial \varphi}{\partial t}\right)^2 + \left(\frac{\partial z}{\partial t}\right)^2}\cdot dt \tag{1.13}$$

and, based on the expressions (1.8) and (1.11), the differential arc length $d\ell$ in the spherical coordinate system (r, ϑ, φ) is given by:

$$d\ell = \sqrt{\left(\frac{\partial r}{\partial t}\right)^2 + \left(r\cdot\frac{\partial \vartheta}{\partial t}\right)^2 + \left(r\cdot\sin\vartheta\cdot\frac{\partial \varphi}{\partial t}\right)^2}\cdot dt \tag{1.14}$$

If the 3D curve in the Cartesian coordinate system (x, y, z) is parametrically described by the equations $y = y(x)$ and $z = z(x)$, then the differential arc length $d\ell$ is given by the following expression:

$$d\ell = \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\cdot dx \tag{1.15}$$

1.3. Differential Vector Element of an Oriented 3D Surface

The differential vector element of an oriented surface in an orthogonal 3D coordinate system (u_1, u_2, u_3) is given by the expression:

$$d\vec{S} = h_2\cdot h_3\cdot du_2\cdot du_3\cdot \vec{\mathrm{e}}_1 + h_1\cdot h_3\cdot du_1\cdot du_3\cdot \vec{\mathrm{e}}_2 + h_1\cdot h_2\cdot du_1\cdot du_2\cdot \vec{\mathrm{e}}_3 \tag{1.16}$$

From expressions (1.16) and (1.3), it follows directly that the differential vector element of an oriented 3D surface in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$d\vec{S} = \vec{n}\cdot dS = dy\cdot dz\cdot \vec{\mathrm{i}} + dx\cdot dz\cdot \vec{\mathrm{j}} + dx\cdot dy\cdot \vec{\mathrm{k}} \tag{1.17}$$

whereas from expressions (1.16) and (1.4), it follows that the differential vector element of an oriented 3D surface in the cylindrical coordinate system (r, φ, z) is given by the expression:

$$d\vec{S} = \vec{n}\cdot dS = r\cdot d\varphi\cdot dz\cdot \vec{\mathrm{e}}_{\mathrm{r}} + dr\cdot dz\cdot \vec{\mathrm{e}}_{\varphi} + r\cdot dr\cdot d\varphi\cdot \vec{\mathrm{e}}_{\mathrm{z}} \tag{1.18}$$

and from expressions (1.16) and (1.5), it follows that the differential vector element of an oriented 3D surface in the spherical coordinate system (r, ϑ, φ) is given by the expression:

$$d\vec{S} = \vec{n}\cdot dS = r^2\cdot\sin\vartheta\cdot d\varphi\cdot d\vartheta\cdot \vec{\mathrm{e}}_{\mathrm{r}} + r\cdot\sin\vartheta\cdot dr\cdot d\varphi\cdot \vec{\mathrm{e}}_{\vartheta} + r\cdot dr\cdot d\vartheta\cdot \vec{\mathrm{e}}_{\varphi} \tag{1.19}$$

where $\vec{n}$ is the unit normal vector to the oriented surface at the point of interest.

Let the parametric representation of the 3D surface in the Cartesian coordinate system (x, y, z) be given by:

$$x = x(u, v) \quad ; \quad y = y(u, v) \quad ; \quad z = z(u, v) \tag{1.20}$$

The operational expression for the differential surface area dS of a surface S in the Cartesian coordinate system, as derived from expressions (1.20) and (1.16), is given by the following equation:

$$dS = \sqrt{n_x^2 + n_y^2 + n_z^2}\cdot du\cdot dv \tag{1.21}$$

where:

$$\vec{n} = \{n_x, n_y, n_z\} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} \tag{1.22}$$

If a 3D surface in the Cartesian coordinate system (x, y, z) is defined by the equation $z = z(x, y)$, then the differential surface area dS is given by the following expression:

$$dS = \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2} \cdot dx \cdot dy \tag{1.23}$$

1.4. Differential Volume Element

The differential volume element in an orthogonal 3D coordinate system (u_1, u_2, u_3) is given by the expression:

$$dV = h_1 \cdot h_2 \cdot h_3 \cdot du_1 \cdot du_2 \cdot du_3 \tag{1.24}$$

From expressions (1.24) and (1.3), it follows directly that the differential volume element in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$dV = dx \cdot dy \cdot dz \tag{1.25}$$

whereas from expressions (1.24) and (1.4), it follows that the differential volume element in the cylindrical coordinate system (r, φ, z) is given by the expression:

$$dV = r \cdot dr \cdot d\varphi \cdot dz \tag{1.26}$$

and from expressions (1.24) and (1.5), it follows that the differential volume element in the spherical coordinate system (r, ϑ, φ) is given by the expression:

$$dV = r^2 \cdot \sin\vartheta \cdot dr \cdot d\vartheta \cdot d\varphi \tag{1.27}$$

1.5. Hamiltonian Operator

The Hamiltonian operator (nabla operator) is a vector differential operator that, in an orthogonal 3D coordinate system (u_1, u_2, u_3), is given by the following expression:

$$\nabla \equiv \vec{\nabla} = \vec{e}_1 \cdot \frac{1}{h_1} \cdot \frac{\partial}{\partial u_1} + \vec{e}_2 \cdot \frac{1}{h_2} \cdot \frac{\partial}{\partial u_2} + \vec{e}_3 \cdot \frac{1}{h_3} \cdot \frac{\partial}{\partial u_3} \tag{1.28}$$

From expressions (1.28) and (1.3), it follows directly that the nabla operator in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$\nabla \equiv \vec{\nabla} = \vec{i} \cdot \frac{\partial}{\partial x} + \vec{j} \cdot \frac{\partial}{\partial y} + \vec{k} \cdot \frac{\partial}{\partial z} \tag{1.29}$$

whereas from expressions (1.28) and (1.4), it follows that the nabla operator in the cylindrical coordinate system (r, φ, z) is given by the following expression:

$$\nabla \equiv \vec{\nabla} = \vec{e}_r \cdot \frac{\partial}{\partial r} + \vec{e}_\varphi \cdot \frac{1}{r} \cdot \frac{\partial}{\partial \varphi} + \vec{e}_z \cdot \frac{\partial}{\partial z} \tag{1.30}$$

and from expressions (1.28) and (1.5), it follows that the nabla operator in the spherical coordinate system (r, ϑ, φ) is given by the following expression:

$$\nabla \equiv \vec{\nabla} = \vec{e}_r \cdot \frac{\partial}{\partial r} + \vec{e}_\vartheta \cdot \frac{1}{r} \cdot \frac{\partial}{\partial \vartheta} + \vec{e}_\varphi \cdot \frac{1}{r \cdot \sin\vartheta} \cdot \frac{\partial}{\partial \varphi} \tag{1.31}$$

1.6. Gradient of a Scalar Field

The gradient of a scalar field is a vector field that is collinear with the normal to the level surface of the scalar field and points in the direction of the steepest increase of the field. A level surface is a surface on which the scalar function has a constant value.

The gradient of the scalar field Ψ in an orthogonal 3D coordinate system (u_1, u_2, u_3) is given by the following expression:

$$\operatorname{grad}\Psi = \nabla\Psi = \frac{1}{h_1} \cdot \frac{\partial \Psi}{\partial u_1} \cdot \vec{e}_1 + \frac{1}{h_2} \cdot \frac{\partial \Psi}{\partial u_2} \cdot \vec{e}_2 + \frac{1}{h_3} \cdot \frac{\partial \Psi}{\partial u_3} \cdot \vec{e}_3 \tag{1.32}$$

From expressions (1.32) and (1.3), it follows directly that the gradient of the scalar field $\Psi(x, y, z)$ in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$\operatorname{grad}\Psi = \nabla\Psi = \frac{\partial \Psi}{\partial x} \cdot \vec{i} + \frac{\partial \Psi}{\partial y} \cdot \vec{j} + \frac{\partial \Psi}{\partial z} \cdot \vec{k} \tag{1.33}$$

whereas from expressions (1.32) and (1.4), it follows that the gradient of the scalar field $\Psi(r, \varphi, z)$ in the cylindrical coordinate system (r, φ, z) is given by the following expression:

$$\operatorname{grad}\Psi = \nabla\Psi = \frac{\partial \Psi}{\partial r} \cdot \vec{e}_r + \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \varphi} \cdot \vec{e}_\varphi + \frac{\partial \Psi}{\partial z} \cdot \vec{e}_z \tag{1.34}$$

and from expressions (1.32) and (1.5), it follows that the gradient of the scalar field $\Psi(r, \vartheta, \varphi)$ in the spherical coordinate system (r, ϑ, φ) is given by the following expression:

$$\operatorname{grad}\Psi = \nabla\Psi = \frac{\partial \Psi}{\partial r} \cdot \vec{e}_r + \frac{1}{r} \cdot \frac{\partial \Psi}{\partial \vartheta} \cdot \vec{e}_\vartheta + \frac{1}{r \cdot \sin\vartheta} \cdot \frac{\partial \Psi}{\partial \varphi} \cdot \vec{e}_\varphi \tag{1.35}$$

1.7. Directional Derivative of a Scalar Field

The directional derivative of a scalar field Ψ in the direction of a vector $\vec{s}$ is given by the following expression:

$$\frac{\partial \Psi}{\partial s} \equiv \frac{\partial \Psi}{\partial \vec{s}} = \vec{s}_0 \cdot \nabla\Psi = \vec{s}_0 \cdot \operatorname{grad}\Psi \tag{1.36}$$

where $\vec{s}_0$ is the unit vector in the direction of the vector $\vec{s}$, whereas the gradient of the scalar field Ψ is defined by expressions (1.32) - (1.35).

It follows from expression (1.36) that the directional derivative of the scalar field Ψ in the direction of the vector $\vec{s}$ is equal to the scalar projection of the gradient of Ψ onto the vector $\vec{s}$.

From expression (1.36), it follows that the gradient of the scalar field Ψ can also be expressed as:

$$\operatorname{grad}\Psi = \nabla\Psi = \frac{\partial \Psi}{\partial n} \cdot \vec{n} \tag{1.37}$$

where:

$\vec{n}$ - the unit normal vector to the level surface of the scalar field Ψ, directed along the direction of the steepest increase of Ψ,

$\dfrac{\partial \Psi}{\partial n}$ - the directional derivative of scalar field Ψ in the direction of the vector $\vec{n}$.

1.8. Directional Derivative of a Vector Field

The directional derivative of the vector field $\vec{a}$ in the direction of the vector $\vec{s}$ is given by the following expression:

$$\frac{\partial \vec{a}}{\partial s} \equiv \frac{\partial \vec{a}}{\partial \vec{s}} = \left(\vec{s}_0 \cdot \nabla\right)\vec{a} \tag{1.38}$$

where $\vec{s}_0$ is the unit vector in the direction of the vector $\vec{s}$.

From expression (1.38), it follows that:

$$\left(\vec{s} \cdot \nabla\right)\vec{a} = s \cdot \frac{\partial \vec{a}}{\partial s} \tag{1.39}$$

where s is the magnitude of the vector $\vec{s}$.

1.9. Divergence of a Vector Field

The divergence of a vector field is a scalar field that represents the net rate at which the vector field flows out of an infinitesimal volume around a point. It quantifies the strength of a source or sink of the field's radial component at that point. In other words, divergence represents the volume density of the outward flux of a vector field from an infinitesimal volume surrounding the point.

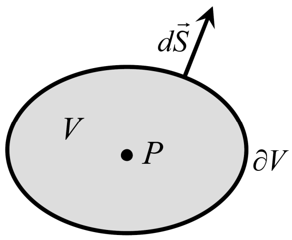


Figure 1.1. Volume V around point P, bounded by an oriented closed surface ∂V

Let the point P be located inside a volume V, which is bounded by an oriented closed surface ∂V (Figure 1.1). The divergence of the vector field $\vec{a}$ at point P is defined by the following expression:

$$\operatorname{div} \vec{a} = \nabla \cdot \vec{a} = \lim_{V \to 0} \frac{\oint\limits_{\partial V} \vec{a} \cdot d\vec{S}}{V} \tag{1.40}$$

According to expression (1.40), the divergence of the vector field $\vec{a}$ can be expressed as the scalar product of the nabla operator and the vector field $\vec{a}$. If the vector field $\vec{a}$ represents the surface density of some physical quantity, then $\nabla \cdot \vec{a}$ represents the corresponding volume density of that quantity.

From the definition of divergence (1.40), it follows that the divergence of the vector field $\vec{a} = \{a_1, a_2, a_3\}$ in an orthogonal 3D coordinate system is given by the following expression:

$$\operatorname{div} \vec{a} = \nabla \cdot \vec{a} = \frac{1}{h_1 \cdot h_2 \cdot h_3} \cdot \left[\frac{\partial\left(h_2 \cdot h_3 \cdot a_1\right)}{\partial u_1} + \frac{\partial\left(h_1 \cdot h_3 \cdot a_2\right)}{\partial u_2} + \frac{\partial\left(h_1 \cdot h_2 \cdot a_3\right)}{\partial u_3}\right] \tag{1.41}$$

From expressions (1.41) and (1.3), it follows directly that the divergence of the vector field $\vec{a} = \{a_x, a_y, a_z\}$ in the Cartesian coordinate system (x, y, z) is given by the expression:

$$\operatorname{div} \vec{a} = \nabla \cdot \vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} \tag{1.42}$$

whereas from expressions (1.41) and (1.4), it follows that the divergence of the vector field $\vec{a} = \{a_r, a_\varphi, a_z\}$ in the cylindrical coordinate system (r, φ, z) is given by the expression:

$$\operatorname{div}\vec{a} = \nabla\cdot\vec{a} = \frac{1}{r}\cdot\frac{\partial(r\cdot a_{\mathrm{r}})}{\partial r} + \frac{1}{r}\cdot\frac{\partial a_{\varphi}}{\partial\varphi} + \frac{\partial a_{\mathrm{z}}}{\partial z} \tag{1.43}$$

and from expressions (1.41) and (1.5), it follows that the divergence of the vector field $\vec{a} = \{a_{\mathrm{r}}, a_{\vartheta}, a_{\varphi}\}$ in the spherical coordinate system (r, ϑ, φ) is given by the expression:

$$\operatorname{div}\vec{a} = \nabla\cdot\vec{a} = \frac{1}{r^2}\cdot\frac{\partial\left(r^2\cdot a_{\mathrm{r}}\right)}{\partial r} + \frac{1}{r\cdot\sin\vartheta}\cdot\frac{\partial(\sin\vartheta\cdot a_{\vartheta})}{\partial\vartheta} + \frac{1}{r\cdot\sin\vartheta}\cdot\frac{\partial a_{\varphi}}{\partial\varphi} \tag{1.44}$$

1.10. Curl of a Vector Field

The curl of a vector field is a vector field that represents the circulation density at each point in space. At a given point, the curl is represented by a vector whose direction indicates the axis of rotation (following the right-hand rule), and whose magnitude corresponds to the maximum circulation per unit area. The magnitude of the curl at a point provides information about the intensity of the local rotational field source.

Let the point P lie on a surface S bounded by an oriented closed curve C (Figure 1.2). Let the orientations of the surface S and the curve C be consistent with the right-hand rule.

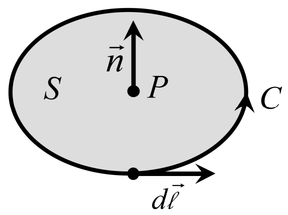


Figure 1.2. The surface S, on which the point P lies, bounded by the closed curve C

The curl of the vector field $\vec{a}$ at the point P, which lies on the surface S, is defined by the following expressions:

$$\operatorname{curl}\vec{a} = \nabla\times\vec{a} \quad ; \quad \vec{n}\cdot\operatorname{curl}\vec{a} = \lim_{S\to 0}\frac{\oint_C \vec{a}\cdot d\vec{\ell}}{S} \tag{1.45}$$

where $\vec{n}$ is the unit normal vector to the oriented surface S at the point P. According to expression (1.45), the curl can be expressed as the cross product of the nabla operator and the vector field $\vec{a}$.

From the definition of curl (1.45), it follows that the curl of the vector field $\vec{a} = \{a_1, a_2, a_3\}$ in an orthogonal 3D coordinate system (u_1, u_2, u_3) is given by the following expression:

$$\operatorname{curl}\vec{a} = \nabla\times\vec{a} = \frac{1}{h_1\cdot h_2\cdot h_3}\cdot\begin{vmatrix} h_1\cdot\vec{e}_1 & h_2\cdot\vec{e}_2 & h_3\cdot\vec{e}_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1\cdot a_1 & h_2\cdot a_2 & h_3\cdot a_3 \end{vmatrix} \tag{1.46}$$

From expressions (1.46) and (1.3), it follows directly that the curl of the vector field $\vec{a} = \{a_{\mathrm{x}}, a_{\mathrm{y}}, a_{\mathrm{z}}\}$ in the Cartesian coordinate system (x, y, z) is given by the following expression:

$$\operatorname{curl}\vec{a} = \nabla\times\vec{a} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_{\mathrm{x}} & a_{\mathrm{y}} & a_{\mathrm{z}} \end{vmatrix} \tag{1.47}$$

whereas from expressions (1.47) and (1.4), it follows that the curl of the vector field $\vec{a} = \{a_r, a_\varphi, a_z\}$ in the cylindrical coordinate system (r, φ, z) is given by the following expression:

$$\text{curl}\,\vec{a} = \nabla \times \vec{a} = \frac{1}{r} \cdot \begin{vmatrix} \vec{e}_r & r \cdot \vec{e}_\varphi & \vec{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \varphi} & \dfrac{\partial}{\partial z} \\ a_r & r \cdot a_\varphi & a_z \end{vmatrix} \tag{1.48}$$

and from expressions (1.41) and (1.5), it follows that the curl of the vector field $\vec{a} = \{a_r, a_\vartheta, a_\varphi\}$ in the spherical coordinate system (r, ϑ, φ) is given by the following expression:

$$\text{curl}\,\vec{a} = \nabla \times \vec{a} = \frac{1}{r^2 \cdot \sin\vartheta} \cdot \begin{vmatrix} \vec{e}_r & r \cdot \vec{e}_\vartheta & r \cdot \sin\vartheta \cdot \vec{e}_\varphi \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \vartheta} & \dfrac{\partial}{\partial \varphi} \\ a_r & r \cdot a_\vartheta & r \cdot \sin\vartheta \cdot a_\varphi \end{vmatrix} \tag{1.49}$$

1.11. Laplacian of a Scalar Field and Laplacian of a Vector Field

The Laplacian of the scalar field Ψ is given by the following expression:

$$\Delta \Psi = \nabla^2 \Psi = (\nabla \cdot \nabla)\Psi = \nabla \cdot (\nabla \Psi) = \text{div}\,(\text{grad}\,\Psi) \tag{1.50}$$

where Δ is the Laplace differential operator.

The Laplacian of the scalar field Ψ in an orthogonal 3D coordinate system (u_1, u_2, u_3) is given by the following expression:

$$\Delta \Psi = \frac{1}{h_1 \cdot h_2 \cdot h_3} \cdot \left[\frac{\partial}{\partial u_1}\left(p_1 \cdot \frac{\partial \Psi}{\partial u_1} \right) + \frac{\partial}{\partial u_2}\left(p_2 \cdot \frac{\partial \Psi}{\partial u_2} \right) + \frac{\partial}{\partial u_3}\left(p_3 \cdot \frac{\partial \Psi}{\partial u_3} \right) \right] \tag{1.51}$$

where:

$$p_1 = \frac{h_2 \cdot h_3}{h_1} \quad ; \quad p_2 = \frac{h_1 \cdot h_3}{h_2} \quad ; \quad p_3 = \frac{h_1 \cdot h_2}{h_3} \tag{1.52}$$

From expressions (1.51), (1.52), and (1.3), it follows that the Laplacian of the scalar field Ψ in the rectangular coordinate system (x, y, z) is given by the following expression:

$$\Delta \Psi = \frac{\partial^2 \Psi}{\partial x^2} + \frac{\partial^2 \Psi}{\partial y^2} + \frac{\partial^2 \Psi}{\partial z^2} \tag{1.53}$$

whereas from expressions (1.51), (1.52) and (1.4), it follows that the Laplacian of the scalar field Ψ in the cylindrical coordinate system (r, φ, z) is given by the following expression:

$$\Delta \Psi = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2} \cdot \frac{\partial^2 \Psi}{\partial \varphi^2} + \frac{\partial^2 \Psi}{\partial z^2} \tag{1.54}$$

and from expressions (1.51), (1.52) and (1.5), it follows that the Laplacian of the scalar field Ψ in the spherical coordinate system (r, ϑ, φ) is given by the following expression:

$$\Delta \Psi = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \Psi}{\partial r} \right) + \frac{1}{r^2 \cdot \sin\vartheta} \cdot \frac{\partial}{\partial \vartheta}\left(\sin\vartheta \cdot \frac{\partial \Psi}{\partial \vartheta} \right) + \frac{1}{r^2 \cdot \sin^2\vartheta} \cdot \frac{\partial^2 \Psi}{\partial \varphi^2} \tag{1.55}$$

The following expression is valid for the Laplacian of the vector field $\vec{a}$:

$$\Delta\vec{a} = (\nabla\cdot\nabla)\vec{a} = \nabla\cdot(\nabla\cdot\vec{a}) - \nabla\times(\nabla\times\vec{a}) = \text{grad (div } \vec{a}) - \text{curl (curl } \vec{a}) \tag{1.56}$$

1.12. Types of Vector Fields

Conservative vector field is a vector field that is the gradient of some scalar field:

$$\vec{a} = -\nabla\Psi = -\text{ grad}\,\Psi \tag{1.57}$$

where the scalar field Ψ is the potential of the conservative vector field $\vec{a}$.

A conservative vector field is always an irrotational vector field, and the vector field $\vec{a}$ is an *irrotational* vector field if:

$$\nabla\times\vec{a} = 0 \tag{1.58}$$

A vector field $\vec{a}$ is a *solenoidal* vector field if its divergence is equal to zero:

$$\nabla\cdot\vec{a} = 0 \tag{1.59}$$

1.13. Ostrogradsky-Green-Gauss Integral Theorem

The Ostrogradsky-Green-Gauss theorem is the first major integral theorem of vector analysis, also known as the Ostrogradsky-Gauss theorem, Green-Gauss theorem, Gauss's theorem or the divergence theorem. This theorem relates the volume integral of the divergence of a vector field $\vec{a}$ over volume V to the surface integral of the second kind of the same vector field $\vec{a}$ over the closed surface ∂V that bounds the volume V (Figure 1.3), and can be written as:

$$\oint_{\partial V}\vec{a}\cdot d\vec{S} = \int_{V}(\nabla\cdot\vec{a})\cdot dV = \int_{V}\text{div}\,\vec{a}\cdot dV \tag{1.60}$$

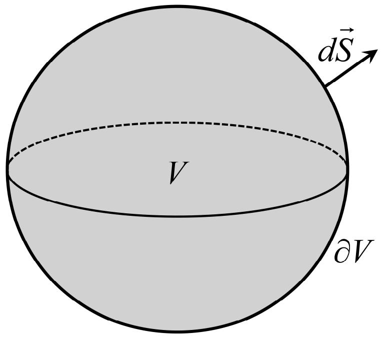


Figure 1.3. Volume V bounded by an oriented closed surface ∂V

According to expression (1.60), the Ostrogradsky-Green-Gauss theorem is a tool for converting between surface integrals of the second kind and volume integrals. This theorem transforms the flux of a vector field $\vec{a}$ through a closed surface ∂V into the volume integral of the divergence of the vector field $\vec{a}$ within the enclosed volume V. It follows directly that the flux of the vector field through the closed surface ∂V is a consequence of the existence of sources of the vector field $\vec{a}$ inside the volume V.

If the volume V tends to zero around point P (Figure 1.1), then the Ostrogradsky-Green-Gauss theorem takes the differential form given by expression (1.40), which is the definition of the divergence of the vector field $\vec{a}$.

1.14. Stokes' Integral Theorem

Stokes' theorem is the second important integral theorem of vector analysis. This theorem relates the line integral of the second kind of a vector field $\vec{a}$ along a closed curve *C*, which bounds the surface *S*, to the surface integral of the curl of that vector field $\vec{a}$ over the surface *S* (Figure 1.4). It can be expressed as follows:

$$\oint_C \vec{a} \cdot d\vec{\ell} = \int_S (\nabla \times \vec{a}) \cdot d\vec{S} = \int_S \operatorname{curl} \vec{a} \cdot d\vec{S} \tag{1.61}$$

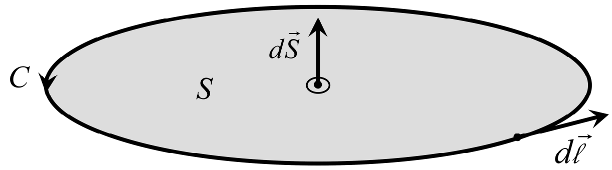


Figure 1.4. Oriented surface S bounded by an oriented closed curve C

According to equation (1.61), Stokes' integral theorem serves as a tool for converting between line integrals of the second kind and surface integrals of the second kind. Specifically, the theorem relates the circulation of a vector field $\vec{a}$ along a closed oriented curve *C* to the flux of the curl of that vector field $\vec{a}$ through the surface *S*, oriented according to the right-hand rule. It follows that the circulation of a vector field $\vec{a}$ along a closed oriented curve *C* implies the presence of rotational sources of the vector field $\vec{a}$ within the curve *C*.

If the surface *S* shrinks to zero around the point *P* (Figure 1.2), Stokes' theorem reduces to the differential form given by equation (1.45), which represents the definition of the curl of the vector field $\vec{a}$.

1.15. Solved Example

Example 1.1. For a scalar field $\Psi = x^3 \cdot y^2 \cdot z,$ determine the magnitude and direction of the gradient at the point $T(2, 3, 4)$ in the Cartesian coordinate system.

Solution:

According to (1.33), in the Cartesian coordinate system, the gradient of a scalar field Ψ is given by the expression:

$$\operatorname{grad} \Psi = \nabla \Psi = \frac{\partial \Psi}{\partial x} \cdot \vec{\mathrm{i}} + \frac{\partial \Psi}{\partial y} \cdot \vec{\mathrm{j}} + \frac{\partial \Psi}{\partial z} \cdot \vec{\mathrm{k}} \tag{1.62}$$

The partial derivatives of a scalar field Ψ are given by the following expressions:

$$\frac{\partial \Psi}{\partial x} = 3 \cdot x^2 \cdot y^2 \cdot z \quad ; \quad \frac{\partial \Psi}{\partial y} = 2 \cdot x^3 \cdot y \cdot z \quad ; \quad \frac{\partial \Psi}{\partial z} = x^3 \cdot y^2 \tag{1.63}$$

and the values of these derivatives at the given point *T* are:

$$\left.\frac{\partial \Psi}{\partial x}\right|_{\mathrm{T}} = 432 \quad ; \quad \left.\frac{\partial \Psi}{\partial y}\right|_{\mathrm{T}} = 192 \quad ; \quad \left.\frac{\partial \Psi}{\partial z}\right|_{\mathrm{T}} = 72 \tag{1.64}$$

The gradient of the scalar field Ψ at the point *T* is given by:

$$\nabla \Psi\big|_{\mathrm{T}} = 432 \cdot \vec{\mathrm{i}} + 192 \cdot \vec{\mathrm{j}} + 72 \cdot \vec{\mathrm{k}} \tag{1.65}$$

so the magnitude of the gradient is:

$$\left|(\nabla \Psi)_{T}\right| = \sqrt{432^2 + 192^2 + 72^2} = 478.1966123 \tag{1.66}$$

The direction of the gradient vector can be expressed in terms of the cosines of the angles it makes with the coordinate axes:

$$(\nabla \Psi)_0 = \cos\alpha \cdot \vec{i} + \cos\beta \cdot \vec{j} + \cos\gamma \cdot \vec{k} \tag{1.67}$$

and at the given point *T*, the following holds:

$$\cos\alpha = \frac{432}{478.1966123} = 0.9033941 \quad \Rightarrow \quad \alpha = 25.3921431^{o} \tag{1.68}$$

$$\cos\beta = \frac{192}{478.1966123} = 0.4015085 \quad \Rightarrow \quad \beta = 66.3274846^{o} \tag{1.69}$$

$$\cos\gamma = \frac{72}{478.1966123} = 0.1505657 \quad \Rightarrow \quad \gamma = 81.3402898^{o} \tag{1.70}$$

2. THE MOST IMPORTANT ELECTROMAGNETIC QUANTITIES

For the sake of clarity, it is helpful to list the most important electromagnetic quantities, along with their symbols and units of measurement, at the very beginning of the textbook. These include:

- $\vec{E}$ - electric field intensity vector, (V/m),
- $\vec{J}$ - vector of the surface density of the electric current, (A/m^2),
- $\vec{D}$ - electric displacement vector; electric flux density vector, (C/m^2),
- $\vec{P}$ - electric polarization vector, (C/m^2),
- $\vec{H}$ - magnetic field intensity vector, (A/m),
- $\vec{B}$ - magnetic flux density vector; magnetic induction vector (T ≡ Vs/m^2),
- $\vec{M}$ - magnetization vector, (A/m),
- $\vec{J}_{\rm s}$ - vector of the surface density of the source (impressed) electric current, (A/m^2),
- $g_{\rm st} = -\nabla \cdot \vec{J}_{\rm s}$ - volume density of the source leakage electric current*, (A/m^3),
- $\vec{A}$ - magnetic vector potential, (Tm ≡ Vs/m),
- φ - electric scalar potential, (V),
- $\varphi_{\rm m}$ - magnetic scalar potential, (A),
- ρ - volume electric charge density, (C/m^3),
- $\rho_{\rm s}$ - volume density of the source (impressed) electric charge, (C/m^3),
- σ - surface electric charge density, (C/m^2),
- λ - linear electric charge density, (C/m),
- q, Q - electric charge, (C),
- τ - linear density of the electric current, (A/m),
- κ - electrical conductivity of a medium†, (S/m),
- $\varepsilon = \varepsilon_0 \cdot \varepsilon_{\rm r}$ - permittivity of a medium, (F/m),
- $\varepsilon_0 = 8{,}8541878128 \times 10^{-12}$ F/m - vacuum permittivity,
- $\varepsilon_{\rm r} \geq 1$ - relative permittivity of a medium,
- $\mu = \mu_0 \cdot \mu_{\rm r}$ - magnetic permeability of a medium, (H/m ≡ Vs/Am),
- $\mu_0 = 4 \cdot \pi \times 10^{-7}$ H/m - vacuum magnetic permeability,
- $\mu_{\rm r}$ - magnetic permeability of a medium.

The quantity $g_{\rm st}$ was introduced into electromagnetism by the author of this textbook to provide a correct formulation of Maxwell's equations and the equations for the electric scalar potential in a conducting medium that contains impressed electric currents. It represents the volume density of the source leakage electric current injected into the surrounding medium by the source. An analogous quantity appears in fluid dynamics. This interpretation is consistent with the physical meaning of vector field divergence.

* A new physical quantity in electromagnetics, introduced by the author of this textbook.
† In electromagnetic theory, the symbol σ is commonly used to denote the electrical conductivity of a medium. However, in this textbook, the symbol σ is used to represent surface electric charge density.

3. CONSTITUTIVE RELATIONS

The constitutive equations describe the relationships between two or more electromagnetic field vectors through the properties of the medium. The following three constitutive equations are commonly used:

$$\vec{D} = \varepsilon_0 \cdot \vec{E} + \vec{P} \tag{3.1}$$

$$\vec{J}_\mathrm{c} = \kappa \cdot \vec{E} \tag{3.2}$$

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}\right) \tag{3.3}$$

where:

$\vec{J}_\mathrm{c}$ - the vector of the surface density of the conduction electric current,

$\vec{P}$ - the electric polarization vector,

$\vec{M}$ - the magnetization vector.

Therefore, in the general case, the following holds:

$$\vec{D} = \vec{D}\left(\vec{E}\right) \tag{3.4}$$

$$\vec{J}_\mathrm{c} = \vec{J}_\mathrm{c}\left(\vec{E}\right) \tag{3.5}$$

$$\vec{B} = \vec{B}\left(\vec{H}\right) \tag{3.6}$$

A medium can be:

- Isotropic or anisotropic,
- Linear or nonlinear,
- Homogeneous or inhomogeneous (heterogeneous).

If a medium is anisotropic, its properties vary with direction, whereas an isotropic medium has the same properties in all directions. For example, cold-rolled sheet metal exhibits different magnetic permeability along the rolling direction compared to the direction perpendicular to it.

If the medium is nonlinear, its characteristics depend on the intensity of the electric or magnetic field. For example, the magnetic nonlinearity of a ferromagnetic medium is described by a magnetic hysteresis curve. In contrast, if the medium is linear, its characteristic is independent of the field intensity, and the hysteresis curve reduces to a straight line.

If the medium is homogeneous, it has the same properties at every point. In contrast, the properties of an inhomogeneous medium vary from point to point.

If the medium is anisotropic, its properties are represented by tensors*, and the vectors on the left- and right-hand sides of the constitutive relations are not collinear. In that case, the following equations hold:

$$\begin{Bmatrix} D_\mathrm{x} \\ D_\mathrm{y} \\ D_\mathrm{z} \end{Bmatrix} = \begin{bmatrix} \varepsilon_{\mathrm{x,x}} & \varepsilon_{\mathrm{x,y}} & \varepsilon_{\mathrm{x,z}} \\ \varepsilon_{\mathrm{y,x}} & \varepsilon_{\mathrm{y,y}} & \varepsilon_{\mathrm{y,z}} \\ \varepsilon_{\mathrm{z,x}} & \varepsilon_{\mathrm{z,y}} & \varepsilon_{\mathrm{z,z}} \end{bmatrix} \cdot \begin{Bmatrix} E_\mathrm{x} \\ E_\mathrm{y} \\ E_\mathrm{z} \end{Bmatrix} \tag{3.7}$$

$$\begin{Bmatrix} J_\mathrm{cx} \\ J_\mathrm{cy} \\ J_\mathrm{cz} \end{Bmatrix} = \begin{bmatrix} \kappa_{\mathrm{x,x}} & \kappa_{\mathrm{x,y}} & \kappa_{\mathrm{x,z}} \\ \kappa_{\mathrm{y,x}} & \kappa_{\mathrm{y,y}} & \kappa_{\mathrm{y,z}} \\ \kappa_{\mathrm{z,x}} & \kappa_{\mathrm{z,y}} & \kappa_{\mathrm{z,z}} \end{bmatrix} \cdot \begin{Bmatrix} E_\mathrm{x} \\ E_\mathrm{y} \\ E_\mathrm{z} \end{Bmatrix} \tag{3.8}$$

* A tensor is an algebraic object that describes a multilinear relationship between sets of algebraic quantities associated with a vector space, whereas a matrix is a second-order tensor.

$$\begin{Bmatrix} B_x \\ B_y \\ B_z \end{Bmatrix} = \begin{bmatrix} \mu_{x,x} & \mu_{x,y} & \mu_{x,z} \\ \mu_{y,x} & \mu_{y,y} & \mu_{y,z} \\ \mu_{z,x} & \mu_{z,y} & \mu_{z,z} \end{bmatrix} \cdot \begin{Bmatrix} H_x \\ H_y \\ H_z \end{Bmatrix} \tag{3.9}$$

where:

D_x, D_y, D_z - the components of the electric displacement vector,

E_x, E_y, E_z - the components of the electric field intensity vector,

J_{cx}, J_{cy}, J_{cz} - the components of the vector of the surface density of the conduction electric current,

B_x, B_y, B_z - the components of the magnetic flux density vector,

H_x, H_y, H_z - the components of the magnetic field intensity vector.

In equations (3.7) - (3.9), the properties of the medium are described by tensors (3×3 matrices). The permittivity, electrical conductivity, and magnetic permeability tensors are symmetric, which means, for example, that $\varepsilon_{x,y} = \varepsilon_{y,x}$. By rotating the coordinate system, these tensors can be transformed into diagonal matrices.

For a nonlinear medium, the history of events within the medium also plays an important role, and in that case, the following equations hold:

$$\varepsilon = \varepsilon(E) \quad ; \quad \kappa = \kappa(E) \quad ; \quad \mu = \mu(H) \tag{3.10}$$

If the medium is both nonlinear and anisotropic, its characteristics are described by tensor functions, i.e., the components of the relevant tensors depend on the intensity of the corresponding field.

If the medium is isotropic, its properties are represented by scalar constants or scalar functions. In a linear and isotropic medium, the constitutive equations can be written as:

$$\vec{D} = \varepsilon \cdot \vec{E} \;\; (D = \varepsilon \cdot E) \; ; \quad \vec{J} = \kappa \cdot \vec{E} \;\; (J = \kappa \cdot E) \quad ; \quad \vec{B} = \mu \cdot \vec{H} \;\; (B = \mu \cdot H) \tag{3.11}$$

In this case, the medium properties are scalar constants or scalar functions. If the medium is linear, isotropic, and homogeneous, its properties are scalar constants.

With respect to linearity, isotropy, and homogeneity, the properties of media of the same type can still differ from one another.

4. VECTOR OF THE SURFACE DENSITY OF THE TOTAL ELECTRIC CURRENT

In a stationary medium, the vector of the surface density of the total electric current* can be expressed as:

$$\vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \vec{J}_{\text{conv}} + \frac{\partial \vec{D}}{\partial t} \tag{4.1}$$

where:

$\vec{J}_{\text{tot}}$ - the vector of the surface density of the total electric current,

$\vec{J}_{\text{s}}$ - the vector of the surface density of the source (impressed) electric current,

$\vec{J}_{\text{c}}$ - the vector of the surface density of the conduction electric current,

$\vec{J}_{\text{conv}}$ - the vector of the surface density of the convection electric current,

$\vec{J}_{\text{disp}}$ - the vector of the surface density of the displacement electric current.

The displacement electric current arises from the time variation of the electric displacement vector, and its surface density is described by the partial derivative of the electric displacement vector with respect to time *t*:

$$\vec{J}_{\text{disp}} = \frac{\partial \vec{D}}{\partial t} \tag{4.2}$$

According to equation (3.2), the surface density of the conduction electric current can be expressed using the following constitutive relation:

$$\vec{J}_{\text{c}} = \kappa \cdot \vec{E} \tag{4.3}$$

which represents Ohm's law in differential form. In a linear, isotropic, and homogeneous (LIH) medium, the electrical conductivity κ is a scalar constant.

Surface density of the convection electric current can be written as:

$$\vec{J}_{\text{conv}} = \rho \cdot \vec{v}_{\text{c}} \tag{4.4}$$

and represents the surface density of the electric current resulting from the motion of electric charge with volume density ρ at velocity $\vec{v}_{\text{c}}$, under the influence of forces that are not electromagnetic in nature.

Conduction electric current is the movement of charged particles (electrons, positive and negative ions, etc.) under the action of electromagnetic forces. Convection electric current, on the other hand, refers to the movement of charged particles caused by forces that are not electromagnetic in nature. Convection currents typically occur in gases and liquids.

This textbook does not address electric currents in gases and liquids; therefore, problems involving convection electric currents will not be considered. When convection electric currents are excluded, the vector of the surface density of the total electric current in *stationary media* can be described by the following expression:

$$\vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \vec{J}_{\text{disp}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \frac{\partial \vec{D}}{\partial t} \tag{4.5}$$

---

* The first complication in the physically correct naming of linear, surface, and volume densities of the electric current is the use of the term surface current density to denote the linear density of the surface electric current, measured perpendicular to the current direction - where surface electric current refers to an electric current flowing along a surface. Surface current density is a vector quantity whose direction coincides with that of the surface electric current. The second complication is the inconsistent use of the terms electric current density and volume electric current density to describe surface or volume densities of the electric current. To avoid physical ambiguity, this textbook adopts longer, unambiguous names for all three types of electric current densities.

If convection electric currents are excluded from consideration, the surface density of the total electric current in a *slowly moving medium* - moving at a constant relative velocity $\vec{v}$ with respect to the observer – can be described by the following expression:

$$\vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \vec{J}_{\text{disp}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \frac{d\vec{D}}{dt} \tag{4.6}$$

where, in comparison to expression (4.5), the partial time derivative is replaced by the full time derivative, given by the following expression:

$$\frac{d\vec{D}}{dt} = \frac{\partial D}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial\vec{D}}{\partial y}\cdot\frac{\partial y}{\partial t} + \frac{\partial\vec{D}}{\partial z}\cdot\frac{\partial z}{\partial t} + \frac{\partial\vec{D}}{\partial t} = \left(\vec{v}\cdot\nabla\right)\vec{D} + \frac{\partial\vec{D}}{\partial t} \tag{4.7}$$

where the nabla operator in the Cartesian coordinate system is given by the following expression:

$$\nabla \equiv \vec{\nabla} = \frac{\partial}{\partial x}\cdot\vec{\mathrm{i}} + \frac{\partial}{\partial y}\cdot\vec{\mathrm{j}} + \frac{\partial}{\partial z}\cdot\vec{\mathrm{k}} \tag{4.8}$$

Furthermore:

$$\nabla\times\left(\vec{v}\times\vec{D}\right) = \vec{v}\cdot\left(\nabla\cdot\vec{D}\right) - \left(\vec{v}\cdot\nabla\right)\vec{D} \tag{4.9}$$

where $\vec{v}$ is is the vector of the constant velocity relative to the observer.

From expressions (4.9) and (4.7), the following expression can be readily obtained:

$$\frac{d\vec{D}}{dt} = -\nabla\times\left(\vec{v}\times\vec{D}\right) + \vec{v}\cdot\left(\nabla\cdot\vec{D}\right) + \frac{\partial\vec{D}}{\partial t} \tag{4.10}$$

It follows that the surface density of the total electric current in a slowly moving medium, which moves with a constant relative velocity $\vec{v}$ with respect to the observer, can be written as:

$$\vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \frac{d\vec{D}}{dt} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} - \nabla\times\left(\vec{v}\times\vec{D}\right) + \vec{v}\cdot\left(\nabla\cdot\vec{D}\right) + \frac{\partial\vec{D}}{\partial t} \tag{4.11}$$

For the vector of the surface density of the total electric current, the continuity equation is valid and can be written as:

$$\operatorname{div}\vec{J}_{\text{tot}} = \nabla\cdot\vec{J}_{\text{tot}} = 0 \tag{4.12}$$

which is consistent with Maxwell's first differential equation, from which Maxwell's third differential equation can be derived.

5. PHASOR TRANSFORMATION OF SINUSOIDAL QUANTITIES

In both electric circuit analysis and electromagnetism, sinusoidal (time-harmonic) quantities commonly appear. These include scalar functions of time, vector functions of time, scalar fields, and vector fields. Sinusoidal functions are described by sine or cosine functions.

Sinusoidal quantities in the time domain can be mapped to the phasor domain using the phasor transform, developed in the late 19th century by American mathematician and electrical engineer Charles Proteus Steinmetz while working at General Electric. He was inspired by Heaviside's operator calculus, which closely resembles the Laplace transform.

Let the sinusoidal electric voltage be described by the following scalar function of time:

$$u = u(t) = U_{\max} \cdot \cos(\omega \cdot t + \theta) = \sqrt{2} \cdot U \cdot \cos(\omega \cdot t + \theta) \tag{5.1}$$

where:

u - the instantaneous value of the electric voltage, (V),

t - the time, (s),

$U_{\max}$ - the maximum value (peak value) of the electric voltage, (V),

U - the root-mean-square (RMS; effective) electric voltage, (V),

ω - the angular frequency, (rad/s),

θ - the phase angle, (rad).

Euler's formula, which relates sine and cosine functions to complex exponential functions, can be written as:

$$e^{j\cdot(\omega\cdot t+\theta)} = \cos(\omega \cdot t + \theta) + j \cdot \sin(\omega \cdot t + \theta) \tag{5.2}$$

where $j = \sqrt{-1}$ is the imaginary unit.

Thus, the cosine is the real part, and the sine is the imaginary part of the exponential function:

$$\cos(\omega \cdot t + \theta) = \mathrm{Re}\left[e^{j\cdot(\omega\cdot t+\theta)}\right] \quad ; \quad \sin(\omega \cdot t + \theta) = \mathrm{Im}\left[e^{j\cdot(\omega\cdot t+\theta)}\right] \tag{5.3}$$

It follows that:

$$u = \sqrt{2} \cdot U \cdot \cos(\omega \cdot t + \theta) = \mathrm{Re}\left[\sqrt{2} \cdot U \cdot e^{j\cdot(\omega\cdot t+\theta)}\right] = \mathrm{Re}\left[\sqrt{2} \cdot \overline{U} \cdot e^{j\cdot\omega\cdot t}\right] = \mathrm{Re}\left[\tilde{U}_{\max}\right] \tag{5.4}$$

$$\overline{U} = U \cdot e^{j\cdot\theta} \tag{5.5}$$

$$\overline{U}_{\max} = \sqrt{2} \cdot \overline{U} = U_{\max} \cdot e^{j\cdot\theta} \tag{5.6}$$

$$\tilde{U}_{\max} = \sqrt{2} \cdot \overline{U} \cdot e^{j\cdot\omega\cdot t} = \overline{U}_{\max} \cdot e^{j\cdot\omega\cdot t} \tag{5.7}$$

where:

$\overline{U}$ - the electric voltage phasor whose modulus is the RMS (effective) electric voltage,

$\overline{U}_{\max}$ - the electric voltage phasor whose modulus is the maximum value of the electric voltage,

$\tilde{U}_{\max}$ - the rotating phasor whose modulus is the maximum value of the electric voltage.

A *phasor* is a complex number (scalar) expressed in Euler notation. Although a phasor is sometimes referred to as a vector, this terminology is not appropriate – especially because this textbook also uses phasors of vector quantities. A phasor is, in fact, a phase vector. The rotating phasor is a complex function in Euler notation. However, rotating phasors will not be used in this textbook.

The electric voltage phasor u (corresponding to a sinusoidal scalar function) can be defined in four different ways:

- The electric voltage is described in the time domain by a cosine function, and the modulus of the phasor $\overline{U}$ is the RMS (effective) value of the electric voltage;
- The electric voltage is described in the time domain by a sine function, and the modulus of the phasor $\overline{U}$ is the RMS (effective) value of the electric voltage;
- The electric voltage is described in the time domain by a cosine function, and the modulus of the phasor $\overline{U}_{\max}$ is the maximum value of the electric voltage;
- The electric voltage is described in the time domain by a sine function, and the modulus of the phasor $\overline{U}_{\max}$ is the maximum value of the electric voltage.

This textbook adopts the first definition of the phasor of a sinusoidal scalar function, which states that the sinusoidal function in the time domain is described by a cosine function, and the modulus of the phasor corresponds to the RMS (effective) value of the sinusoidal function.

The phasor representation of sinusoidal quantities enables the transformation of integro-differential equations in the time domain into algebraic equations in the phasor domain. For the electric voltage described by expressions (5.1) and (5.5), the following relation holds:

$$u \quad \rightarrow \quad \overline{U} \tag{5.8}$$

$$\frac{du}{dt} \quad \rightarrow \quad j \cdot \omega \cdot \overline{U} \tag{5.9}$$

$$\frac{d^2u}{dt^2} \quad \rightarrow \quad (j \cdot \omega)^2 \cdot \overline{U} = -\omega^2 \cdot \overline{U} \tag{5.10}$$

$$\int u \cdot dt \quad \rightarrow \quad \frac{1}{j \cdot \omega} \cdot \overline{U} = -\frac{j}{\omega} \cdot \overline{U} \tag{5.11}$$

because it is:

$$\frac{de^{j \cdot \omega \cdot t}}{dt} = j \cdot \omega \cdot e^{j \cdot \omega \cdot t} \quad ; \quad \frac{d^2 e^{j \cdot \omega \cdot t}}{dt^2} = (j \cdot \omega)^2 \cdot e^{j \cdot \omega \cdot t} \tag{5.12}$$

$$\int e^{j \cdot \omega \cdot t} \cdot dt = \frac{e^{j \cdot \omega \cdot t}}{j \cdot \omega} + C \tag{5.13}$$

Let the sinusoidal vector (a vector function of time) in the time domain be described in the Cartesian coordinate system by the following expression:

$$\vec{a} = a_{\mathrm{x}} \cdot \vec{\mathrm{i}} + a_{\mathrm{y}} \cdot \vec{\mathrm{j}} + a_{\mathrm{z}} \cdot \vec{\mathrm{k}} = \left\{ a_{\mathrm{x}}, a_{\mathrm{y}}, a_{\mathrm{z}} \right\} \tag{5.14}$$

where:

$$a_{\mathrm{x}} = \sqrt{2} \cdot A_{\mathrm{x}} \cdot \cos(\omega \cdot t + \theta_{\mathrm{x}}) \tag{5.15}$$

$$a_{\mathrm{y}} = \sqrt{2} \cdot A_{\mathrm{y}} \cdot \cos(\omega \cdot t + \theta_{\mathrm{y}}) \tag{5.16}$$

$$a_{\mathrm{z}} = \sqrt{2} \cdot A_{\mathrm{z}} \cdot \cos(\omega \cdot t + \theta_{\mathrm{z}}) \tag{5.17}$$

where A_{x}, A_{y} i A_{z} are the RMS (effective) values of the components of the vector $\vec{a}$.

Thus, the components of the vector $\vec{a}$ are sinusoidal scalar functions of time, which can be represented by phasors in the phasor domain. This means that a sinusoidal vector can be mapped from the time domain to the phasor domain as follows:

$$\underline{\vec{a}} = \overline{A}_x \cdot \vec{i} + \overline{A}_y \cdot \vec{j} + \overline{A}_z \cdot \vec{k} = \left\{ \overline{A}_x , \overline{A}_y , \overline{A}_z \right\} \tag{5.18}$$

where:

$\underline{\vec{a}}$ - the *phasor of the vector* $\vec{a}$,

$\overline{A}_x$ - the phasor of the *x*-component of the vector $\vec{a}$,

$\overline{A}_y$ - the phasor of the *y*-component of the vector $\vec{a}$,

$\overline{A}_z$ - the phasor of the *z*-component of the vector $\vec{a}$.

The phasor of a vector is a complex vector. In this textbook, vector phasors are used in which the moduli of the components correspond to the RMS (effective) values of the associated cosine functions.

In a cylindrical coordinate system, the phasor of the vector $\vec{a}$ is described by the following expression:

$$\underline{\vec{a}} = \overline{A}_r \cdot \vec{e}_r + \overline{A}_\varphi \cdot \vec{e}_\varphi + \overline{A}_z \cdot \vec{e}_z = \left\{ \overline{A}_r , \overline{A}_\varphi , \overline{A}_z \right\} \tag{5.19}$$

whereas in a spherical coordinate system, the phasor of the vector $\vec{a}$ is described by the following expression:

$$\underline{\vec{a}} = \overline{A}_r \cdot \vec{e}_r + \overline{A}_\vartheta \cdot \vec{e}_\vartheta + \overline{A}_\varphi \cdot \vec{e}_\varphi = \left\{ \overline{A}_r , \overline{A}_\vartheta , \overline{A}_\varphi \right\} \tag{5.20}$$

Phasors of sinusoidal scalar fields formally have the same form as phasors of sinusoidal scalar functions that depend only on time, but the magnitudes and phase angles of the phasors of sinusoidal scalar fields depend on the spatial coordinates of the chosen coordinate system.

Similarly, phasors of sinusoidal vector fields have the same formal structure as phasors of time-dependent sinusoidal vector functions, but their magnitudes and phase angles vary with spatial coordinates.

Since sinusoidal functions can be expressed using either sine or cosine functions of time, it is useful to know the following two trigonometric identities:

$$\sin(\omega \cdot t + \theta) = \cos(\omega \cdot t + \theta - \pi/2) = \cos\left(\omega \cdot t + \theta - 90^{\circ}\right) \tag{5.21}$$

$$\cos(\omega \cdot t + \theta) = \sin(\omega \cdot t + \theta + \pi/2) = \sin\left(\omega \cdot t + \theta + 90^{\circ}\right) \tag{5.22}$$

6. MAXWELL'S EQUATIONS IN THE TIME DOMAIN

Maxwell's equations are the fundamental equations of classical electromagnetism, or classical electrodynamics. They can be expressed in differential form (valid at a single point) and in integral form (valid over a 3D region). The equations are named after Scottish physicist James Clerk Maxwell, whose most significant works are his two publications: *A Dynamical Theory of the Electromagnetic Field* (1864) and the two-volume *A Treatise on Electricity and Magnetism* (1873). Building on previous theoretical and experimental knowledge, Maxwell developed electromagnetic theory, unifying the theories of electricity and magnetism. Maxwell's equations reveal the wave nature of the electromagnetic field, and his electromagnetic theory of light demonstrates that light is an electromagnetic wave of very high frequency.

After Maxwell's death, his ideas and equations were refined and modified to make them simpler and more accessible. The most notable contributors to this process were the German physicist Heinrich Rudolf Hertz, the Irish physicist George Francis FitzGerald, the British physicist Oliver Joseph Lodge and the English self-taught electromagnetist, mathematician, and physicist Oliver Heaviside. Heaviside referred to himself, FitzGerald, and Lodge as "The Maxwellians".

In his electromagnetic theory, Maxwell formulated a system of twenty differential equations. These equations were later condensed by Oliver Heaviside in 1884 into the four well-known vector differential equations. Independently, American physicist Josiah Willard Gibbs and German physicist Heinrich Rudolf Hertz arrived at the same idea of reducing Maxwell's system to four vector equations. It is therefore not surprising that the four differential equations – now universally known as Maxwell's equations – were once also referred to as the Hertz-Heaviside equations, the Maxwell-Hertz equations, or the Maxwell-Heaviside equations.

6.1. Maxwell's Differential Equations in a Slowly Moving Perfect Dielectric

A perfect dielectric, or insulator, is a dielectric material with zero electrical conductivity ($\kappa = 0$). In other words, a perfect dielectric is a non-conducting medium.

Maxwell's differential equations in a slowly moving perfect dielectric containing sources, which moves at a constant relative velocity $\vec{v}$ with respect to the observer, can be written as follows:

$$\text{curl}\,\vec{H} = \nabla\times\vec{H} = \vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \frac{d\vec{D}}{dt} \tag{6.1}$$

$$\text{curl}\,\vec{E} = \nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \tag{6.2}$$

$$\text{div}\,\vec{D} = \nabla\cdot\vec{D} = \rho_{\text{s}} \tag{6.3}$$

$$\text{div}\vec{B} = \nabla\cdot\vec{B} = 0 \tag{6.4}$$

where:

$\vec{H}$ - the magnetic field intensity vector,

$\vec{E}$ - the electric field intensity vector,

$\vec{D}$ - the electric displacement vector,

$\vec{B}$ - the magnetic flux density vector,

$\vec{J}_{\text{tot}}$ - the vector of the surface density of the total electric current,

$\vec{J}_{\text{s}}$ - the vector of the surface density of the source (impressed) electric current,

ρ_{s} - the volume density of the source (impressed) electric charge.

According to equation (4.10), the following expression holds for the total derivative of the electric displacement vector with respect to time:

$$\frac{d\vec{D}}{dt} = -\nabla\times\left(\vec{v}\times\vec{D}\right) + \vec{v}\cdot\left(\nabla\cdot\vec{D}\right) + \frac{\partial\vec{D}}{\partial t} \tag{6.5}$$

which, in a slowly moving perfect dielectric, according to (6.3) and (6.5), can be written as:

$$\frac{d\vec{D}}{dt} = -\nabla\times\left(\vec{v}\times\vec{D}\right) + \rho_{\mathrm{s}}\cdot\vec{v} + \frac{\partial\vec{D}}{\partial t} \tag{6.6}$$

where $\vec{v}$ is the vector of constant relative velocity with respect to the observer.

Analogous to expression (6.5), the following expression holds for the full derivative of the magnetic flux density vector with respect to time:

$$\frac{d\vec{B}}{dt} = -\nabla\times\left(\vec{v}\times\vec{B}\right) + \vec{v}\cdot\left(\nabla\cdot\vec{B}\right) + \frac{\partial\vec{B}}{\partial t} \tag{6.7}$$

which, in a slowly moving perfect dielectric, according to equations (6.4) and (6.7), can be written as:

$$\frac{d\vec{B}}{dt} = -\nabla\times\left(\vec{v}\times\vec{B}\right) + \frac{\partial\vec{B}}{\partial t} \tag{6.8}$$

where $\vec{v}$ is the vector of constant relative velocity with respect to the observer.

If expressions (6.6) and (6.8) are substituted into Maxwell's differential equations (6.1) and (6.2), then Maxwell's differential equations in a slowly moving perfect dielectric can be written as follows:

$$\nabla\times\vec{H} = \vec{J}_{\mathrm{s}} + \rho_{\mathrm{s}}\cdot\vec{v} + \frac{\partial\vec{D}}{\partial t} - \nabla\times\left(\vec{v}\times\vec{D}\right) \tag{6.9}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} + \nabla\times\left(\vec{v}\times\vec{B}\right) \tag{6.10}$$

$$\nabla\cdot\vec{D} = \rho_{\mathrm{s}} \tag{6.11}$$

$$\nabla\cdot\vec{B} = 0 \tag{6.12}$$

Which, in a linear, isotropic, and homogeneous (LIH) perfect dielectric, can be written as follows:

$$\nabla\times\vec{H} = \vec{J}_{\mathrm{s}} + \rho_{\mathrm{s}}\cdot\vec{v} + \varepsilon\cdot\frac{\partial\vec{E}}{\partial t} - \varepsilon\cdot\nabla\times\left(\vec{v}\times\vec{E}\right) \tag{6.13}$$

$$\nabla\times\vec{E} = -\mu\cdot\frac{\partial\vec{H}}{\partial t} + \mu\cdot\nabla\times\left(\vec{v}\times\vec{H}\right) \tag{6.14}$$

$$\nabla\cdot\vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{6.15}$$

$$\nabla\cdot\vec{H} = 0 \tag{6.16}$$

where μ and ε are the properties of the perfect LIH dielectric.

Since the following relation is valid:

$$\mathrm{div}\left(\mathrm{curl}\,\vec{a}\right) = \nabla\cdot\left(\nabla\times\vec{a}\right) = 0 \quad ; \quad \forall\vec{a} \tag{6.17}$$

It follows from Maxwell's first differential equation (6.1) and equation (6.9) that the continuity equation in a slowly moving perfect dielectric is given by the following expression:

$$\nabla \cdot \left(\vec{J}_{\mathrm{s}} + \frac{d\vec{D}}{dt} \right) = \nabla \cdot \left(\vec{J}_{\mathrm{s}} + \rho_{\mathrm{s}} \cdot \vec{v} + \frac{\partial \vec{D}}{\partial t} \right) = 0 \tag{6.18}$$

from which it follows that:

$$\nabla \cdot \vec{J}_{\mathrm{s}} = -\frac{d(\nabla \cdot \vec{D})}{dt} \tag{6.19}$$

Since, according to expression (6.3), in a perfect dielectric:

$$\nabla \cdot \vec{D} = \rho_{\mathrm{s}} \tag{6.20}$$

from expressions (6.6) and (6.17) - (6.20), it follows that:

$$\nabla \cdot \vec{J}_{\mathrm{s}} = -\frac{d\rho_{\mathrm{s}}}{dt} = -\frac{\partial \rho_{\mathrm{s}}}{\partial t} - \nabla \cdot (\rho_{\mathrm{s}} \cdot \vec{v}) = -\frac{\partial \rho_{\mathrm{s}}}{\partial t} - \vec{v} \cdot (\nabla \rho_{\mathrm{s}}) \tag{6.21}$$

and it represents the continuity equation that connects the sources of the electromagnetic field.

6.2. Maxwell's Differential Equations in a Stationary Perfect Dielectric

From Maxwell's differential equations (6.1) - (6.4) for a slowly moving perfect dielectric containing sources, one can easily obtain the corresponding equations for a stationary perfect dielectric (a stationary system). In this case, the full-time derivative should be replaced by the partial time derivative. Accordingly, Maxwell's differential equations in a stationary perfect dielectric can be written as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{tot}} = \vec{J}_{\mathrm{s}} + \frac{\partial \vec{D}}{\partial t} \tag{6.22}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6.23}$$

$$\nabla \cdot \vec{D} = \rho_{\mathrm{s}} \tag{6.24}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.25}$$

and these are the four well-known Maxwell differential equations.

From Maxwell's differential equations (6.13) - (6.16) in a slowly moving perfect LIH dielectric, the following Maxwell's differential equations in a stationary perfect LIH dielectric can easily be obtained for $\vec{v} = 0$:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \tag{6.26}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{6.27}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{6.28}$$

$$\nabla \cdot \vec{H} = 0 \tag{6.29}$$

In fact, the condition $\vec{v} = 0$ is included in Maxwell's differential equations (6.13) and (6.14).

From expressions (6.20) and (6.21), it follows that the continuity equation for a stationary perfect dielectric is:

$$\nabla\cdot\vec{J}_{\mathrm{s}} = -\frac{\partial\rho_{\mathrm{s}}}{\partial t} \tag{6.30}$$

which expresses the law of conservation of electric charge in mathematical form.

6.3. Maxwell's Differential Equations in a Slowly Moving Conducting Medium

Maxwell differential equations for a slowly moving conducting medium containing sources, moving at constant relative velocity $\vec{v}$ with respect to the observer, take the following form:

$$\nabla\times\vec{H} = \vec{J}_{\mathrm{tot}} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \frac{d\vec{D}}{dt} \tag{6.31}$$

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \tag{6.32}$$

$$\nabla\cdot\left(\vec{J}_{\mathrm{c}} + \frac{d\vec{D}}{dt}\right) = -\nabla\cdot\vec{J}_{\mathrm{s}} = g_{\mathrm{st}} \tag{6.33}$$

$$\nabla\cdot\vec{B} = 0 \tag{6.34}$$

where $\vec{J}_{\mathrm{c}}$ is the vector of the surface density of the conduction electric current, whereas $g_{\mathrm{st}} = -\nabla\cdot\vec{J}_{\mathrm{s}}$ is the volume density of the source leakage electric current*.

There is no free charge in a conducting medium $(\rho_{\mathrm{s}} = 0)$. If a free charge is introduced, it is rapidly dissipated or redistributed over the surface of the conductor. According to equations (6.32) and (6.33), in a conducting medium, the sources of the electric field can be a time-varying magnetic field and a leakage electric current injected by an independent source into the surrounding conducting medium.

A novelty in this textbook is Maxwell’s third differential equation in a slowly moving conducting medium (6.33), which is derived from the first of Maxwell’s differential equations for such a medium, in such a way that:

$$\nabla\cdot\vec{J}_{\mathrm{tot}} = \nabla\cdot\left(\vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \frac{d\vec{D}}{dt}\right) = \nabla\cdot\left(\nabla\times\vec{H}\right) = 0 \tag{6.35}$$

Therefore, the divergence of the curl of any vector field is equal to zero.

Maxwell’s differential equations in a slowly moving perfect dielectric, (6.1) - (6.4), represent a special case of Maxwell’s differential equations in a slowly moving conducting medium, (6.31) - (6.34). This is a physical requirement that must be satisfied. If $\vec{J}_{\mathrm{c}} = 0$ is substituted into equation (6.31), equation (6.1) is obtained. Equations (6.2) and (6.32) are identical, as are equations (6.4) and (6.34). It remains to be shown that equation (6.3) is a special case of equation (6.33). In a slowly moving perfect dielectric, equation (6.33) can be written as:

$$\nabla\cdot\frac{d\vec{D}}{dt} = \frac{d(\nabla\cdot\vec{D})}{dt} = -\nabla\cdot\vec{J}_{\mathrm{s}} \tag{6.36}$$

From (6.30) and (6.36), the following equation follows:

$$\frac{d(\nabla\cdot\vec{D})}{dt} = \frac{d\rho_{\mathrm{s}}}{dt} \quad\Rightarrow\quad \nabla\cdot\vec{D} = \rho_{\mathrm{s}} \tag{6.37}$$

* The components of the vector divergence are the derivatives of the vector components in the Cartesian coordinates *x*, *y*, and *z*. So, it is easy to conclude that the unit of g_{st} is $\mathrm{A/m^3}$, which indicates its physical meaning.

If equations (6.5) and (6.8) are substituted into equations (6.31) and (6.32), Maxwell's differential equations in a slowly moving conducting medium can then be written as:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{D}\right) \tag{6.38}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla \times \left(\vec{v} \times \vec{B}\right) \tag{6.39}$$

$$\nabla \cdot \left(\vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} \right) = g_{\mathrm{st}} \tag{6.40}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.41}$$

which, in a slowly moving conducting LIH medium, can be written as:

$$\nabla \times \left(\vec{H} + \varepsilon \cdot \vec{v} \times \vec{E}\right) = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} + \varepsilon \cdot \vec{v} \cdot \left(\nabla \cdot \vec{E}\right) + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} - \varepsilon \cdot \nabla \times \left(\vec{v} \times \vec{E}\right) \tag{6.42}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} + \mu \cdot \nabla \times \left(\vec{v} \times \vec{H}\right) \tag{6.43}$$

$$\nabla \cdot \left(\kappa \cdot \vec{E} + \varepsilon \cdot \vec{v} \cdot \left(\nabla \cdot \vec{E}\right) + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \right) = g_{\mathrm{st}} \tag{6.44}$$

$$\nabla \cdot \vec{H} = 0 \tag{6.45}$$

where μ, κ, and ε are the physical properties of the conducting LIH medium.

6.4. Maxwell's Differential Equations in a Stationary Conducting Medium

From Maxwell's differential equations (6.31) - (6.34) for a slowly moving conducting medium, the corresponding equations for a stationary conducting medium (a stationary system) can be readily obtained. In this case, the full-time derivative is replaced by the partial time derivative. Thus, Maxwell's differential equations in a stationary conducting medium can be written as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{tot}} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \frac{\partial \vec{D}}{\partial t} \tag{6.46}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{6.47}$$

$$\nabla \cdot \left(\vec{J}_{\mathrm{c}} + \frac{\partial \vec{D}}{\partial t} \right) = g_{\mathrm{st}} \tag{6.48}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.49}$$

Maxwell's differential equations in a stationary perfect dielectric (6.22) - (6.25) represent a special case of Maxwell's differential equations in a stationary conducting medium (6.46) - (6.49).

From Maxwell's differential equations (6.46) - (6.49), Maxwell's differential equations for a stationary conducting LIH medium can be readily obtained and written as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \tag{6.50}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{6.51}$$

$$\nabla \cdot \left(\kappa \cdot \vec{E} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \right) = \left(\kappa + \varepsilon \cdot \frac{\partial}{\partial t} \right) \left(\nabla \cdot \vec{E} \right) = g_{\mathrm{st}} \tag{6.52}$$

$$\nabla \cdot \vec{H} = 0 \tag{6.53}$$

where μ, κ, and ε are the physical properties of the conducting LIH medium.

6.5. Maxwell's Differential Equations in a Slowly Moving Good Conductor

In a good conductor, displacement electric currents can generally be neglected; that is, the following approximation is used

$$\frac{d\vec{D}}{dt} = 0 \tag{6.54}$$

because:

$$J_{\mathrm{c}} >> \frac{dD}{dt} \tag{6.55}$$

From expressions (6.31) - (6.34), (6.54), and (6.8), Maxwell’s differential equations for a slowly moving good conductor can be readily obtained and written as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{tot}} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} \tag{6.56}$$

$$\nabla \times \vec{E} = -\frac{d\vec{B}}{dt} = -\frac{\partial \vec{B}}{\partial t} + \nabla \times \left(\vec{v} \times \vec{B} \right) \tag{6.57}$$

$$\nabla \cdot \vec{J}_{\mathrm{c}} = g_{\mathrm{st}} \tag{6.58}$$

$$\nabla \cdot \vec{B} = 0 \tag{6.59}$$

which, in a slowly moving good LIH conductor, can be written as:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} \tag{6.60}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{d\vec{H}}{dt} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} - \mu \cdot \nabla \times \left(\vec{v} \times \vec{H} \right) \tag{6.61}$$

$$\nabla \cdot \vec{E} = \frac{g_{\mathrm{st}}}{\kappa} \tag{6.62}$$

$$\nabla \cdot \vec{H} = 0 \tag{6.63}$$

6.6. Maxwell's Differential Equations in a Stationary Good Conductor

From Maxwell's differential equations (6.56) - (6.59) for a slowly moving conducting medium, the corresponding equations for a stationary good conductor (stationary system) can be readily obtained. In equation (6.57), the full time derivative should be replaced by the partial time derivative, or equivalently, the condition $\vec{v}=0$ should be applied. Thus, Maxwell's differential equations for a stationary good conductor can be written as:

$$\nabla\times\vec{H}=\vec{J}_{\mathrm{tot}}=\vec{J}_{\mathrm{s}}+\vec{J}_{\mathrm{c}} \tag{6.64}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t} \tag{6.65}$$

$$\nabla\cdot\vec{J}_{\mathrm{c}}=g_{\mathrm{st}} \tag{6.66}$$

$$\nabla\cdot\vec{B}=0 \tag{6.67}$$

which, in a stationary good LIH conductor, can be written as:

$$\nabla\times\vec{H}=\vec{J}_{\mathrm{s}}+\kappa\cdot\vec{E} \tag{6.68}$$

$$\nabla\times\vec{E}=-\mu\cdot\frac{\partial\vec{H}}{\partial t} \tag{6.69}$$

$$\nabla\cdot\vec{E}=\frac{g_{\mathrm{st}}}{\kappa} \tag{6.70}$$

$$\nabla\cdot\vec{H}=0 \tag{6.71}$$

6.7. Maxwell's Integral Equations in a Stationary Perfect Dielectric

Maxwell's differential equations in a stationary perfect dielectric are given by the well-known expressions (6.22) - (6.25). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a stationary perfect dielectric can be obtained:

$$\oint_C\vec{H}\cdot d\vec{\ell}=\int_S\vec{J}_{\mathrm{s}}\cdot d\vec{S}+\int_S\frac{\partial\vec{D}}{\partial t}\cdot d\vec{S}=i_{\mathrm{tot}}=i_{\mathrm{s}}+i_{\mathrm{disp}} \tag{6.72}$$

$$e=\oint_C\vec{E}\cdot d\vec{\ell}=-\int_S\frac{\partial\vec{B}}{\partial t}\cdot d\vec{S}=-\frac{\partial\Phi}{\partial t} \tag{6.73}$$

$$\oint_{\partial V}\vec{D}\cdot d\vec{S}=\int_V\rho_{\mathrm{s}}\cdot dV=q_{\mathrm{s}} \tag{6.74}$$

$$\oint_{\partial V}\vec{B}\cdot d\vec{S}=0 \tag{6.75}$$

where:

i_{tot} - the total electric current flowing through the surface S (Figure 1.4),

i_{s} - the source electric current flowing through the surface S (Figure 1.4),

i_{disp} - the displacement electric current flowing through the surface S (Figure 1.4),

e - the electromotive force (EMF) induced in the closed contour C (Figure 1.4),

Φ - the magnetic flux through surface S (Figure 1.4),

q_{s} - the source electric charge inside volume V (Figure 1.3).

According to expression (6.72):

$$i_{\mathrm{s}} = \int_S \vec{J}_{\mathrm{s}} \cdot d\vec{S} \tag{6.76}$$

$$i_{\mathrm{disp}} = \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} \tag{6.77}$$

According to expression (6.73):

$$e = \oint_C \vec{E} \cdot d\vec{\ell} \tag{6.78}$$

$$\Phi = \int_S \vec{B} \cdot d\vec{S} \tag{6.79}$$

According to expression (6.74):

$$q_{\mathrm{s}} = \int_V \rho_{\mathrm{s}} \cdot dV \tag{6.80}$$

Maxwell’s first integral equation in a stationary perfect dielectric (6.72) represents the generalized Ampère’s law, which states that the circulation of the magnetic field intensity vector around a closed curve C is equal to the total electric current flowing through an arbitrary surface S bounded by the curve C, in accordance with the right-hand rule.

Maxwell’s second integral equation in a stationary perfect dielectric (6.73) represents Faraday’s law of electromagnetic induction, which states that the circulation of the electric field vector around a closed curve C is equal to the negative time derivative of the magnetic flux passing through an arbitrary surface S bounded by the curve C, in accordance with the right-hand rule. According to Faraday’s law of electromagnetic induction in a stationary system, the electromotive force (EMF) induced in a closed contour C is equal to the negative time derivative of the magnetic flux through the surface S bounded by that contour, also respecting the right-hand rule.

Maxwell’s third integral equation in a stationary perfect dielectric (6.74) represents Gauss’s law for electricity, which states that the flux of the electric displacement vector through the closed surface S enclosing a volume V is equal to the total electric charge contained within that volume.

Maxwell’s fourth integral equation in a stationary perfect dielectric (6.75) represents Gauss’s law for magnetism, which states that the magnetic flux through the closed surface S enclosing a volume V is equal to zero. This implies that magnetic monopoles do not exist.

The well-known differential continuity equation in the case of a stationary perfect dielectric is given by expression (6.30). By applying the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and illustrated in Figure 1.3, to this equation, the following integral form of the continuity equation for a stationary perfect dielectric can be obtained:

$$\int_V g_{\mathrm{st}} \cdot dV = -\int_V \left(\nabla \cdot \vec{J}_{\mathrm{s}}\right) \cdot dV = \frac{\partial q_s}{\partial t} \tag{6.81}$$

In the case of a perfect dielectric, the volume density of the leakage electric current injected by the source into the surrounding medium is equal to the volume density of the displacement electric current injected by the source into the surrounding dielectric.

6.8. Maxwell's Integral Equations in a Slowly Moving Perfect Dielectric

Maxwell's differential equations in a slowly moving perfect dielectric are given by expressions (6.9) - (6.12). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving perfect dielectric are obtained:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_{\mathrm{s}} + \rho_{\mathrm{s}} \cdot \vec{v} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \vec{D} \right) \cdot d\vec{\ell} \tag{6.82}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} = -\frac{d\Phi}{dt} \tag{6.83}$$

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho_{\mathrm{s}} \cdot dV = q_{\mathrm{s}} \tag{6.84}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{6.85}$$

Maxwell's second integral equation in a slowly moving perfect dielectric, given by expression (6.83), represents Faraday's law of electromagnetic induction in a slowly moving medium. According to this law, the electromotive force (EMF) induced in a closed contour that is moving through a time-varying electromagnetic field or undergoing deformation can be expressed as:

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\frac{d\Phi}{dt} = -\frac{\partial \Phi}{\partial t} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} = e_{\mathrm{tr}} + e_{\mathrm{mo}} \tag{6.86}$$

where e_{tr} is the induced EMF due to the transformation (time-dependent change of the magnetic field):

$$e_{\mathrm{tr}} = -\frac{\partial \Phi}{\partial t} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \tag{6.87}$$

whereas e_{mo} is the induced EMF due to relative motion and/or deformation of the closed contour:

$$e_{\mathrm{mo}} = \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} \tag{6.88}$$

The differential continuity equation for a slowly moving perfect dielectric is given by expression (6.21). By applying the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and illustrated in Figure 1.3, to that equation, the following integral form of the continuity equation in a slowly moving perfect dielectric can be obtained:

$$\int_V g_{\mathrm{st}} \cdot dV = -\int_V \left(\nabla \cdot \vec{J}_{\mathrm{s}} \right) \cdot dV = \frac{d q_s}{d t} \tag{6.89}$$

6.9. Maxwell's Integral Equations in a Slowly Moving Conducting Medium

Maxwell's differential equations in a slowly moving conducting medium are given by expressions (6.38) - (6.41). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving conducting medium can be obtained:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \vec{J}_c + \vec{v} \cdot \left(\nabla \cdot \vec{D} \right) + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \vec{D} \right) \cdot d\vec{\ell} \tag{6.90}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} = -\frac{d\Phi}{dt} \tag{6.91}$$

$$\oint_{\partial V} \left(\vec{J}_c + \vec{v} \cdot \left(\nabla \cdot \vec{D} \right) + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = \int_V g_{st} \cdot dV \tag{6.92}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{6.93}$$

6.10. Maxwell's Integral Equations in a Stationary Conducting Medium

Maxwell’s differential equations in a slowly moving conducting medium are given by expressions (6.38) - (6.41). By applying Stokes’ integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving conducting medium can be obtained:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = i_s + i_c + i_{disp} \tag{6.94}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{6.95}$$

$$\oint_{\partial V} \left(\vec{J}_c + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = \int_V g_{st} \cdot dV = i_{st} \tag{6.96}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{6.97}$$

where:

i_c - the conduction electric current flowing through the surface S (Figure 1.4),

i_{st} - the source leakage electric current flowing into the surrounding conducting medium within the considered volume V.

From expression (6.96), it follows that the electric current flowing out of the volume V through the closed surface ∂V is equal to the electric current originating within the volume V (Figure 6.1).

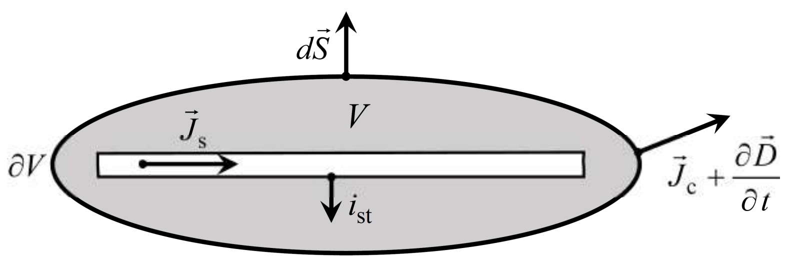


Figure 6.1. Graphical illustration of the expression (6.96)

The preceding integral equations can also be obtained from Maxwell's integral equations for a slowly moving conducting medium, given by expressions (6.90) - (6.93), by setting $\vec{v} = 0$.

From Maxwell's third integral equation (6.96), it follows that the sum of the conduction and displacement electric currents flowing out of volume V is equal to the total (conduction and displacement) leakage electric current injected by the independent source into the surrounding stationary conducting medium.

6.11. Maxwell's Integral Equations in a Slowly Moving Good Conductor

Maxwell's differential equations in a slowly moving good conductor are given by expressions (6.56) - (6.59). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving good conductor can be obtained:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_\mathrm{s} + \vec{J}_\mathrm{c} \right) \cdot d\vec{S} = i_\mathrm{s} + i_\mathrm{c} \tag{6.98}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} \tag{6.99}$$

$$\oint_{\partial V} \vec{J}_\mathrm{c} \cdot d\vec{S} = \int_V g_\mathrm{st} \cdot dV = i_\mathrm{st} \tag{6.100}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{6.101}$$

From Maxwell's third integral equation (6.100), it follows that the conduction electric current flowing out of volume V is equal to the conduction leakage current injected by the independent source into the surrounding slowly moving good conductor.

6.12. Maxwell's Integral Equations in a Stationary Good Conductor

Maxwell's differential equations in a stationary good conductor are given by expressions (6.64) - (6.67). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a stationary good conductor can be obtained:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_\mathrm{s} + \vec{J}_\mathrm{c} \right) \cdot d\vec{S} = i_\mathrm{s} + i_\mathrm{c} \tag{6.102}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} \tag{6.103}$$

$$\oint_{\partial V} \vec{J} \cdot d\vec{S} = \int_V g_\mathrm{st} \cdot dV = i_\mathrm{st} \tag{6.104}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{6.105}$$

The preceding integral equations can also be obtained from Maxwell's integral equations in a slowly moving good conductor, given by expressions (6.98) - (6.101), by applying the condition $\vec{v} = 0$.

7. MAXWELL'S EQUATIONS IN THE PHASOR DOMAIN

If the electromagnetic field is sinusoidal (time-harmonic), Maxwell's differential and integral equations can be transformed from the time domain to the phasor domain. The phasor representation of sinusoidal scalar and vector fields is discussed in detail in Chapter 5.

7.1. Maxwell's Differential Equations in a Slowly Moving Conducting Medium

Maxwell's differential equations in a slowly moving conducting medium, in the time domain, are given by expressions (6.38) - (6.41), and their phasor-domain equivalents can be expressed as:

$$\nabla\times\underline{\vec{H}}=\underline{\vec{J}}_{\mathrm{s}}+\underline{\vec{J}}_{\mathrm{c}}+\vec{v}\cdot\left(\nabla\cdot\underline{\vec{D}}\right)+j\cdot\omega\cdot\underline{\vec{D}}-\nabla\times\left(\vec{v}\times\underline{\vec{D}}\right) \tag{7.1}$$

$$\nabla\times\underline{\vec{E}}=-\ j\cdot\omega\cdot\underline{\vec{B}}+\nabla\times\left(\vec{v}\times\underline{\vec{B}}\right) \tag{7.2}$$

$$\nabla\cdot\left(\underline{\vec{J}}_{\mathrm{c}}+\vec{v}\cdot\left(\nabla\cdot\underline{\vec{D}}\right)+j\cdot\omega\cdot\underline{\vec{D}}\right)=-\nabla\cdot\underline{\vec{J}}_{\mathrm{s}}=\overline{g}_{\mathrm{st}} \tag{7.3}$$

$$\nabla\cdot\underline{\vec{B}}=0 \tag{7.4}$$

where:

$\underline{\vec{H}}$ - the phasor of the magnetic field intensity vector,

$\underline{\vec{E}}$ - the phasor of the electric field intensity vector,

$\underline{\vec{D}}$ - the phasor of the electric displacement vector,

$\underline{\vec{B}}$ - the phasor of the magnetic flux density vector,

$\underline{\vec{J}}_{\mathrm{s}}$ - the phasor of the vector of the surface density of the source (impressed) electric current,

$\underline{\vec{J}}_{\mathrm{c}}$ - the phasor of the vector of the surface density of the conduction electric current,

$\overline{g}_{\mathrm{st}}$ - the phasor of the volume density of the leakage electric current injected by an independent source into the surrounding conducting medium,

ω - the angular frequency,

$\vec{v}$ - the vector of constant velocity relative to the observer,

j - the imaginary unit.

Maxwell's differential equation (7.3) also represents the differential form of the continuity equation in a slowly moving conducting medium.

7.2. Maxwell's Differential Equations in a Slowly Moving Good Conductor

Maxwell's differential equations for a slowly moving good conductor in the phasor domain can be obtained from those for a slowly moving conducting medium in the phasor domain by incorporating the condition $\underline{\vec{D}}=0$ into expressions (7.1) and (7.3), as a consequence of neglecting displacement electric currents. The resulting differential equations are as follows:

$$\nabla\times\underline{\vec{H}}=\underline{\vec{J}}_{\mathrm{s}}+\underline{\vec{J}}_{\mathrm{c}} \tag{7.5}$$

$$\nabla\times\underline{\vec{E}}=-\,j\cdot\omega\cdot\underline{\vec{B}}+\nabla\times\left(\vec{v}\times\underline{\vec{B}}\right) \tag{7.6}$$

$$\nabla\cdot\underline{\vec{J}}_{\mathrm{c}}=\overline{g}_{\mathrm{st}} \tag{7.7}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{7.8}$$

which, in the case of a slowly moving good LIH conductor, can be written as:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + \kappa \cdot \underline{\vec{E}} \tag{7.9}$$

$$\nabla \times \underline{\vec{E}} = -j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} + \mu \cdot \nabla \times \left(\vec{v} \times \underline{\vec{H}}\right) \tag{7.10}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{g}_{\mathrm{st}}}{\kappa} \tag{7.11}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{7.12}$$

Maxwell's differential equation (7.3) also represents the differential form of the continuity equation in a slowly moving good conductor, while equation (7.11) serves the same role in the case of a slowly moving good LIH conductor.

7.3. Maxwell's Differential Equations in a Stationary Conducting Medium

Maxwell's differential equations in a stationary conducting medium in the phasor domain can be obtained from those in a slowly moving conducting medium in the phasor domain by applying the condition $\vec{v} = 0$ in expressions (7.1) - (7.3). The resulting differential equations are as follows:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + \underline{\vec{J}}_{\mathrm{c}} + j \cdot \omega \cdot \underline{\vec{D}} \tag{7.13}$$

$$\nabla \times \underline{\vec{E}} = -j \cdot \omega \cdot \underline{\vec{B}} \tag{7.14}$$

$$\nabla \cdot \left(\underline{\vec{J}}_{\mathrm{c}} + j \cdot \omega \cdot \underline{\vec{D}}\right) = \overline{g}_{\mathrm{st}} \tag{7.15}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{7.16}$$

which, in the case of a stationary conducting LIH medium, can be written as:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + \left(\kappa + j \cdot \omega \cdot \varepsilon\right) \cdot \underline{\vec{E}} \tag{7.17}$$

$$\nabla \times \underline{\vec{E}} = -j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} \tag{7.18}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{g}_{\mathrm{st}}}{\kappa + j \cdot \omega \cdot \varepsilon} \tag{7.19}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{7.20}$$

where $\kappa + j \cdot \omega \cdot \varepsilon$ is the complex conductivity of the conducting LIH medium.*

* In electromagnetic theory, the symbol $\overline{\kappa} = \sigma + j \cdot \omega \cdot \varepsilon$ is commonly used to denote the complex conductivity of a medium, where σ represents the electrical conductivity of the medium. However, this notation is avoided in this textbook, since the symbol σ is reserved for surface electric charge density.

7.4. Maxwell's Differential Equations in a Stationary Good Conductor

Maxwell's differential equations in a stationary good conductor in the phasor domain can be obtained from those for a stationary conducting medium in the phasor domain by setting $\underline{\vec{D}} = 0$ in expressions (7.13) and (7.15), as a consequence of neglecting displacement electric currents. The resulting differential equations are as follows:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\rm s} + \underline{\vec{J}}_{\rm c} \tag{7.21}$$

$$\nabla \times \underline{\vec{E}} = -\, j \cdot \omega \cdot \underline{\vec{B}} \tag{7.22}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{g}_{\rm st}}{\kappa} \tag{7.23}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{7.24}$$

which, in the case of a stationary good LIH conductor, can be written as:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\rm s} + \kappa \cdot \underline{\vec{E}} \tag{7.25}$$

$$\nabla \times \underline{\vec{E}} = -\, j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} \tag{7.26}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{g}_{\rm st}}{\kappa} \tag{7.27}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{7.28}$$

7.5. Maxwell's Differential Equations in a Slowly Moving Perfect Dielectric

Maxwell's differential equations in a slowly moving perfect dielectric, expressed in the time domain by equations (6.9) - (6.12), can be transformed into the following phasor-domain form:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\rm s} + \overline{\rho}_{\rm s} \cdot \vec{v} + j \cdot \omega \cdot \underline{\vec{D}} - \nabla \times \left(\vec{v} \times \underline{\vec{D}} \right) \tag{7.29}$$

$$\nabla \times \underline{\vec{E}} = -\, j \cdot \omega \cdot \underline{\vec{B}} + \nabla \times \left(\vec{v} \times \underline{\vec{B}} \right) \tag{7.30}$$

$$\nabla \cdot \underline{\vec{D}} = \overline{\rho}_{\rm s} \tag{7.31}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{7.32}$$

where $\overline{\rho}_{\rm s}$ is the phasor of the volume density of the source (impressed) electric charge.

Taking the divergence of both sides of Maxwell's differential equation (7.29) yields the following:

$$\nabla \cdot \underline{\vec{D}} = -\frac{\nabla \cdot \underline{\vec{J}}_{\rm s} + \nabla \cdot \left(\overline{\rho}_{\rm s} \cdot \vec{v} \right)}{j \cdot \omega} \tag{7.33}$$

By substituting expression (7.33) into expression (7.31), the following differential form of the continuity equation in a slowly moving perfect dielectric can be obtained:

$$\overline{g}_{\rm st} = -\nabla \cdot \underline{\vec{J}}_{\rm s} = j \cdot \omega \cdot \overline{\rho}_{\rm s} + \nabla \cdot \left(\overline{\rho}_{\rm s} \cdot \vec{v} \right) = j \cdot \omega \cdot \overline{\rho}_{\rm s} + \vec{v} \cdot \left(\nabla \overline{\rho}_{\rm s} \right) \tag{7.34}$$

From Maxwell's differential equations (7.29) - (7.32), the following Maxwell's differential equations for a slowly moving perfect LIH dielectric can be readily obtained:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_s + \overline{\rho}_s \cdot \vec{v} + j \cdot \omega \cdot \varepsilon \cdot \underline{\vec{E}} - \varepsilon \cdot \nabla \times \left(\vec{v} \times \underline{\vec{E}} \right) \tag{7.35}$$

$$\nabla \times \underline{\vec{E}} = - j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} + \mu \cdot \nabla \times \left(\vec{v} \times \underline{\vec{H}} \right) \tag{7.36}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{\rho}_s}{\varepsilon} \tag{7.37}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{7.38}$$

7.6. Maxwell's Differential Equations in a Stationary Perfect Dielectric

Maxwell’s differential equations in a stationary perfect dielectric in the phasor domain can be obtained from those in a slowly moving perfect dielectric in the phasor domain by applying the condition $\vec{v} = 0$ to expressions (7.29) and (7.30). The resulting differential equations are as follows:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_s + j \cdot \omega \cdot \underline{\vec{D}} \tag{7.39}$$

$$\nabla \times \underline{\vec{E}} = - j \cdot \omega \cdot \underline{\vec{B}} \tag{7.40}$$

$$\nabla \cdot \underline{\vec{D}} = \overline{\rho}_s \tag{7.41}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{7.42}$$

which, in the case of a stationary perfect LIH dielectric, can be written as:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_s + j \cdot \omega \cdot \varepsilon \cdot \underline{\vec{E}} \tag{7.43}$$

$$\nabla \times \underline{\vec{E}} = - j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} \tag{7.44}$$

$$\nabla \cdot \underline{\vec{E}} = \frac{\overline{\rho}_s}{\varepsilon} \tag{7.45}$$

$$\nabla \cdot \underline{\vec{H}} = 0 \tag{7.46}$$

If the condition $\vec{v} = 0$ is applied to the differential continuity equation for a slowly moving perfect dielectric in the phasor domain, given by expression (7.34), the following continuity equation for a stationary perfect dielectric in the phasor domain can be obtained:

$$\overline{g}_{st} = - \nabla \cdot \underline{\vec{J}}_s = j \cdot \omega \cdot \overline{\rho}_s \tag{7.47}$$

7.7. Maxwell's Integral Equations in a Slowly Moving Conducting Medium

Maxwell’s differential equations in a slowly moving conducting medium are given by expressions (7.1) - (7.4). By applying Stokes’ integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving conducting medium can be obtained:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_s + \underline{\vec{J}}_c + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}} \right) + j \cdot \omega \cdot \underline{\vec{D}} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \underline{\vec{D}} \right) \cdot d\vec{\ell} \tag{7.48}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = - j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \underline{\vec{B}} \right) \cdot d\vec{\ell} \tag{7.49}$$

$$\oint_{\partial V} \left(\underline{\vec{J}} + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}}\right) + j \cdot \omega \cdot \underline{\vec{D}}\right) \cdot d\vec{S} = \int_V \underline{g}_{\mathrm{st}} \cdot dV \tag{7.50}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.51}$$

7.8. Maxwell's Integral Equations in a Slowly Moving Good Conductor

Maxwell's integral equations in a slowly moving good conductor in the phasor domain can be obtained from those for a slowly moving conducting medium in the phasor domain by setting $\underline{\vec{D}} = 0$ in expressions (7.48) and (7.50), as a consequence of neglecting displacement electric currents. The resulting integral equations are as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \underline{\vec{J}}_{\mathrm{s}} \cdot d\vec{S} + \int_S \underline{\vec{J}}_{\mathrm{c}} \cdot d\vec{S} = \underline{I}_{\mathrm{s}} + \underline{I}_{\mathrm{c}} \tag{7.52}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\, j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \underline{\vec{B}}\right) \cdot d\vec{\ell} \tag{7.53}$$

$$\oint_{\partial V} \underline{\vec{J}} \cdot d\vec{S} = \int_V \underline{g}_{\mathrm{st}} \cdot dV = \underline{I}_{\mathrm{st}} \tag{7.54}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.55}$$

where:

$\underline{I}_{\mathrm{s}}$ - the phasor of the source electric current flowing through the surface *S* (Figure 1.4),

$\underline{I}_{\mathrm{c}}$ - the phasor of the conduction electric current flowing through the surface *S* (Figure 1.4),

$\underline{I}_{\mathrm{st}}$ - the phasor of the source leakage electric current flowing into the surrounding conducting medium within the considered volume *V*.

7.9. Maxwell's Integral Equations in a Stationary Conducting Medium

Maxwell's integral equations in a stationary conducting medium in the phasor domain can be obtained from those for a slowly moving conducting medium in the phasor domain by setting $\vec{v} = 0$ in expressions (7.48) - (7.51). The resulting integral equations are as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \underline{\vec{J}}_{\mathrm{s}} \cdot d\vec{S} + \int_S \underline{\vec{J}}_{\mathrm{c}} \cdot d\vec{S} + j \cdot \omega \cdot \int_S \underline{\vec{D}} \cdot d\vec{S} = \underline{I}_{\mathrm{s}} + \underline{I}_{\mathrm{c}} + \underline{I}_{\mathrm{disp}} \tag{7.56}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\, j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} \tag{7.57}$$

$$\oint_{\partial V} \left(\underline{\vec{J}}_{\mathrm{c}} + j \cdot \omega \cdot \underline{\vec{D}}\right) \cdot d\vec{S} = \int_V \underline{g}_{\mathrm{st}} \cdot dV = \underline{I}_{\mathrm{st}} \tag{7.58}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.59}$$

where $\underline{I}_{\mathrm{disp}}$ is the phasor of the displacement electric current flowing through the surface *S* (Figure 1.4).

7.10. Maxwell's Integral Equations in a Stationary Good Conductor

Maxwell's integral equations in a stationary good conductor in the phasor domain can be obtained from those for a slowly moving good conductor by setting $\vec{v}=0$ in expression (7.53). The resulting integral equations are as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \underline{\vec{J}}_{\mathrm{s}} \cdot d\vec{S} + \int_S \underline{\vec{J}}_{\mathrm{c}} \cdot d\vec{S} = \bar{I}_{\mathrm{s}} + \bar{I}_{\mathrm{c}} \tag{7.60}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} \tag{7.61}$$

$$\oint_{\partial V} \underline{\vec{J}} \cdot d\vec{S} = \int_V \bar{g}_{\mathrm{st}} \cdot dV = \bar{I}_{\mathrm{st}} \tag{7.62}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.63}$$

7.11. Maxwell's Integral Equations in a Slowly Moving Perfect Dielectric

Maxwell's differential equations in a slowly moving perfect dielectric are given by expressions (7.29) - (7.32). By applying Stokes' integral theorem, as defined in expression (1.61) and illustrated in Figure 1.4, to the first two equations, and the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and shown in Figure 1.3, to the remaining two equations, the following Maxwell integral equations for a slowly moving perfect dielectric can be obtained:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_{\mathrm{s}} + \bar{\rho}_{\mathrm{s}} \cdot \vec{v} + j \cdot \omega \cdot \underline{\vec{D}}\right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \underline{\vec{D}}\right) \cdot d\vec{\ell} \tag{7.64}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \underline{\vec{B}}\right) \cdot d\vec{\ell} \tag{7.65}$$

$$\oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} = \int_V \bar{\rho}_{\mathrm{s}} \cdot dV = \overline{Q}_{\mathrm{s}} \tag{7.66}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.67}$$

where $\overline{Q}_{\mathrm{s}}$ is the phasor of the source electric charge within volume V (Figure 1.3).

The differential continuity equation in a slowly moving perfect dielectric in the phasor domain is given by expression (7.34). By applying the Ostrogradsky-Green-Gauss integral theorem, as defined in expression (1.60) and illustrated in Figure 1.3, to this expression, the following integral form of the continuity equation in a slowly moving perfect dielectric in the phasor domain can be obtained:

$$\oint_{\partial V} \left(\bar{\rho}_{\mathrm{s}} \cdot \vec{v}\right) \cdot d\vec{S} = \int_V \bar{g}_{\mathrm{st}} \cdot dV - j \cdot \omega \cdot \int_V \bar{\rho}_{\mathrm{s}} \cdot dV = \bar{I}_{\mathrm{st}} - j \cdot \omega \cdot \overline{Q}_{\mathrm{s}} \tag{7.68}$$

7.12. Maxwell's Integral Equations in a Stationary Perfect Dielectric

Maxwell's integral equations in a stationary perfect dielectric in the phasor domain can be obtained from those for a slowly moving perfect dielectric by setting $\vec{v}=0$ in expressions (7.64) and (7.65). The resulting integral equations are as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \underline{\vec{J}}_{\mathrm{s}} \cdot d\vec{S} + j \cdot \omega \cdot \int_S \underline{\vec{D}} \cdot d\vec{S} = \bar{I}_{\mathrm{s}} + \bar{I}_{\mathrm{disp}} \tag{7.69}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -\, j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} \tag{7.70}$$

$$\oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} = \int_V \overline{\rho}_\mathrm{s} \cdot dV = \overline{Q}_\mathrm{s} \tag{7.71}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{7.72}$$

The integral continuity equation for a stationary perfect dielectric in the phasor domain can be obtained from the corresponding equation for a slowly moving perfect dielectric, given by expression (7.68), by substituting $\vec{v} = 0$ into it. The resulting integral continuity equation is:

$$\bar{I}_\mathrm{st} = \int_V \overline{g}_\mathrm{st} \cdot dV = j \cdot \omega \cdot \int_V \overline{\rho}_\mathrm{s} \cdot dV = j \cdot \omega \cdot \oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} = j \cdot \omega \cdot \overline{Q}_\mathrm{s} \tag{7.73}$$

which asserts that the electric current introduced by an independent source into volume V corresponds to the displacement current generated by that source within the same volume.

8. LORENTZ FORCE

Maxwell's equations, the Lorentz force law, and Newton's second law together provide a complete classical description of the dynamics of charged particles and the electromagnetic field.

A point electric charge q, moving with velocity $\vec{v}$ in an electromagnetic field as observed from a stationary reference frame, is subjected to the Lorentz force:

$$\vec{F} = q \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{8.1}$$

where:

$\vec{E}$ - the electric field intensity,

$\vec{B}$ - the magnetic flux density.

The Lorentz force is the electromagnetic force acting on an electric charge q, and it is equal to the sum of the electric force $\vec{F}_{\mathrm{e}}$ acting on the charge:

$$\vec{F}_{\mathrm{e}} = q \cdot \vec{E} \tag{8.2}$$

and the magnetic force $\vec{F}_{\mathrm{m}}$ acting on the same charge moving with velocity $\vec{v}$ in a magnetic field:

$$\vec{F}_{\mathrm{m}} = q \cdot \left(\vec{v} \times \vec{B}\right) \tag{8.3}$$

The magnetic force acts on a charge only when the charge is moving within a magnetic field, whereas the electric force acts on a charge both when it is at rest and when it is moving within an electric field.

In the general case, the differential form of the Lorentz force law is given by the following expression:

$$\vec{f} = \rho \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \tag{8.4}$$

where $\vec{f}$ is the volume force density vector acting on an electric charge with volume density ρ, moving in an electromagnetic field. Since the surface density of the electric current resulting from the motion of an electric charge can be described by the expression:

$$\vec{J} = \rho \cdot \vec{v} \tag{8.5}$$

It follows that the volume force density may also be expressed as:

$$\vec{f} = \rho \cdot \vec{E} + \vec{J} \times \vec{B} \tag{8.6}$$

which leads to the integral form of the generalized Lorentz force law:

$$\vec{F} = \int_V \vec{f} \cdot dV = \int_V \rho \cdot \left(\vec{E} + \vec{v} \times \vec{B}\right) \cdot dV = \int_V \left(\rho \cdot \vec{E} + \vec{J} \times \vec{B}\right) \cdot dV \tag{8.7}$$

9. ELECTROMAGNETIC POTENTIALS

To represent the electromagnetic field in a linear, isotropic, and homogeneous (LIH) medium, *auxiliary functions* known as *electromagnetic potentials* are also employed. Several different pairs of electromagnetic potentials can be defined, each associated with a different gauge conditions. Each pair consists of one vector potential and one scalar potential.

The most commonly used pair of electromagnetic potentials consists of the magnetic vector potential $\vec{A}$ and the electric scalar potential φ, which, in accordance with Maxwell's equations, are defined in a slowly moving reference frame by the following expressions:

$$\vec{B} = \nabla \times \vec{A} \tag{9.1}$$

$$\vec{E} = -\nabla\varphi - \frac{d\vec{A}}{dt} \tag{9.2}$$

whereas in a stationary reference frame, they are defined by the following expressions:

$$\vec{B} = \nabla \times \vec{A} \tag{9.3}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t} \tag{9.4}$$

Electromagnetic potentials are not unique functions. If a correction is added to one potential, a corresponding correction must be applied to the other in order to keep the magnetic flux density and the electric field intensity unchanged.

Let us consider a stationary system where:

$$\vec{B} = \nabla \times \vec{A} = \nabla \times \left(\vec{A} + \vec{A}_{\mathrm{c}}\right) \tag{9.5}$$

$$\vec{E} = -\nabla\varphi - \frac{\partial\vec{A}}{\partial t} = -\nabla\left(\varphi + \varphi_{\mathrm{c}}\right) - \frac{\partial\left(\vec{A} + \vec{A}_{\mathrm{c}}\right)}{\partial t} \tag{9.6}$$

From expression (9.5), it follows that the correction to the magnetic vector potential is given by the gradient of a scalar field:

$$\vec{A}_{\mathrm{c}} = \nabla\Psi \tag{9.7}$$

because:

$$\nabla \times \vec{A}_c = \nabla \times \left(\nabla\Psi\right) = 0 \tag{9.8}$$

and from expression (9.6), it follows that the correction to the electric scalar potential is:

$$\varphi_{\mathrm{c}} = -\frac{\partial\Psi}{\partial t} \tag{9.9}$$

9.1. Electromagnetic Potentials in a Stationary Perfect Dielectric

According to (6.26) and (6.28), Maxwell's first and third differential equations in a stationary perfect LIH dielectric can be written as:

$$\nabla \times \vec{B} = \mu \cdot \vec{J}_{\mathrm{s}} + \mu \cdot \varepsilon \cdot \frac{\partial\vec{E}}{\partial t} \tag{9.10}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{9.11}$$

where $\vec{J}_{\mathrm{s}}$ and ρ_{s} are the source densities of the electromagnetic field, which satisfy the continuity equation (6.30).

If the definitions of the electromagnetic potentials (9.3) and (9.4) are substituted into Maxwell's equations (9.10) and (9.11), the following expressions are obtained:

$$\nabla\times\left(\nabla\times\vec{A}\right)=\mu\cdot\vec{J}_{\mathrm{s}}-\mu\cdot\varepsilon\cdot\frac{\partial}{\partial t}\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right) \tag{9.12}$$

$$\nabla\cdot\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right)=-\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{9.13}$$

It holds that:

$$\nabla\times\left(\nabla\times\vec{A}\right)=\nabla\left(\nabla\cdot\vec{A}\right)-\left(\nabla\cdot\nabla\right)\vec{A}=\nabla\left(\nabla\cdot\vec{A}\right)-\Delta\vec{A} \tag{9.14}$$

and expressions (9.12) and (9.13) then take the following form:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\left(\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}\right)=-\mu\cdot\vec{J}_{\mathrm{s}} \tag{9.15}$$

$$\Delta\varphi+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)=-\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{9.16}$$

If the Lorenz gauge condition*:

$$\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}=0 \tag{9.17}$$

is inserted into expressions (9.15) and (9.16), the following inhomogeneous undamped wave equations for the potentials are obtained:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\cdot\vec{J}_{\mathrm{s}} \tag{9.18}$$

$$\Delta\varphi-\mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{9.19}$$

This is a complete system of equations if the densities of sources $\vec{J}_{\mathrm{s}}$ and ρ_{s} are known.

The Lorenz gauge condition ensures that the source of the magnetic vector potential is solely the surface density of the source electric current, and that the source of the electric scalar potential is solely the volume density of the source electric charge. Electromagnetic potentials that satisfy the Lorenz gauge condition are referred to as Lorenz potentials.

From the inhomogeneous undamped wave equations (9.18) and (9.19) in the time domain, the inhomogeneous Helmholtz differential equations can be easily derived in the phasor domain:

$$\Delta\underline{\vec{A}}-\bar{\gamma}^2\cdot\underline{\vec{A}}=-\mu\cdot\underline{\vec{J}}_{\mathrm{s}} \tag{9.20}$$

$$\Delta\bar{\varphi}-\bar{\gamma}^2\cdot\bar{\varphi}=-\frac{\bar{\rho}_{\mathrm{s}}}{\varepsilon} \tag{9.21}$$

where:

$$\bar{\gamma}^2=-k^2=-\omega^2\cdot\mu\cdot\varepsilon \quad ; \quad k=\omega\cdot\sqrt{\mu\cdot\varepsilon} \quad ; \quad \bar{\gamma}=j\cdot k \tag{9.22}$$

The real constant k is the *wave number*, and the complex constant $\bar{\gamma}$ is the *propagation constant*. In the case of a perfect dielectric, the wave number is real, whereas in a conducting medium, the wave number is complex.

* The Lorenz gauge condition is named after the Danish physicist and mathematician Ludwig Valentin Lorenz.

9.2. Electromagnetic Potentials in a Stationary Conducting Medium

According to (6.50) and (6.52), Maxwell's first and third differential equations in a stationary, conducting LIH medium can be written as:

$$\nabla\times\vec{B}=\mu\cdot\vec{J}_{\mathrm{s}}+\mu\cdot\kappa\cdot\vec{E}+\mu\cdot\varepsilon\cdot\frac{\partial\vec{E}}{\partial t} \tag{9.23}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\nabla\cdot\vec{E}\right)=-\nabla\cdot\vec{J}_{\mathrm{s}}=g_{\mathrm{st}} \tag{9.24}$$

where equation (9.24) also represents the continuity equation in a stationary, conducting LIH medium.

If the definitions of the electromagnetic potentials (9.3) and (9.4) are substituted into Maxwell's differential equations (9.23) and (9.24), the following expressions are obtained:

$$\nabla\times\left(\nabla\times\vec{A}\right)=\nabla\left(\nabla\cdot\vec{A}\right)-\Delta\vec{A}=\mu\cdot\vec{J}_{\mathrm{s}}-\mu\cdot\kappa\cdot\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right)-\mu\cdot\varepsilon\cdot\frac{\partial}{\partial t}\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right) \tag{9.25}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\nabla\cdot\left(\nabla\varphi+\frac{\partial\vec{A}}{\partial t}\right)=-g_{\mathrm{st}} \tag{9.26}$$

which can be transformed into the following form:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}-\nabla\left(\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}+\mu\cdot\kappa\cdot\varphi\right)=-\mu\cdot\vec{J}_{\mathrm{s}} \tag{9.27}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\Delta\varphi+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)\right)=-g_{\mathrm{st}} \tag{9.28}$$

If the Lorenz gauge condition in a conducting LIH medium:

$$\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}+\mu\cdot\kappa\cdot\varphi=0 \tag{9.29}$$

is inserted into expressions (9.27) and (9.28), the final form of the inhomogeneous differential equations for the electromagnetic potentials in a conducting medium is obtained:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}=-\mu\cdot\vec{J}_{\mathrm{s}} \tag{9.30}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\Delta\varphi-\mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\varphi}{\partial t}\right)=-g_{\mathrm{st}} \tag{9.31}$$

which, in a good LIH conductor where displacement electric currents can be neglected, reduce to the so-called inhomogeneous diffusion equations for the electromagnetic potentials:

$$\Delta\vec{A}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}=-\mu\cdot\vec{J}_{\mathrm{s}} \tag{9.32}$$

$$\Delta\varphi-\mu\cdot\kappa\cdot\frac{\partial\varphi}{\partial t}=-\frac{g_{\mathrm{st}}}{\kappa} \tag{9.33}$$

In a source-free, stationary, conducting LIH medium, the inhomogeneous differential equations (9.30) and (9.31) reduce to homogeneous damped wave equations for the potentials:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}=0 \tag{9.34}$$

$$\Delta\varphi-\mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\varphi}{\partial t}=0 \tag{9.35}$$

From the inhomogeneous differential equations (9.30) and (9.31) in the time domain, the corresponding inhomogeneous Helmholtz differential equations can be derived in the phasor domain:

$$\Delta\underline{\vec{A}}-\overline{\gamma}^2\cdot\underline{\vec{A}}=-\mu\cdot\underline{\vec{J}}_s \tag{9.36}$$

$$\Delta\overline{\varphi}-\overline{\gamma}^2\cdot\overline{\varphi}=-\frac{\overline{g}_{\mathrm{st}}}{\kappa+j\cdot\omega\cdot\varepsilon} \tag{9.37}$$

where:

$$\overline{\gamma}^2=-\overline{k}^2=j\cdot\omega\cdot\mu\cdot(\kappa+j\cdot\omega\cdot\varepsilon) \quad ; \quad \overline{\gamma}=j\cdot\overline{k} \tag{9.38}$$

From the inhomogeneous differential equations (9.32) and (9.33) in the time domain, the following inhomogeneous Helmholtz differential equations can be derived in the phasor domain:

$$\Delta\underline{\vec{A}}-\overline{\gamma}^2\cdot\underline{\vec{A}}=-\mu\cdot\underline{\vec{J}}_s \tag{9.39}$$

$$\Delta\overline{\varphi}-\overline{\gamma}^2\cdot\overline{\varphi}=-\frac{\overline{g}_{\mathrm{st}}}{\kappa} \tag{9.40}$$

where:

$$\overline{\gamma}^2=-\overline{k}^2=j\cdot\omega\cdot\mu\cdot\kappa \quad ; \quad \overline{\gamma}=j\cdot\overline{k} \tag{9.41}$$

From expressions (9.38) and (9.41), it follows that in a conducting medium, both the propagation constant $\overline{\gamma}$ and the wave number $\overline{k}$ are complex constants.

10. WAVE FIELD EQUATIONS IN SOURCE-FREE LIH MEDIA

According to (6.50) - (6.54), in a source-free stationary conducting LIH medium, Maxwell's differential equations can be written as:

$$\nabla\times\vec{H}=\kappa\cdot\vec{E}+\varepsilon\cdot\frac{\partial\vec{E}}{\partial t} \tag{10.1}$$

$$\nabla\times\vec{E}=-\mu\cdot\frac{\partial\vec{H}}{\partial t} \tag{10.2}$$

$$\nabla\cdot\vec{E}=0 \tag{10.3}$$

$$\nabla\cdot\vec{H}=0 \tag{10.4}$$

It holds that:

$$\nabla\times\left(\nabla\times\vec{H}\right)=\nabla\left(\nabla\cdot\vec{H}\right)-\Delta\vec{H}=-\Delta\vec{H} \tag{10.5}$$

$$\nabla\times\left(\nabla\times\vec{E}\right)=\nabla\left(\nabla\cdot\vec{E}\right)-\Delta\vec{E}=-\Delta\vec{E} \tag{10.6}$$

and from expressions (10.1), (10.2), (10.5), and (10.6), it follows that:

$$\Delta\vec{H}=-\kappa\cdot\left(\nabla\times\vec{E}\right)-\varepsilon\cdot\frac{\partial(\nabla\times\vec{E})}{\partial t} \tag{10.7}$$

$$\Delta\vec{E}=\mu\cdot\frac{\partial(\nabla\times\vec{H})}{\partial t} \tag{10.8}$$

After substituting expression (10.2) into (10.7) and expression (10.1) into (10.8), the following homogeneous damped wave field equations in a source-free conducting LIH medium are obtained:

$$\Delta\vec{E}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{E}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{E}}{\partial t}=0 \tag{10.9}$$

$$\Delta\vec{H}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{H}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{H}}{\partial t}=0 \tag{10.10}$$

which, in a source-free stationary perfect LIH dielectric, reduce to homogeneous undamped wave field equations:

$$\Delta\vec{E}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{E}}{\partial t^2}=0 \tag{10.11}$$

$$\Delta\vec{H}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{H}}{\partial t^2}=0 \tag{10.12}$$

whereas the homogeneous damped wave field equations (10.9) and (10.10), in a source-free stationary good LIH conductor and neglecting displacement electric currents, reduce to homogeneous diffusion equations:

$$\Delta\vec{E}-\mu\cdot\kappa\cdot\frac{\partial\vec{E}}{\partial t}=0 \quad ; \quad \Delta\vec{H}-\mu\cdot\kappa\cdot\frac{\partial\vec{H}}{\partial t}=0 \tag{10.13}$$

11. CONDITIONS AT THE BOUNDARY BETWEEN TWO STATIONARY MEDIA

On the boundary surface S between two homogeneous stationary media (Figure 11.1), the boundary conditions must be satisfied for the field quantities $\vec{E}$, $\vec{H}$, $\vec{D}$, $\vec{B}$, and $\vec{J}$.

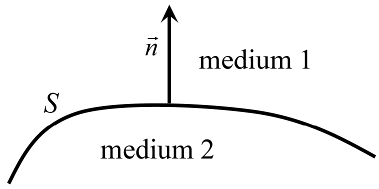


Figure 11.1. Boundary surface between two homogeneous stationary media

The boundary conditions between two stationary media follow from Maxwell's integral equations for a stationary medium and depend on the properties of each medium. In the general case, the boundary conditions can be expressed in vector form as [2]:

$$\vec{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \tag{11.1}$$

$$\vec{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{K} \tag{11.2}$$

$$\vec{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\sigma \tag{11.3}$$

$$\vec{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0 \tag{11.4}$$

$$\vec{n}\cdot\left(\vec{J}_1-\vec{J}_2\right)+\nabla_{\mathrm{tp}}\cdot\vec{K}=-\frac{\partial\sigma}{\partial t} \tag{11.5}$$

where:

K - the linear density of the surface electric current* on the boundary surface S,

$\vec{K}$ - the vector of the linear density of the surface electric current,

σ - the surface density of the free electric charge on the boundary surface S,

$\nabla_{\mathrm{tp}}\cdot\vec{K}$ - the 2D divergence of the vector $\vec{K}$ in the tangential plane, which appears only if one of the two media is superconducting.

If the normal and tangential components of the field vectors are used instead of the vectors themselves, then the general boundary conditions (11.1) - (11.5) can be expressed in scalar form as:

$$E_{\mathrm{t}1}=E_{\mathrm{t}2} \tag{11.6}$$

$$H_{\mathrm{t}1}=H_{\mathrm{t}2}+K_{\mathrm{t}} \tag{11.7}$$

$$D_{\mathrm{n}1}=D_{\mathrm{n}2}+\sigma \tag{11.8}$$

$$B_{\mathrm{n}1}=B_{\mathrm{n}2} \tag{11.9}$$

$$J_{\mathrm{n}1}=J_{\mathrm{n}2}-\frac{\partial\sigma}{\partial t}-\nabla_{\mathrm{tr}}\cdot\vec{K} \tag{11.10}$$

---

* This physical quantity is commonly known as surface current density, but this abbreviated term is not used in this textbook to avoid confusion.

where:

E_{t1}, E_{t2}, H_{t1}, and H_{t2} are the tangential components of the field vectors,

D_{n1}, D_{n2}, B_{n1}, B_{n2}, J_{n1}, and J_{n2} are the normal components of the field vectors.

A surface electric current of linear density $\vec{K}$ is generated by the electromagnetic field at the boundary between two media only when one of the media is a superconductor. However, a surface electric current can also be produced by an independent source located at the boundary, in which case we refer to it as an impressed surface electric current.

11.1. Conditions at the Boundary Between Two Conducting Media

Let both stationary media be conducting (Figure 11.1), and let an impressed surface electric current exist on the boundary surface. Assume that neither medium is superconducting. Then the boundary conditions can be written as:

$$\vec{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \tag{11.11}$$

$$\vec{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{K} \tag{11.12}$$

$$\vec{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\sigma \tag{11.13}$$

$$\vec{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0 \tag{11.14}$$

$$\vec{n}\cdot\left(\vec{J}_1-\vec{J}_2\right)=-\frac{\partial\sigma}{\partial t} \tag{11.15}$$

where $\vec{K}$ is the vector of the linear density of an impressed surface electric current. In this case, there is no surface electric current generated by the electromagnetic field.

From equation (11.13), it follows that a free electric charge with surface density σ accumulates at the boundary between two conducting media placed in an electromagnetic field.

In the static case, the time derivative of the surface electric charge density in boundary condition (11.15) is equal to zero, whereas boundary conditions (11.11) - (11.14) remain unchanged.

11.2. Conditions at the Boundary Between a Conducting Medium and a Perfect Dielectric

Let medium 2 be a stationary conducting medium and medium 1 a stationary perfect dielectric (Figure 11.1), with an impressed surface electric current present on the boundary surface. Assume that neither medium is superconducting. Then the boundary conditions can be written as:

$$\vec{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \tag{11.16}$$

$$\vec{n}\times\left(\vec{H}_1-\vec{H}_2\right)=\vec{K} \tag{11.17}$$

$$\vec{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\sigma \tag{11.18}$$

$$\vec{n}\cdot\left(\vec{B}_1-\vec{B}_2\right)=0 \tag{11.19}$$

$$\vec{n}\cdot\vec{J}_2=\frac{\partial\sigma}{\partial t} \tag{11.20}$$

because:

$$\vec{J}_1 = 0 \tag{11.21}$$

which means that no electric current enters a perfect dielectric. In this case, there is no surface electric current generated by the electromagnetic field.

In the electrostatic case, when the conducting medium 2 is placed in an electrostatic field, the boundary conditions can be written as:

$$\vec{n} \times \vec{E}_1 = 0 \tag{11.22}$$

$$\vec{n} \cdot \vec{D}_1 = \sigma \tag{11.23}$$

because:

$$\vec{E}_2 = 0 \tag{11.24}$$

which means that a conductor in an electrostatic field is equipotential.

The magnetostatic field can be considered independently of the electrostatic field, and boundary conditions (11.17) and (11.19) apply to it.

11.3. Conditions at the Boundary Between Two Perfect Dielectrics

Let both stationary media be perfect dielectrics (Figure 11.1), and assume that there is neither an impressed surface electric current nor an impressed surface electric charge at the boundary between them. Then the boundary conditions can be written as:

$$\vec{n} \times \left(\vec{E}_1 - \vec{E}_2\right) = 0 \tag{11.25}$$

$$\vec{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = 0 \tag{11.26}$$

$$\vec{n} \cdot \left(\vec{D}_1 - \vec{D}_2\right) = 0 \tag{11.27}$$

$$\vec{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \tag{11.28}$$

11.4. Conditions at the Boundary Between a Superconductor and a Perfect Dielectric

Let stationary medium 2 be a superconductor and stationary medium 1 a perfect dielectric. Since a time-varying electromagnetic field cannot penetrate the superconductor, the boundary conditions can be written as:

$$\vec{n} \times \vec{E}_1 = 0 \tag{11.29}$$

$$\vec{n} \times \vec{H}_1 = \vec{K} \tag{11.30}$$

$$\vec{n} \cdot \vec{D}_1 = \sigma \tag{11.31}$$

$$\vec{n} \cdot \vec{B}_1 = 0 \tag{11.32}$$

$$\nabla_{\mathrm{tp}} \cdot \vec{K} = -\frac{\partial \sigma}{\partial t} \tag{11.33}$$

where $\nabla_{\mathrm{tp}} \cdot \vec{K}$ is the 2D divergence of the vector $\vec{K}$ in the tangential plane.

In this case, the following holds:

$$\vec{J}_1 = \vec{J}_2 = 0 \tag{11.34}$$

and both the surface electric current and the surface electric charge at the boundary are generated by the electromagnetic field.

In the static case, the electrostatic and magnetostatic fields are treated as mutually independent. The electric field intensity in a superconductor is zero, but a magnetic field can penetrate the superconductor if a steady electric current flows through it. The boundary conditions can then be written as:

$$\vec{n} \times \vec{E}_1 = 0 \tag{11.35}$$

$$\vec{n} \times \left(\vec{H}_1 - \vec{H}_2\right) = \vec{K} \tag{11.36}$$

$$\vec{n} \cdot \vec{D}_1 = \sigma \tag{11.37}$$

$$\vec{n} \cdot \left(\vec{B}_1 - \vec{B}_2\right) = 0 \tag{11.38}$$

$$\nabla_{\mathrm{tp}} \cdot \vec{K} = 0 \tag{11.39}$$

11.5. Conditions at the Boundary Between Two Media in the Phasor Domain

The previously stated boundary conditions at the interface between two media are valid in the time domain. If the electromagnetic field is sinusoidal (time-harmonic), then the general boundary conditions (11.1) - (11.5) in the phasor domain can be written as:

$$\vec{n} \times \left(\underline{\vec{E}}_1 - \underline{\vec{E}}_2\right) = 0 \tag{11.40}$$

$$\vec{n} \times \left(\underline{\vec{H}}_1 - \underline{\vec{H}}_2\right) = \underline{\vec{K}} \tag{11.41}$$

$$\vec{n} \cdot \left(\underline{\vec{D}}_1 - \underline{\vec{D}}_2\right) = \underline{\sigma} \tag{11.42}$$

$$\vec{n} \cdot \left(\underline{\vec{B}}_1 - \underline{\vec{B}}_2\right) = 0 \tag{11.43}$$

$$\vec{n} \cdot \left(\underline{\vec{J}}_1 - \underline{\vec{J}}_2\right) + \nabla_{\mathrm{tp}} \cdot \underline{\vec{K}} = -\, j \cdot \omega \cdot \underline{\sigma} \tag{11.44}$$

12. ELECTROMAGNETIC ENERGY AND POYNTING'S THEOREM

The electromagnetic field is fully described by Maxwell's equations, which also allow the calculation of the energy stored in the field. Poynting's theorem, representing the law of conservation of electromagnetic energy, has both integral and differential forms and is applicable to time-harmonic electromagnetic fields as well.

12.1. Energy Stored in an Electromagnetic Field

The time derivative of the energy W stored in the electromagnetic field is given by the following expression [6, 12]:

$$\frac{\partial W}{\partial t} = \frac{\partial W_{\mathrm{e}}}{\partial t} + \frac{\partial W_{\mathrm{m}}}{\partial t} = \int_V \left(\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) \cdot dV \tag{12.1}$$

where W_{e} is the energy stored in the electric field, and W_{m} is the energy stored in the magnetic field.

From expression (12.1), it follows that, in the general case, the energy stored in an electric field can be described by the following expression:

$$W_{\mathrm{e}} = \int_0^t \int_V \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} \cdot dV \cdot dt = \int_V \int_0^D \vec{E} \cdot \delta \vec{D} \cdot dV \quad ; \quad \delta \vec{D} = \frac{\partial \vec{D}}{\partial t} \cdot dt \tag{12.2}$$

and the energy stored in the magnetic field is given by:

$$W_{\mathrm{m}} = \int_0^t \int_V \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \cdot dV \cdot dt = \int_V \int_0^B \vec{H} \cdot \delta \vec{B} \cdot dV \quad ; \quad \delta \vec{B} = \frac{\partial \vec{B}}{\partial t} \cdot dt \tag{12.3}$$

The energy stored in an electromagnetic field within a volume V can also be expressed as:

$$W = \int_V w \cdot dV \tag{12.4}$$

where w is the volume energy density stored in the electromagnetic field, which, according to (12.2) and (12.3), is given by the expression:

$$w = \int_0^D \vec{E} \cdot \delta \vec{D} + \int_0^B \vec{H} \cdot \delta \vec{B} \tag{12.5}$$

which, in the special case when the medium is *isotropic and nonlinear*, can be written as:

$$w = \int_0^D E \cdot \delta D + \int_0^B H \cdot \delta B \tag{12.6}$$

whereas, in the special case when the medium is *isotropic and linear*, it can be written as:

$$w = \int_0^D \frac{D}{\varepsilon} \cdot \delta D + \int_0^B \frac{B}{\mu} \cdot \delta B = \frac{1}{2} \cdot \left(E \cdot D + H \cdot B \right) \tag{12.7}$$

and, in the special case when the medium is *anisotropic and linear*, it can be written as:

$$w = \frac{1}{2} \cdot \left(\vec{E} \cdot \vec{D} + \vec{H} \cdot \vec{B} \right) \tag{12.8}$$

It follows that the energy stored in an electromagnetic field in an *anisotropic and linear* medium can be described by the following expression:

$$W = W_{\mathrm{e}} + W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V \left(\vec{E} \cdot \vec{D} \ + \ \vec{H} \cdot \vec{B} \right) \cdot dV \tag{12.9}$$

whereas the energy stored in an electromagnetic field in an *isotropic and linear* medium can be described by the following expression:

$$W = W_{\mathrm{e}} + W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V \left(E \cdot D + H \cdot B \right) \cdot dV = \frac{1}{2} \cdot \int_V \left(\varepsilon \cdot E^2 + \mu \cdot H^2 \right) \cdot dV \tag{12.10}$$

The energy stored in the electromagnetic field of a stationary *perfect dielectric* can also be computed by integrating over the sources and a closed surface that bounds the volume *V*, without integrating over the volume itself. If expressions (9.3) and (9.4) are substituted into expressions (12.2) and (12.3), and Maxwell's differential equations in a stationary perfect dielectric ((6.22) - (6.25)) are taken into account, the following expression for the stored energy in a stationary *nonlinear perfect dielectric* can be obtained:

$$W = \int_V \left(\int_0^{\rho_s} \varphi \cdot \delta\rho_{\mathrm{s}} + \int_0^{A} \vec{J}_{\mathrm{s}} \cdot \delta\vec{A} \right) \cdot dV - \oint_{\partial V} \left(\int_0^{D} \varphi \cdot \delta\vec{D} + \int_0^{A} \vec{H} \times \delta\vec{A} \right) \cdot d\vec{S} \tag{12.11}$$

where:

φ - the electric scalar potential,

$\vec{A}$ - the magnetic vector potential,

$\delta\vec{A} = \dfrac{\partial \vec{A}}{\partial t} \cdot dt$ - the variation of the magnetic vector potential,

$\vec{J}_{\mathrm{s}}$ - the vector of the surface density of the source (impressed) electric current,

ρ_{s} - the volume density of the source (impressed) electric charge,

$\delta\rho_{\mathrm{s}} = \dfrac{\partial \rho_{\mathrm{s}}}{\partial t} \cdot dt$ - the variation of the volume density of the source (impressed) electric charge.

It follows from expression (12.11) that the energy stored in the electromagnetic field of an *unbounded nonlinear* perfect dielectric can be described by the following expression:

$$W = \int_V \left(\int_0^{\rho_s} \varphi \cdot \delta\rho_{\mathrm{s}} + \int_0^{A} \vec{J}_{\mathrm{s}} \cdot \delta\vec{A} \right) \cdot dV \tag{12.12}$$

From expression (12.11), it is straightforward to obtain that the energy stored in the electromagnetic field of a *linear* perfect dielectric can be described by the following expression:

$$W = \frac{1}{2} \cdot \int_V \left(\varphi \cdot \rho_{\mathrm{s}} + \vec{J}_{\mathrm{s}} \cdot \vec{A} \right) \cdot dV - \frac{1}{2} \cdot \oint_{\partial V} \left(\varphi \cdot \vec{D} + \vec{H} \times \vec{A} \right) \cdot d\vec{S} \tag{12.13}$$

from which it follows that the energy stored in the electromagnetic field of an *unbounded linear* perfect dielectric can be described by the following expression:

$$W = \frac{1}{2} \cdot \int_V \left(\varphi \cdot \rho_{\mathrm{s}} + \vec{J}_{\mathrm{s}} \cdot \vec{A} \right) \cdot dV \tag{12.14}$$

According to expressions (12.12) and (12.14), in the case of an unbounded perfect dielectric, the energy stored in the electromagnetic field can be fully computed by integrating over the sources, instead of integrating over the field, i.e., over the entire volume *V*. However, if the perfect dielectric is bounded, then, according to expressions (12.11) and (12.13), the integration must be performed over the closed surface that bounds the volume *V*, and not solely over the sources.

It is important to note that in a conducting medium, the energy stored in the electromagnetic field can never be computed in a way that completely avoids integration over the entire volume *V*.

12.2. Poynting's Theorem in the Time Domain

Poynting's theorem is named after the British physicist John Henry Poynting, who introduced the so-called *Poynting vector*[*] $\vec{\Gamma}$ into electromagnetic theory, which can be written as:

$$\vec{\Gamma} = \vec{E} \times \vec{H} \tag{12.15}$$

where $\vec{E}$ is the electric field intensity vector, and $\vec{H}$ is the magnetic field intensity vector. The direction of the Poynting vector represents the direction of electromagnetic energy flow, or equivalently, the direction of propagation of the electromagnetic wave. Physically, the Poynting vector is the *vector of the surface density of instantaneous electromagnetic power*, and its unit is watts per square meter (W/m²).

According to expressions (6.46) and (6.47), in a stationary conducting medium, at points where no independent sources are present, the following Maxwell differential equations hold:

$$\nabla \times \vec{H} = \vec{J}_{\text{tot}} = \vec{J}_{\text{c}} + \frac{\partial \vec{D}}{\partial t} \tag{12.16}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{12.17}$$

Furthermore, the following holds:

$$\nabla \cdot \vec{\Gamma} = \nabla \cdot \left(\vec{E} \times \vec{H}\right) = \vec{H} \cdot \left(\nabla \times \vec{E}\right) - \vec{E} \cdot \left(\nabla \times \vec{H}\right) \tag{12.18}$$

from which, after substituting expressions (12.6) and (12.17) into expression (12.8), it follows that:

$$\nabla \cdot \vec{\Gamma} = -\vec{E} \cdot \vec{J}_{\text{c}} - \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} - \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{12.19}$$

and this is Poynting's theorem in differential form, which can also be written in the more commonly used form:

$$\nabla \cdot \vec{\Gamma} + \vec{E} \cdot \vec{J}_{\text{c}} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} = 0 \tag{12.20}$$

If the considered volume V contains independent sources (generators) that transfer energy to the electromagnetic field within that volume, then Poynting's theorem in differential form can be written as follows:

$$\frac{dp_{\text{s}}}{dV} = \nabla \cdot \vec{\Gamma} + \vec{E} \cdot \vec{J}_{\text{c}} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \tag{12.21}$$

where:

p_{s} - the instantaneous electromagnetic power delivered to volume V by independent sources,

$\frac{dp_{\text{s}}}{dV}$ - the volume density of the instantaneous electromagnetic power from independent sources.

If the Ostrogradsky-Green-Gauss integral theorem, given by expression (1.60) and illustrated in Figure 1.3, is applied to expression (12.21), Poynting's theorem in integral form can be obtained, which can be written as follows:

$$p_{\text{s}} = \oint_S \vec{\Gamma} \cdot d\vec{S} + \int_V \vec{E} \cdot \vec{J}_{\text{c}} \cdot dV + \int_V \left(\vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \right) \cdot dV \tag{12.22}$$

[*] The Poynting vector is sometimes referred to as the Umov-Poynting vector, after John Henry Poynting and the Russian physicist and mathematician Nikolay Alekseevich Umov.

or, more concisely:

$$p_{\mathrm{s}} = p_{\mathrm{out}} + p_{\mathrm{J}} + \frac{\partial W}{\partial t} \tag{12.23}$$

where:

p_{out} - the instantaneous electromagnetic power flowing out of volume V,

p_{J} - the instantaneous Joule loss power in volume V,

W - the electromagnetic energy stored in volume V,

$\dfrac{\partial W}{\partial t}$ - the time derivative of the energy W, as described by expression (12.1),

$S \equiv \partial V$ - the closed surface enclosing volume V.

According to (12.22) and (12.23), the instantaneous electromagnetic power flowing out of volume V through the closed surface S is given by the following expression:

$$p_{\mathrm{out}} = \int_V \left(\nabla \cdot \vec{\Gamma}\right) \cdot dV = \oint_S \vec{\Gamma} \cdot d\vec{S} \tag{12.24}$$

and the instantaneous power of Joule losses in the considered volume V is given by the following expression:

$$p_{\mathrm{J}} = \int_V \vec{E} \cdot \vec{J}_{\mathrm{c}} \cdot dV \tag{12.25}$$

where, by definition, Joule losses are heat losses that occur in ohmic resistance when an electric current flows through it. This means that Joule losses in the considered conducting medium of volume V are caused by the electric current flowing through that medium. It is easy to conclude that, in the special case when the medium is a perfect dielectric, Joule losses do not exist.

According to expression (12.23), the instantaneous electromagnetic power delivered to volume V by independent sources is equal to the sum of the instantaneous electromagnetic power flowing out of volume V, the instantaneous power of Joule losses within volume V, and the time derivative of the energy stored in the electromagnetic field within volume V.

The instantaneous electromagnetic power flowing into volume V through the closed surface S is given by the following expression:

$$p_{\mathrm{in}} = -\int_V \left(\nabla \cdot \vec{\Gamma}\right) \cdot dV = -\oint_S \vec{\Gamma} \cdot d\vec{S} = -\, p_{\mathrm{out}} \tag{12.26}$$

and expression (12.23) can be rewritten in the following form:

$$p_{\mathrm{s}} + p_{\mathrm{in}} = p_{\mathrm{J}} + \frac{\partial W}{\partial t} \tag{12.27}$$

which means that the sum of the instantaneous electromagnetic power delivered to volume V by independent sources and the instantaneous electromagnetic power flowing into volume V through the closed surface S is equal to the sum of the instantaneous power of Joule losses within volume V and the time derivative of the energy stored in the electromagnetic field within volume V.

If the medium is ***linear and isotropic***, then according to expressions (12.10) and (12.22), Poynting's theorem in integral form can be written as:

$$p_{\mathrm{s}} = \oint_S \vec{\Gamma} \cdot d\vec{S} + \int_V \kappa \cdot E^2 \cdot dV + \frac{1}{2} \cdot \int_V \frac{\partial \left(\varepsilon \cdot E^2 + \mu \cdot H^2\right)}{\partial t} \cdot dV \tag{12.28}$$

Therefore, Poynting's theorem in integral form describes the *flow of instantaneous electromagnetic power*[*].

---

[*] The flow of instantaneous electromagnetic power is sometimes referred to as the flow of electromagnetic energy.

12.3. Complex Poynting's Theorem

The Poynting vector is defined in the time domain by expression (12.1) and, in its physical meaning, represents the vector of the surface density of instantaneous electromagnetic power. If the electromagnetic field is sinusoidal (time-harmonic), a phasor transformation is applied. In the phasor domain, the complex Poynting vector is given by the following expression:

$$\underline{\vec{\Gamma}} = \underline{\vec{E}} \times \underline{\vec{H}}^{*} \tag{12.29}$$

where:

$\underline{\vec{E}}$ - the phasor of the electric field intensity vector,

$\underline{\vec{H}}$ - the phasor of the magnetic field intensity vector,

$\underline{\vec{H}}^{*}$ - the complex conjugate of the phasor of the magnetic field intensity vector $\underline{\vec{H}}$,

* - the symbol denotes complex conjugation of complex scalars and complex vectors.

The real part of the complex Poynting vector is equal to the average value of the Poynting vector in the time domain:

$$\vec{\Gamma}_{\mathrm{av}} = \frac{1}{T} \cdot \int_{0}^{\mathrm{T}} \vec{\Gamma} \cdot dt = \frac{1}{T} \cdot \int_{0}^{\mathrm{T}} \left(\vec{E} \times \vec{H} \right) \cdot dt = \mathrm{Re}\left(\underline{\vec{\Gamma}} \right) = \mathrm{Re}\left(\underline{\vec{E}} \times \underline{\vec{H}}^{*} \right) \tag{12.30}$$

where $T = 1/f$ is the time period of the sinusoidal function, whereas ff is the frequency of the sinusoidal electromagnetic field.

In its physical meaning, the complex Poynting vector represents the *vector of the surface density of the complex apparent electromagnetic power*, and its unit is volt-amperes per square meter (VA/m²).

According to expressions (7.13) and (7.14), in a stationary conducting medium, at points where no independent sources are present, the following Maxwell's differential equations hold for the sinusoidal electromagnetic field in the phasor domain:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{c}} + j \cdot \omega \cdot \underline{\vec{D}} \tag{12.31}$$

$$\nabla \times \underline{\vec{E}} = - j \cdot \omega \cdot \underline{\vec{B}} \tag{12.32}$$

From Maxwell's equation (12.31), it follows that:

$$\nabla \times \underline{\vec{H}}^{*} = \underline{\vec{J}}_{\mathrm{c}}^{*} - j \cdot \omega \cdot \underline{\vec{D}}^{*} \tag{12.33}$$

Furthermore, the following holds:

$$\nabla \cdot \underline{\vec{\Gamma}} = \nabla \cdot \left(\underline{\vec{E}} \times \underline{\vec{H}}^{*} \right) = \underline{\vec{H}}^{*} \cdot \left(\nabla \times \underline{\vec{E}} \right) - \underline{\vec{E}} \cdot \left(\nabla \times \underline{\vec{H}}^{*} \right) \tag{12.34}$$

Substituting expressions (12.32) and (12.33) into expression (12.34) yields:

$$\nabla \cdot \underline{\vec{\Gamma}} = - \underline{\vec{E}} \cdot \underline{\vec{J}}_{\mathrm{c}}^{*} - j \cdot \omega \cdot \left(\underline{\vec{B}} \cdot \underline{\vec{H}}^{*} - \underline{\vec{E}} \cdot \underline{\vec{D}}^{*} \right) \tag{12.35}$$

and this is the complex Poynting's theorem in differential form, which can also be written as:

$$\nabla \cdot \underline{\vec{\Gamma}} + \underline{\vec{E}} \cdot \underline{\vec{J}}_{\mathrm{c}}^{*} + j \cdot \omega \cdot \left(\underline{\vec{B}} \cdot \underline{\vec{H}}^{*} - \underline{\vec{E}} \cdot \underline{\vec{D}}^{*} \right) = 0 \tag{12.36}$$

Therefore, the complex Poynting's theorem is the phasor-domain representation of Poynting's theorem.

If the considered volume V contains independent sources (generators) that transfer energy to the electromagnetic field within that volume, then the complex Poynting's theorem in differential form can be written as:

$$\frac{\partial \overline{S}_{\mathrm{ap,s}}}{\partial V} = \nabla \cdot \underline{\vec{\Gamma}} + \underline{\vec{E}} \cdot \underline{\vec{J}}_{\mathrm{c}}^{*} + j \cdot \omega \cdot \left(\underline{\vec{B}} \cdot \underline{\vec{H}}^{*} - \underline{\vec{E}} \cdot \underline{\vec{D}}^{*} \right) \tag{12.37}$$

where:

$\overline{S}_{\mathrm{ap,s}}$ - the complex apparent electromagnetic power delivered to volume V by sources,

$\frac{\partial \overline{S}_{\mathrm{ap,s}}}{\partial V}$ - the volume density of the complex apparent electromagnetic power $\overline{S}_{\mathrm{ap,s}}$.

If the Ostrogradsky–Green–Gauss integral theorem, given by expression (1.60) and illustrated in Figure 1.3, is applied to expression (12.37), the complex Poynting's theorem in integral form is obtained:

$$\overline{S}_{\mathrm{ap,s}} = \oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} + \int_V \underline{\vec{E}} \cdot \underline{\vec{J}}_{\mathrm{c}}^{*} \cdot dV + j \cdot \omega \cdot \int_V \left(\underline{\vec{B}} \cdot \underline{\vec{H}}^{*} - \underline{\vec{E}} \cdot \underline{\vec{D}}^{*} \right) \cdot dV \tag{12.38}$$

which, in the case when the stationary conducting medium is ***linear and isotropic***, can be written as:

$$\overline{S}_{\mathrm{ap,s}} = \oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} + \int_V \kappa \cdot E_{\mathrm{ef}}^2 \cdot dV + j \cdot \omega \cdot \int_V \left(\mu \cdot H_{\mathrm{ef}}^2 - \varepsilon \cdot E_{\mathrm{ef}}^2 \right) \cdot dV \tag{12.39}$$

where:

E_{ef} - the RMS (effective) value of the sinusoidal electric field intensity,

H_{ef} - the RMS (effective) value of the sinusoidal magnetic field intensity.

The complex Poynting's theorem in integral form (12.39) can be written in condensed form as:

$$\overline{S}_{\mathrm{ap,s}} = \overline{S}_{\mathrm{ap,out}} + P + j \cdot \omega \cdot 2 \cdot \left(W_{\mathrm{m,av}} - W_{\mathrm{e,av}} \right) = \overline{S}_{\mathrm{ap,out}} + P + j \cdot Q \tag{12.40}$$

where:

$\overline{S}_{\mathrm{ap,out}}$ - the complex apparent electromagnetic power flowing out of volume V,

P - the active (average) electromagnetic power in the considered volume V,

$W_{\mathrm{m,av}}$ - the average value of the energy stored in a magnetic field,

$W_{\mathrm{e,av}}$ - the average value of energy stored in an electric field,

Q - the reactive electromagnetic power in the considered volume V,

ω - the angular frequency of a sinusoidal electromagnetic field.

According to expressions (12.39) and (12.40), the complex apparent electromagnetic power flowing out of volume V is given by the following expression:

$$\overline{S}_{\mathrm{ap,out}} = \oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} \tag{12.41}$$

whereas the active electromagnetic power P is equal to the average power of Joule losses within a linear and isotropic conducting medium:

$$P = P_{\mathrm{J,av}} = \int_V \underline{\vec{E}} \cdot \underline{\vec{J}}_{\mathrm{c}}^{*} \cdot dV = \int_V \kappa \cdot E_{\mathrm{ef}}^2 \cdot dV = \frac{1}{T} \cdot \int_0^T p_{\mathrm{J}} \cdot dt \tag{12.42}$$

where p_{J} is the instantaneous power of Joule losses within a linear and isotropic conducting medium:

$$p_{\mathrm{J}} = \int_V \kappa \cdot E^2 \cdot dV \quad ; \quad E_{\mathrm{ef}}^2 = \frac{1}{T} \cdot \int_0^T E^2 \cdot dt \tag{12.43}$$

whereas E is the instantaneous value of the sinusoidal electric field intensity.

According to expressions (12.39) and (12.40), the average value of the energy stored in the magnetic field within volume V is given by the following expression:

$$W_{\mathrm{m,av}} = \frac{1}{2} \cdot \int_V \underline{\vec{B}} \cdot \underline{\vec{H}}^* \cdot dV = \frac{1}{2} \cdot \int_V \mu \cdot H_{\mathrm{ef}}^2 \cdot dV = \frac{1}{T} \cdot \int_0^T W_{\mathrm{m}} \cdot dt \tag{12.44}$$

where W_{m} is the instantaneous value of the energy stored in the magnetic field within a linear and isotropic medium:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V \mu \cdot H^2 \cdot dV \tag{12.45}$$

whereas H is the instantaneous value of the sinusoidal magnetic field intensity.

According to expressions (12.39) and (12.40), the average value of the energy stored in the electric field within the volume V is described by the expression:

$$W_{\mathrm{e,av}} = \frac{1}{2} \cdot \int_V \underline{\vec{E}} \cdot \underline{\vec{D}}^* \cdot dV = \frac{1}{2} \cdot \int_V \varepsilon \cdot E_{\mathrm{ef}}^2 \cdot dV = \frac{1}{T} \cdot \int_0^T W_{\mathrm{e}} \cdot dt \tag{12.46}$$

where W_{e} is the instantaneous value of the energy stored in the electric field within a linear and isotropic medium:

$$W_{\mathrm{e}} = \frac{1}{2} \cdot \int_V \varepsilon \cdot E^2 \cdot dV \tag{12.47}$$

whereas E is the instantaneous value of the sinusoidal electric field intensity.

According to expressions (12.39) and (12.40), the reactive electromagnetic power within volume V is given by the following expression:

$$Q = \omega \cdot 2 \cdot \left(W_{\mathrm{m,av}} - W_{\mathrm{e,av}}\right) \tag{12.48}$$

The complex Poynting's theorem in integral form, in a stationary conducting linear and isotropic medium, as described by expression (12.40), can also be written in the following form:

$$\overline{S}_{\mathrm{ap,s}} + \overline{S}_{\mathrm{ap,in}} = \overline{S}_{\mathrm{ap}} = P + j \cdot Q \quad ; \quad \overline{S}_{\mathrm{ap,in}} = -\overline{S}_{\mathrm{ap,out}} \tag{12.49}$$

where:

$\overline{S}_{\mathrm{ap,in}}$ - the complex apparent electromagnetic power entering the volume V,

$\overline{S}_{\mathrm{ap}}$ - the total complex apparent electromagnetic power received by volume V from independent sources within the volume and from the surrounding space.

From expression (12.49), it follows that for a sinusoidal electromagnetic field in a stationary, conducting, linear, and isotropic medium, the total complex apparent electromagnetic power received by volume V from an independent source located within the volume and from the surrounding space through the surface is equal to the sum of the active electromagnetic power P consumed within volume V and the reactive electromagnetic power Q within volume V.

12.4. Complex Apparent Power in Sinusoidal Electric Circuits

The complex Poynting's theorem (12.49) is also valid for sinusoidal linear electric circuits. Let us consider a series connection of a resistor with resistance R, an inductor with inductance L, and a capacitor with capacitance C (Figure 12.1). Let a sinusoidal voltage u, with angular frequency ω, be applied to this series RLC circuit, and let a sinusoidal current i flow through it. In this case, the instantaneous power p is given by the following expression:

$$p = u \cdot i \tag{12.50}$$

According to the complex Poynting's theorem (12.49), in the phasor domain, the complex apparent power is given by the following expression:

$$\overline{S}_{\text{ap}} = \overline{U} \cdot \overline{I}^* = \overline{Z} \cdot I^2 = P + j \cdot 2 \cdot \omega \cdot \left(W_{\text{m,av}} - W_{\text{e,av}}\right) \tag{12.51}$$

where:

$\overline{U}$ - the branch voltage phasor across the RLC series circuit,

$\overline{I}$ - the phasor of the electric current flowing through the branch,

$I = I_{\text{ef}}$ - the modulus of the electric current phasor; the effective (RMS) value of the sinusoidal electric current,

$\overline{Z}$ - the complex impedance of a series RLC circuit,

$W_{\text{m,av}}$ - the average value of the energy stored in the magnetic field of an inductor,

$W_{\text{e,av}}$ - the average value of the energy stored in the electric field of a capacitor.

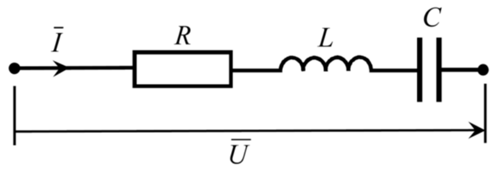


Figure 12.1. Series RLC circuit powered by a sinusoidal electric current

The average value of the energy stored in the magnetic field of an inductor is described by the expression:

$$W_{\text{m,av}} = \frac{L \cdot I^2}{2} = \frac{1}{T} \cdot \int_0^T W_{\text{m}} \cdot dt = \frac{1}{T} \cdot \int_0^T \frac{L \cdot i^2}{2} \cdot dt \tag{12.52}$$

where the instantaneous value of the energy stored in the magnetic field of an inductor, through which a sinusoidal electric current i flows, is given by the following expression:

$$W_{\text{m}} = \frac{L \cdot i^2}{2} \tag{12.53}$$

The average value of the energy stored in the electric field of a capacitor is given by the following expression:

$$W_{\text{e,av}} = \frac{Q_{\text{c}}^2}{2 \cdot C} = \frac{1}{T} \cdot \int_0^T W_{\text{e}} \cdot dt = \frac{1}{T} \cdot \int_0^T \frac{q_{\text{c}}^2}{2 \cdot C} \cdot dt \tag{12.54}$$

where Q_{c} is the effective (RMS) value of the sinusoidal charge on the electric capacitor, whereas q_{c} is the instantaneous value of the charge on the electric capacitor.

From the definition of electric current $i=dq/dt$ in the time domain, it follows that in the phasor domain:

$$Q_{\text{c}}^2 = \overline{Q}_{\text{c}} \cdot \overline{Q}_{\text{c}}^* = \frac{\overline{I}}{j \cdot \omega} \cdot \frac{\overline{I}^*}{-j \cdot \omega} = \frac{\overline{I} \cdot \overline{I}^*}{\omega^2} = \frac{I^2}{\omega^2} \tag{12.55}$$

If expression (12.55) is substituted into expression (12.54), a new expression for the average value of the energy stored in the electric field of the capacitor is obtained:

$$W_{\text{e,av}} = \frac{I^2}{2 \cdot \omega^2 \cdot C} \tag{12.56}$$

If expressions (12.52) and (12.56) are inserted into expression (12.51), the following expression for the complex apparent power of a series RLC circuit is obtained:

$$\overline{S}_{\mathrm{ap}} = \overline{Z} \cdot I^2 = I^2 \cdot R + j \cdot \left(\omega \cdot L - \frac{1}{\omega \cdot C} \right) \cdot I^2 \tag{12.57}$$

where the active power of the RLC circuit is given by the following expression:

$$P = I^2 \cdot R \tag{12.58}$$

According to expression (12.57), the complex impedance of a series RLC circuit can be written as:

$$\overline{Z} = \frac{\overline{U}}{\overline{I}} = R + j \cdot \left(\omega \cdot L - \frac{1}{\omega \cdot C} \right) = R + j \cdot \left(X_{\mathrm{L}} - X_{\mathrm{C}} \right) \tag{12.59}$$

where:

$$X_{\mathrm{L}} = \omega \cdot L \tag{12.60}$$

is known as inductive reactance, whereas:

$$X_{\mathrm{C}} = \frac{1}{\omega \cdot C} \tag{12.61}$$

is known as capacitive reactance.

Therefore, from the expression for the complex apparent power (12.51), the expression (12.59) is obtained for a series RLC circuit, which is the well-known expression for the complex impedance of a series RLC circuit.

12.5. Transmission of Electromagnetic Energy from the Source to the Load

Let the source and the load be connected by a two-wire superconducting transmission line (Figure 12.2). Inside a superconductor, there is neither electromagnetic field nor energy. The energy is transmitted through the surrounding space from the source to the load. Let the surrounding space be a perfect dielectric. If displacement electric currents are neglected, the same principles apply to both time-constant and time-varying voltage sources.

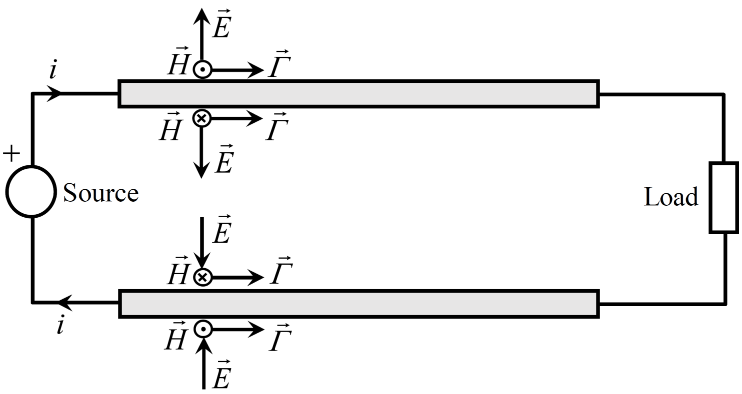


Figure 12.2. Transmission of electromagnetic energy from source to load

If the source and load are connected by a two-wire conductor with finite electrical conductivity, then in addition to the normal (transverse) component of the electric field intensity – which ensures the transmission of electric energy through the surrounding dielectric from the source to the load – there also exists a tangential (longitudinal) component of the electric field intensity vector. This component generates a Poynting vector component that is perpendicular to the outer surface of the conductor and directed inward (Figure 12.3). It causes Joule losses and contributes to the energy stored in the electromagnetic field inside the conductor.

Good conductors with finite electrical conductivity store negligible amounts of electric energy, and the magnetic energy stored within them is insignificant compared to the Joule losses. Therefore, electrical energy is primarily transmitted from the source to the load through the surrounding insulator (dielectric). As such, both superconductors and good conductors with finite electrical conductivity serve the important role of directing energy from the source to the load.

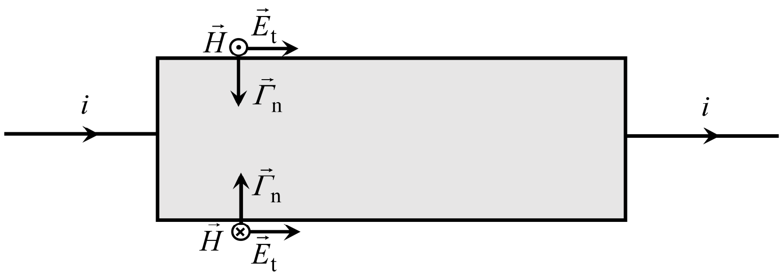


Figure 12.3. The part of electromagnetic energy entering a conductor with finite conductivity

Displacement electric currents exist in the case of a time-varying voltage source and give rise to the so-called reactive electrical energy stored in the dielectric. If the dielectric is imperfect, the transverse electric current also has an active component, which causes Joule losses in the lossy dielectric.

12.6. Charging and Discharging a Parallel-Plate Electric Capacitor

The charging of the electric capacitor is illustrated in Figure 12.4. During the charging process, electrical energy flows from the surrounding space into the dielectric between the conducting plates of the capacitor. The discharging of the electric capacitor is shown in Figure 12.5. During discharge, electrical energy flows from the dielectric between the plates back into the surrounding space.

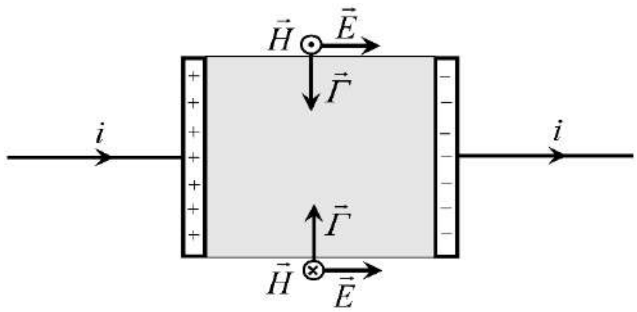


Figure 12.4. Charging a parallel-plate electric capacitor

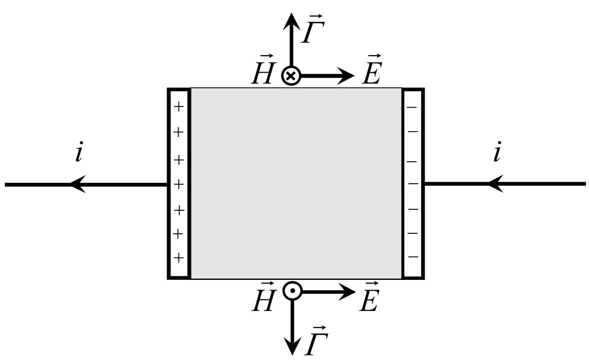


Figure 12.5. Discharging a parallel-plate electric capacitor

During the charging and discharging of an electric capacitor, the direction of the electric field intensity depends on the polarity of the electric charge on the conducting plates, whereas the direction of the magnetic field intensity depends on the direction of the displacement electric current flowing through the dielectric between the plates.

13. CLASSIFICATION OF ELECTROMAGNETIC FIELDS

Electromagnetic fields are classified in various ways in the literature. They can generally be categorized as follows:

- Static fields (electrostatic, magnetostatic, and stationary current fields),
- Quasistatic fields (electroquasistatic, magnetoquasistatic, and fully quasistatic fields),
- Magnetodynamic fields,
- Hybrid or Darwin fields,
- Dynamic fields.

Dynamic problems are electromagnetic problems described by the complete set of Maxwell's equations, without any approximations or simplifications. The hybrid or Darwin electromagnetic field is derived from Maxwell's equations by partially neglecting the displacement electric current.

Quasistatic electromagnetic problems are dynamic electromagnetic problems that can be treated as static problems without introducing significant numerical error. The entire region where the electromagnetic field exists must be taken into account. If the region under consideration is unbounded, the quasistatic conditions are the most restrictive.

It is important to define the criterion of quasistaticity in an unbounded region: *Quasistatic electromagnetic fields are those which, like static fields, are described by Poisson's differential equations for the Lorenz potentials*. The solutions to Poisson's differential equations are undamped and undelayed Lorenz potentials – changes in the field appear simultaneously at all points within the unbounded region.

If the region in which the electromagnetic field exists is bounded, then a disturbance at any point within the region must be felt almost simultaneously at all other points in the region for the quasistatic condition to be satisfied.

In the literature, the electromagnetic field in good conductors, described by the diffusion equations for the Lorenz potentials (9.32) and (9.33), is sometimes referred to as a quasistatic field. However, such a field does not satisfy the previously introduced definition of quasistaticity in an unbounded region. The diffusion equations for the Lorenz potentials are derived from Maxwell's differential equations by neglecting the displacement current. This kind of neglect leads to a quasistatic field – described by Poisson's differential equations for the Lorenz potentials – only in a perfect dielectric.

In the literature, the electromagnetic field in good conductors described by the diffusion equations for the Lorenz potentials is also referred to as a magnetodynamic field, and this terminology is adopted in this textbook as well. In a magnetodynamic time-harmonic field, the potentials in the phasor domain are attenuated and phase-shifted. The phase shift in the phasor domain corresponds to a time delay in the time domain.

14. ELECTROSTATIC FIELD

A static electric field, also known as an electrostatic field, is a field created by stationary electric charges in a perfect dielectric. The differential equations governing the electrostatic field are derived from Maxwell's differential equations for a stationary perfect dielectric (6.22) - (6.25), taking into account that in this case there is no magnetic field and that the time derivatives of electromagnetic quantities are zero.

From Maxwell's differential equations (6.23) and (6.24), the integral form of the electrostatic field equations can be readily derived, and can be written as follows:

$$\nabla \times \vec{E} = 0 \tag{14.1}$$

$$\nabla \cdot \vec{D} = \rho_\mathrm{s} \equiv \rho \tag{14.2}$$

In electrostatics, for the sake of simplicity, the symbol ρ is used to denote the volume density of the source (impressed) electric charge.

From Maxwell's integral equations for a stationary perfect dielectric (6.73) and (6.74), the integral form of the electrostatic field equations can be readily derived and written as follows:

$$\oint_C \vec{E} \cdot d\vec{\ell} = 0 \tag{14.3}$$

$$\oint_S \vec{D} \cdot d\vec{S} = \int_V \rho \cdot dV = Q_\mathrm{enc} \tag{14.4}$$

where Q_enc is the charge within a volume V, which is enclosed by the surface $S \equiv \partial V$. Expressions (14.2) and (14.4) represent Gauss's law for the electrostatic field in its differential and integral forms, respectively.

From expression (9.4), it follows that the electric field intensity, which is a conservative vector field, can be expressed as the negative gradient of the electric scalar potential:

$$\vec{E} = -\nabla \varphi = -\operatorname{grad} \varphi \tag{14.5}$$

According to expression (9.19), in a stationary perfect LIH dielectric, the potential distribution is governed by Poisson's differential equation:

$$\Delta \varphi = -\frac{\rho_\mathrm{s}}{\varepsilon} = -\frac{\rho}{\varepsilon} \quad ; \quad \rho_\mathrm{s} \equiv \rho \tag{14.6}$$

and for a perfect LIH dielectric, according to expression (3.1), the constitutive equation can be written as:

$$\vec{D} = \varepsilon_0 \cdot \varepsilon_\mathrm{r} \cdot \vec{E} = \varepsilon \cdot \vec{E} \tag{14.7}$$

where ε is the permittivity of the medium, ε_0 is the permittivity of vacuum, and ε_r is the relative permittivity of the medium.

The general solution of Poisson's differential equation is the sum of the general solution of the Laplace differential equation (the homogeneous form of Poisson's equation) and a particular solution of Poisson's differential equation, where the particular solution can be written as follows:

$$\varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \int_V \frac{\rho \cdot dV}{r} \tag{14.8}$$

where r is the distance between the source point and the field point.

14.1. Electric Scalar Potential Due to a Point Electric Charge

For a point electric charge Q, located at the origin of a spherical coordinate system, the following holds:

$$\rho = Q \cdot \delta(\vec{r}) \tag{14.9}$$

where $\delta(\vec{r})$ is the Dirac delta function associated with a point located at the origin of the spherical coordinate system.

If expression (14.9) is substituted into expression (14.6), the following Poisson's differential equation is obtained for a point charge in a stationary perfect LIH dielectric:

$$\Delta\varphi = -\frac{Q}{\varepsilon} \cdot \delta(\vec{r}) \tag{14.10}$$

and its particular solution can be written as:

$$\varphi = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r} \tag{14.11}$$

where r is the distance of the field point from the location of the electric charge at the origin of the spherical coordinate system.

Expression (14.11) describes the distribution of the electric scalar potential in an unbounded perfect LIH dielectric produced by a point electric charge Q. In this case, the potential distribution is centrally symmetric with respect to the origin of the coordinate system.

If a point electric charge Q is located at point $T_s = (x_s, y_s, z_s)$ in the Cartesian coordinate system (Figure 14.1), then Poisson's differential equation can be written as:

$$\Delta\varphi = -\frac{Q}{\varepsilon} \cdot \delta(\vec{R}) \quad ; \quad \vec{R} = \vec{r} - \vec{r}_s \tag{14.12}$$

where:

$\vec{r}$ - the radius vector of the field point $T = (x, y, z)$,

$\vec{r}_s$ - the radius vector of the source point $T_s = (x_s, y_s, z_s)$, at which the electric charge QQ is located.

The particular solution of Poisson's differential equation (14.12) is:

$$\varphi = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot R} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot |\vec{r} - \vec{r}_s|} \tag{14.13}$$

where R is the distance between the source point and the field point.

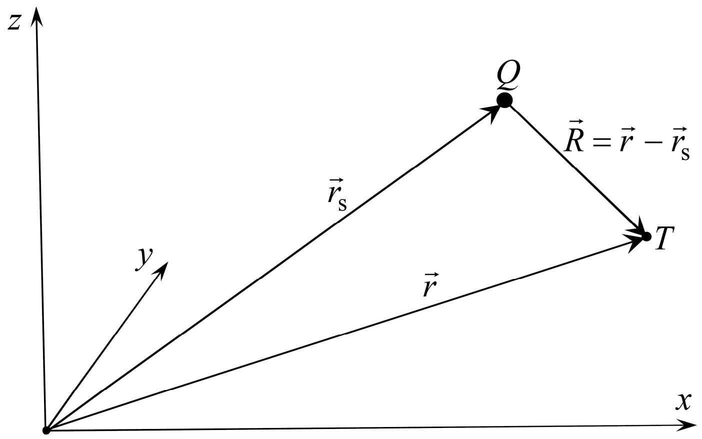


Figure 14.1. Point electric charge in the Cartesian coordinate system

14.2. General Integral Equation of the Electric Scalar Potential

In the general case, the distribution of electric charge may exist within a volume *V* (volume charge with volume density *ρ*), on a surface *S* (surface charge with surface density *σ*), along a curve *C* (line charge with linear density *λ*), and as a set of discrete point charges. Accordingly, the general integral equation of the electric scalar potential can be written as:

$$\varphi = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\left(\int_V \frac{\rho\cdot dV}{r} + \int_S \frac{\sigma\cdot dS}{r} + \int_C \frac{\lambda\cdot d\ell}{r} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{Q_\mathrm{i}}{r_\mathrm{i}}\right) \quad ; \quad \sigma \equiv D_\mathrm{n} \tag{14.14}$$

where *r* is the distance between the source point and the field point.

The general integral equation can also be written in the following form:

$$\varphi = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\left(\int_V \frac{\rho\cdot dV}{R} + \int_S \frac{\sigma\cdot dS}{R} + \int_C \frac{\lambda\cdot d\ell}{R} + \sum_{\mathrm{i}=1}^{\mathrm{n}} \frac{Q_\mathrm{i}}{R_\mathrm{i}}\right) \tag{14.15}$$

where, in this case, $R = \left|\vec{r} - \vec{r}_\mathrm{s}\right|$ is the distance between the source point and the field point.

14.3. Conductor in an Electrostatic Field

A conductor in an electrostatic field has a constant electric potential, whereas the electric field intensity inside the conductor is zero. The entire electric charge is distributed on the surface of the conductor.

If a neutral conducting body (conductor) is placed in an electric field, the field causes a redistribution of the neutral electric charge inside the conductor (Figure 14.2). This phenomenon is known as *electrostatic influence* or *electrostatic induction*.

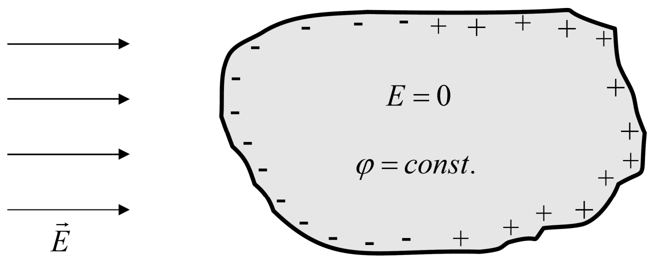


Figure 14.2. Conductor in an electrostatic field

14.4. Coulomb's Law

In 1875, French physicist Charles-Augustin de Coulomb experimentally formulated a law describing the force between two stationary point electric charges. In his experiments, Coulomb used a torsion balance – also known as a torsion pendulum – a device capable of detecting and comparing very small forces. This law became one of the key foundations for James Clerk Maxwell in the development of his electromagnetic theory.

Let there be two isolated point electric charges Q_1 and Q_2 in the air, separated by a distance *r*, and let both charges be positive (Figure 14.3).

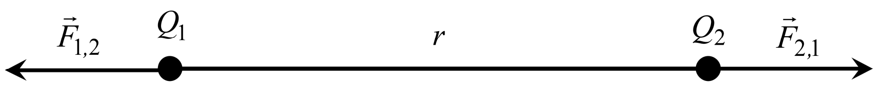


Figure 14.3. Coulomb force between two positive point electric charges

According to Newton's law of action and reaction, the following holds:

$$\vec{F}_{1,2} = -\vec{F}_{2,1} \tag{14.16}$$

where:

$\vec{F}_{1,2}$ - the force with which an electric charge Q_2 acts on an electric charge Q_1,

$\vec{F}_{2,1}$ - the force with which an electric charge Q_1 acts on an electric charge Q_2.

It is a physical fact that like electric charges repel each other, whereas unlike electric charges attract. The amount of force between charges Q_1 and Q_2 is given by the following expression:

$$F = F_{1,2} = F_{2,1} = k_{\mathrm{e}} \cdot \frac{|Q_1 \cdot Q_2|}{r^2} \tag{14.17}$$

where k_e is Coulomb's constant, which in the air (or vacuum) can be expressed as:

$$k_{\mathrm{e}} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \approx 8.9875517923 \times 10^9 \ \ \mathrm{N \cdot m^2/C^2} \tag{14.18}$$

which, for some other perfect dielectric, is given by:

$$k_{\mathrm{e}} = \frac{1}{4 \cdot \pi \cdot \varepsilon} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r} \tag{14.19}$$

The force acting on electric charge Q_2 can also be described by the following vector expression:

$$\vec{F}_{2,1} = k_{\mathrm{e}} \cdot \frac{Q_1 \cdot Q_2}{r^2} \cdot \vec{r}_0 \tag{14.20}$$

where $\vec{r}_0$ is the unit vector directed from charge Q_1 towards charge Q_2 (Figure 14.4).

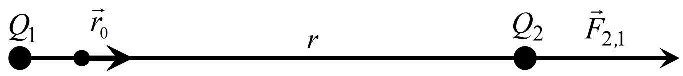


Figure 14.4. Vector representation of Coulomb's force between two like point charges

Based on expressions (14.27) or (14.20), it is easy to see that the force between two point electric charges has the same mathematical form as Newton’s gravitational force. However, the electric force between two charges can be either attractive or repulsive, whereas the gravitational force between two bodies is always attractive. It is important to emphasize that the gravitational force between charged particles is completely negligible compared to the electric force described by Coulomb’s law. This also holds true for elementary particles, such as protons and electrons.

14.5. Electric Field Intensity

Instead of considering the electric force as a direct interaction between two electric charges, one charge is treated as the source of an electric field in the surrounding space. The force acting on the other charge is then viewed as the result of a direct interaction between that charge and the electric field. The electric field intensity is a property of space and, at any point, it is defined as the electric (Coulomb) force acting on a unit positive electric charge placed at that point:

$$\vec{E} = \frac{\vec{F}}{Q_{\mathrm{t}}} \tag{14.21}$$

where:

$\vec{F}$ - the electric, or Coulomb, force on a unit test electric charge,

$\vec{E}$ - the electric field intensity vector,

$Q_t = 1\,\mathrm{C}$ - the unit test electric charge.

The electric field intensity can also be defined as follows:

$$\vec{E} = \lim_{q \to 0} \frac{\vec{F}}{q} \tag{14.22}$$

where q is the test point electric charge.

The electrostatic field can be graphically represented using electric field lines, which are directed imaginary curves indicating the direction of the electric force acting on a positive point charge. The electric field intensity vector is tangent to these lines. Electric field lines originate from positive electric charges and terminate at negative electric charges. Their shape illustrates the direction of the electric field in the region of space under consideration (Figures 14.5 - 14.7).

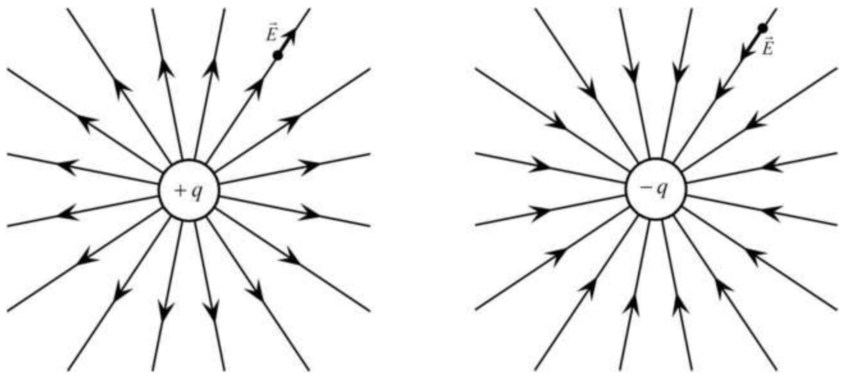


Figure 14.5. Field lines of positive and negative isolated point electric charges in an unbounded perfect dielectric

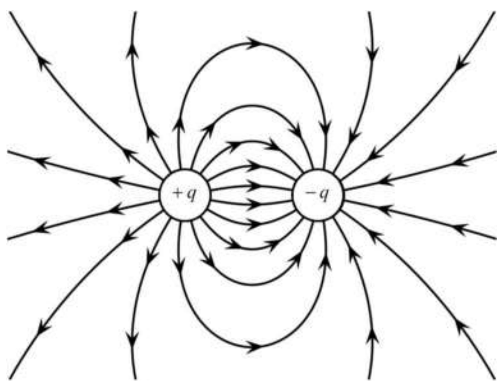


Figure 14.6. Field lines between two point electric charges of the equal magnitude and opposite sign

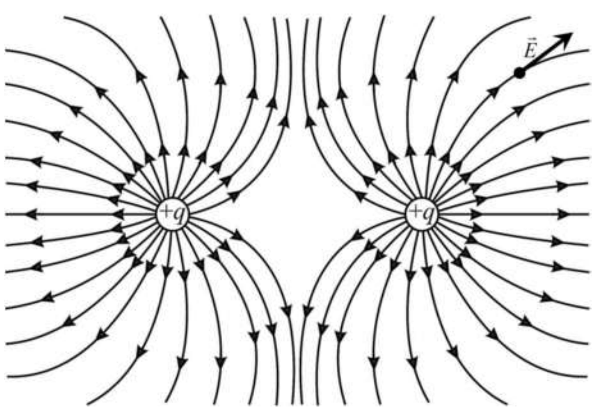


Figure 14.7. Field lines between two equal positive point electric charges

14.6. Calculation of the Electric Field Intensity Using Gauss's Law

Gauss's law in integral form for the electrostatic field is given by expression (14.4). In simple cases exhibiting spherical, cylindrical, or planar symmetry, expressions for the electric field intensity can be derived using Gauss's law.

14.6.1. Electric Field Intensity of an Isolated Point Electric Charge

Let an isolated point electric charge be located in an unbounded perfect LIH dielectric with permittivity ε.

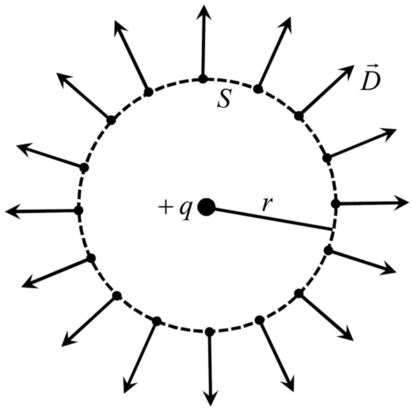


Figure 14.8. Isolated positive point electric charge of magnitude q and spherical surface S

In this case, there is central symmetry, which means that on the spherical surface S (Figure 14.8), whose center coincides with the location of the electric charge $+q$, the electric displacement vector is perpendicular to the closed surface S and has a constant magnitude. In this special case, Gauss's law, given by expression (14.4), can be written as:

$$\oint_S \vec{D} \cdot d\vec{S} = D \cdot S = \varepsilon \cdot E \cdot S = Q_{\text{enc}} = q \tag{14.23}$$

from which it follows that the electric field intensity at a point located a distance r from the charge q is:

$$E = \frac{q}{\varepsilon \cdot S} = \frac{q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} = k_{\text{e}} \cdot \frac{q}{r^2} \tag{14.24}$$

where:

$$S = 4 \cdot \pi \cdot r^2 \tag{14.25}$$

is the surface area of a sphere with radius r.

Expression (14.24) can be derived from Coulomb's law, given by expression (14.20), by including:

$$Q_2 \to q \quad ; \quad Q_2 \to 1\,\text{C} \quad ; \quad \vec{F}_{2,1} \to \vec{E} \tag{14.26}$$

from which follows the vector expression for the electric field intensity of an isolated point charge:

$$\vec{E} = k_{\text{e}} \cdot \frac{q}{r^2} \cdot \vec{r}_0 = \frac{q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \cdot \vec{r}_0 \tag{14.27}$$

where $\vec{r}_0$ is the unit vector defined along the radial direction emanating from the point containing the electric charge q, directed from the charge toward infinity.

14.6.2. Electric Field Intensity of an Isolated Solid Conducting Sphere

Let an isolated solid conducting sphere of radius R, carrying a positive electric charge Q, be placed in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.9).

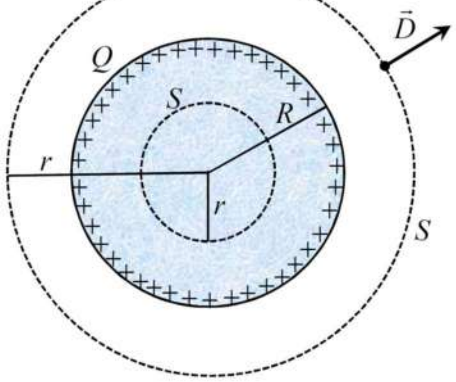


Figure 14.9. An isolated solid conducting sphere in an unbounded perfect LIH dielectric

The entire electric charge Q on the solid conducting sphere, under the influence of electric forces, is confined to its surface and is uniformly distributed. Owing to the spherical symmetry with respect to the center of the sphere, Gauss's law, as given by expression (14.4), can be written as:

$$\oint_S \vec{D} \cdot d\vec{S} = D \cdot S = \varepsilon \cdot E \cdot S = Q_{\text{enc}} = \begin{cases} 0 & \text{for } r < R \\ Q & \text{for } r > R \end{cases} \tag{14.28}$$

from which it follows that inside the solid conducting sphere:

$$E = 0 \quad ; \quad r < R \tag{14.29}$$

whereas outside the conducting sphere, the electric field intensity, taking into account the direction of the field, can be described by the following vector expression:

$$\vec{E} = k_{\text{e}} \cdot \frac{Q}{r^2} \cdot \vec{r}_0 = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \cdot \vec{r}_0 \quad ; \quad r > R \tag{14.30}$$

where $\vec{r}_0$ is the unit vector defined along the radial direction emanating from the center of the conducting sphere, directed from the center toward infinity.

According to expressions (14.28) and (14.30), the electric field intensity inside the solid conducting sphere is zero, whereas outside the sphere it is identical to the field of an isolated point charge Q located at the center of the sphere.

14.6.3. Electric Field Intensity of an Infinitely Long Straight Conductor

Let an infinitely long straight conductor, carrying a positive electric charge with constant linear density λ, be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.10). Based on these assumptions, it follows that:

- There is axial symmetry,
- The electric displacement vector has a constant magnitude over the lateral surface of an imaginary coaxial cylinder along which the integration is performed,
- The electric displacement vector is perpendicular to the conductor and lies in the plane of the bases of the coaxial cylinder.

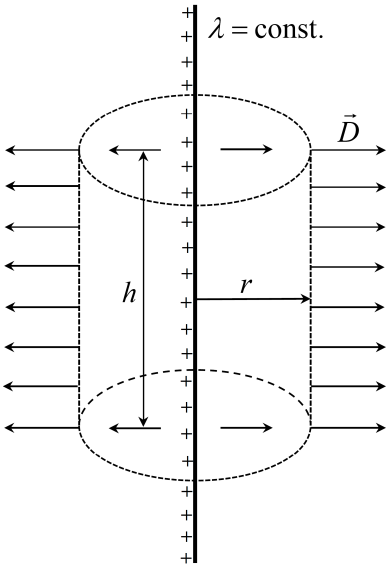


Figure 14.10. An infinitely long straight conductor in an unbounded perfect LIH dielectric charged with a linear charge density λ

In this special case, Gauss's law, as given by expression (14.4), for an imaginary coaxial cylinder can be written as:

$$\oint_S \vec{D} \cdot d\vec{S} = D \cdot 2 \cdot r \cdot \pi \cdot h = \varepsilon \cdot E \cdot 2 \cdot r \cdot \pi \cdot h = Q_{\text{enc}} = \lambda \cdot h \tag{14.31}$$

where E is the intensity of the electric field on the lateral surface (mantle) of the imaginary coaxial cylinder. It is important to note that the electric flux through the bases of the cylinder is equal to zero.

It follows from expression (14.31) that the intensity of the electric field at a point located a distance r from an infinitely long straight conductor can be described by the following expression:

$$E = E_{\text{r}} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon \cdot r} \tag{14.32}$$

and the vector expression for the electric field intensity in a cylindrical coordinate system is:

$$\vec{E} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon \cdot r} \cdot \vec{e}_{\text{r}} \tag{14.33}$$

where $\vec{e}_{\text{r}}$ is the unit vector of the cylindrical coordinate system, in which the infinitely long straight conductor lies along the z-axis.

14.6.4. Electric Field Intensity of an Infinitely Long Straight Conducting Cylinder

Let an infinitely long straight conducting cylinder of radius R, charged with a positive charge of constant linear density λ, be placed in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.11). The cylinder may be either solid or hollow, as this does not affect the distribution of the electric field intensity in the unbounded space under consideration.

Based on the previous considerations in subchapters 14.6.2 and 14.6.3, it is easy to conclude that inside the conducting cylinder:

$$E = 0 \quad ; \quad r < R \tag{14.34}$$

whereas outside the cylinder, the electric field intensity – taking into account the direction of the field – can be described by the vector expression (14.33), which is also valid for an infinitely long straight conductor charged with a constant linear charge density λ.

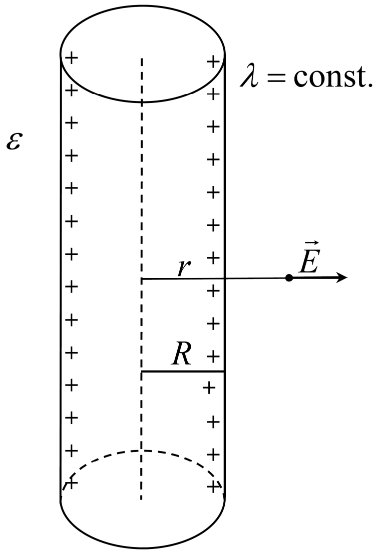


Figure 14.11. An infinitely long straight conducting cylinder in an unbounded perfect LIH dielectric charged with a linear charge density λ

14.6.5. Electric Field Intensity of a Uniformly Charged Plane

Let a plane, charged with a positive charge of constant surface density σ, be placed in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.12). Let σ represent the total surface charge density on both sides of the plane.

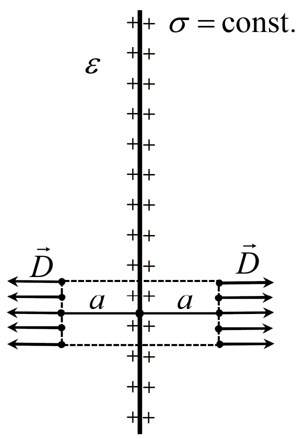


Figure 14.12. A plane in an unbounded perfect LIH dielectric charged with a surface charge density σ

In this case, there is planar symmetry, which means that the electric displacement vector is perpendicular to the plane. In this special case, Gauss's law, as given by expression (14.4), for an imaginary Gaussian cylinder whose bases are parallel to the plane, can be written as:

$$\oint_S \vec{D} \cdot d\vec{S} = D \cdot 2 \cdot S_{\text{base}} = \varepsilon \cdot E \cdot 2 \cdot S_{\text{base}} = Q_{\text{enc}} = \sigma \cdot S_{\text{base}} \tag{14.35}$$

where S_{base} is the area of the base of the imaginary integrating cylinder. The electric flux through the lateral surface (mantle) of the imaginary integrating cylinder is equal to zero.

From expression (14.35), it follows that the electric field intensity created by a surface electric charge on a plane can be described by the following expression:

$$E = \frac{\sigma}{2 \cdot \varepsilon} = \text{const.} \tag{14.36}$$

Therefore, the electric field intensity in the entire space has a constant magnitude, and the electric field intensity vector is perpendicular to the plane and directed away from the plane toward infinity.

Expression (14.36), which describes the electric field intensity of a charged plane, as well as expression (14.32), which describes the electric field intensity of an infinitely long charged straight conductor, can be easily obtained by integrating over the source. An important point is that, in both cases, the direction of the electric field intensity is known in advance, which allows the vector integral to be reduced to a scalar integral. In essence, the component of the electric field intensity of a point charge is being integrated. The expression for the electric field intensity of a charged solid conducting sphere (14.30) can also be obtained by integration over the source, although this approach is more complex for that particular case.

If two parallel planes are charged with like charges of constant surface density σσ, then the electric field intensity between the planes is zero (Figure 14.13). However, if two parallel planes are charged with opposite charges of constant surface density, then the electric field intensity outside the planes is zero (Figure 14.14). It is important to note that the principle of superposition applies in both cases.

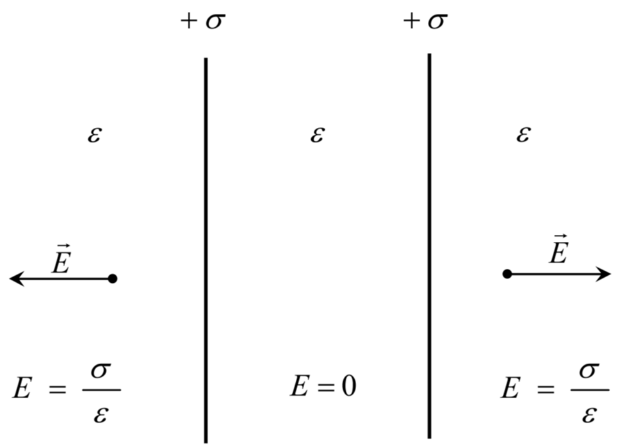


Figure 14.13. Two planes in an unbounded perfect LIH dielectric charged with the like charges of constant surface density σ

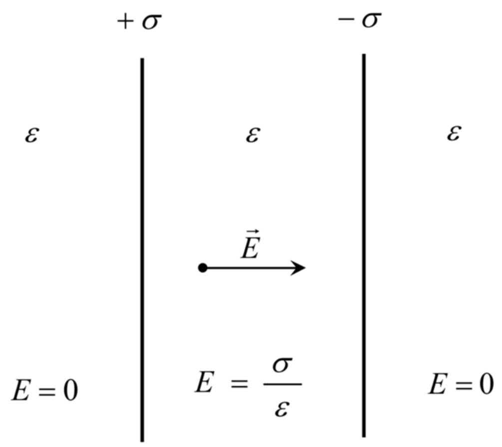


Figure 14.14. Two planes in an unbounded perfect LIH dielectric charged with the opposite charges of constant surface density

14.7. Electric Voltage and Electric Scalar Potential

According to expression (14.5), the electric field intensity and the electric scalar potential are related by:

$$\vec{E} = -\nabla\varphi = -\operatorname{grad}\varphi \tag{14.37}$$

from which it follows that:

$$\varphi = -\int \vec{E}\cdot d\vec{\ell} + C \tag{14.38}$$

For the electric scalar potential, a reference point P can be arbitrarily chosen such that the electric scalar potential at that point is equal to zero.

$$\varphi_{\mathrm{P}} = 0 \text{ V} \tag{14.39}$$

From expression (14.38), it follows that the electric scalar potential at point A can be written as:

$$\varphi_{\mathrm{A}} = -\int_{P}^{A} \vec{E}\cdot d\vec{\ell} = \int_{A}^{P} \vec{E}\cdot d\vec{\ell} \tag{14.40}$$

where the integration can be performed along any path from point A to point P, which is why the electric scalar potential is said to be independent of the integration path.

The electric voltage between points A and B is equal to the work done by the electric force in moving a unit positive charge $Q_{\mathrm{t}} = 1\,\mathrm{C}$ from point A to point B:

$$U_{\mathrm{AB}} = \int_{A}^{B} \vec{F}_{\mathrm{e}}\cdot d\vec{\ell} = \int_{A}^{B} Q_{\mathrm{t}}\cdot\vec{E}\cdot d\vec{\ell} = \int_{A}^{B} \vec{E}\cdot d\vec{\ell} \tag{14.41}$$

This definition of electric voltage is valid in the general case, meaning it also applies to a time-varying electromagnetic field. However, in a time-varying electromagnetic field, the electric voltage is not unique and depends on the integration path from point A to point B. In the case of an electrostatic field, the electric voltage is unique and can be described by the following expression:

$$U_{\mathrm{AB}} = \int_{A}^{B} \vec{E}\cdot d\vec{\ell} = \int_{A}^{P} \vec{E}\cdot d\vec{\ell} + \int_{P}^{B} \vec{E}\cdot d\vec{\ell} = \int_{A}^{P} \vec{E}\cdot d\vec{\ell} - \int_{B}^{P} \vec{E}\cdot d\vec{\ell} = \varphi_{\mathrm{A}} - \varphi_{\mathrm{B}} \tag{14.42}$$

Therefore, in an electrostatic field, the electric voltage between any two points is equal to the difference in electric scalar potentials of those points. Accordingly, it can also be said that in an electrostatic field, the potential of a point is equal to the electric voltage between that point and a reference point:

$$\varphi_{\mathrm{A}} = U_{\mathrm{AP}} = \int_{A}^{P} \vec{E}\cdot d\vec{\ell} \tag{14.43}$$

14.7.1. Electric Scalar Potential of an Isolated Point Electric Charge

Let an isolated point electric charge q be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.8). Assuming that the electric scalar potential at infinity is zero, by integrating the electric field intensity from an arbitrary point A to infinity, according to expression (14.40), the distribution of the electric scalar potential in the LIH dielectric can be written as:

$$\varphi = \varphi_{\mathrm{A}} = \int_{A}^{P} \vec{E}\cdot d\vec{\ell} = \int_{r}^{\infty} E\cdot dr = \frac{q}{4\cdot\pi\cdot\varepsilon}\cdot\int_{r}^{\infty}\frac{dr}{r^2} = \frac{q}{4\cdot\pi\cdot\varepsilon\cdot r} \tag{14.44}$$

where the electric field intensity of the isolated point charge q is given by expression (14.27), and r is the distance from point A to the location of the charge q, i.e., to the center of the spherical coordinate system.

14.7.2. Electric Scalar Potential of an Isolated Solid Conducting Sphere

Let an isolated solid conducting sphere of radius R, carrying a charge Q, be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.9). Assuming that the electric scalar potential at infinity is zero, by integrating the electric field intensity from an arbitrary point A to infinity, according to expression (14.40), the distribution of the electric scalar potential inside the conducting sphere and in the LIH dielectric can be described by the following expressions:

$$\varphi = \varphi_{\mathrm{A}} = \int_A^P \vec{E} \cdot d\vec{\ell} = \int_r^R 0 \cdot dr + \frac{Q}{4 \cdot \pi \cdot \varepsilon} \cdot \int_R^\infty \frac{dr}{r^2} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot R} \quad ; \quad r \leq R \tag{14.45}$$

$$\varphi = \varphi_{\mathrm{A}} = \int_A^R \vec{E} \cdot d\vec{\ell} = \frac{Q}{4 \cdot \pi \cdot \varepsilon} \cdot \int_r^\infty \frac{dr}{r^2} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r} \quad ; \quad r \geq R \tag{14.46}$$

where the electric field intensity is given by expressions (14.29) and (14.30), and r is the distance from point A to the center of the solid conducting sphere, i.e., to the origin of the spherical coordinate system.

Therefore, the electric scalar potential inside a solid conducting sphere is constant and equal to the potential at the surface of the sphere, whereas outside the sphere it decreases with increasing distance of point A from the center of the sphere.

14.7.3. Electric Scalar Potential of an Infinitely Long Straight Conductor

Let an isolated, infinitely long straight conductor, charged with a positive charge of constant linear density λ, be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.10). Assuming that the electric scalar potential is zero at $r = r_0$, by integrating the electric field intensity from an arbitrary point A to the circle $r = r_0$, according to expression (14.40), the distribution of the electric scalar potential in the LIH dielectric is described by the following expression:

$$\varphi = \varphi_{\mathrm{A}} = \int_A^P \vec{E} \cdot d\vec{\ell} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \int_r^{r_0} \frac{dr}{r} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{r}{r_0} \tag{14.47}$$

where the electric field intensity is given by expression (14.33), and r is the shortest distance from point A to the infinitely long straight conductor.

It is easy to conclude from expression (14.47) that, in the case of an infinitely long straight conductor, the reference point cannot be assumed to be at infinity, because the electric scalar potential would then be infinite throughout the entire space.

14.7.4. Electric Scalar Potential of an Infinitely Long Straight Conducting Cylinder

Let an isolated, infinitely long straight conducting cylinder of radius R, charged with a constant linear charge density λ, be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.11). Assuming that the electric scalar potential is zero at $r = r_0$, by integrating the electric field intensity from an arbitrary point A to the circle $r = r_0$, according to expression (14.40), the distribution of the electric scalar potential can be described by the following expressions:

$$\varphi = \varphi_{\mathrm{A}} = \int_A^P \vec{E} \cdot d\vec{\ell} = \int_r^R 0 \cdot dr + \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \int_R^{r_0} \frac{dr}{r} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{R}{r_0} \quad ; \quad r \leq R \tag{14.48}$$

$$\varphi = \varphi_{\mathrm{A}} = \int_A^P \vec{E} \cdot d\vec{\ell} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \int_r^{r_0} \frac{dr}{r} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{r}{r_0} \quad ; \quad r \geq R \tag{14.49}$$

where the electric field intensity is given by expressions (14.33) and (14.34), and r is the shortest distance from point A to the axis of the infinitely long straight conducting cylinder.

14.7.5. Electric Scalar Potential of a Uniformly Charged Plane

Let a plane charged with a constant surface charge density σ be located in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.12). Assuming that the electric scalar potential is zero at $x = x_0$ and $x = -x_0$, by integrating the electric field intensity from an arbitrary point A to the reference point, according to expression (14.40), the distribution of the electric scalar potential can be described by the following expression:

$$\varphi = \varphi_{\mathrm{A}} = \int_A^P \vec{E} \cdot d\vec{\ell} = \int_x^{x_0} E \cdot dx = \frac{\sigma}{2 \cdot \varepsilon} \cdot \left(|x_0| - |x| \right) \tag{14.50}$$

where the electric field intensity is given by expression (14.36), and x is the shortest distance from point A to the charged plane.

Surfaces on which the electric scalar potential is constant are called equipotential surfaces. In a 2D graphical representation of an electrostatic field, these appear as curves of constant potential, known as equipotential lines. In an electrostatic field, electric field lines are perpendicular to equipotential surfaces, which means that in a 2D representation, equipotential lines and electric field lines are also perpendicular to each other. For illustration, Figure 14.15 shows the equipotential lines and electric field lines for two unlike point electric charges of equal magnitude.

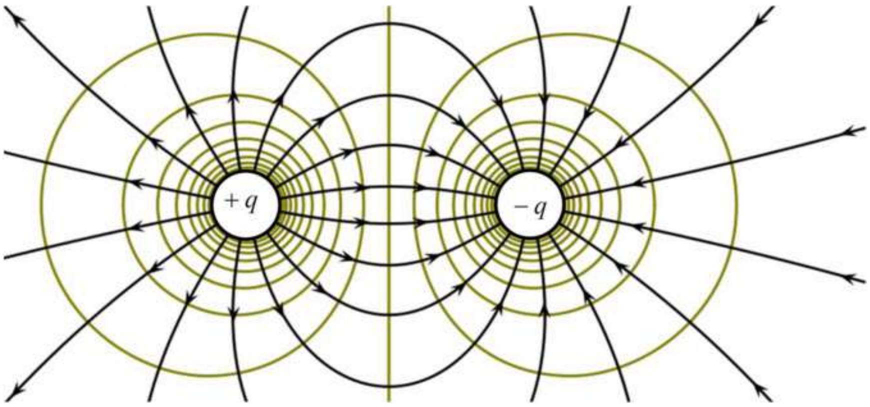


Figure 14.15. Equipotential lines and electric field lines of two unlike charges of equal magnitude

14.8. Capacitance of a Conducting Body

Let an isolated conducting body, charged with a positive charge Q, be located in an unbounded perfect LIH dielectric of permittivity ε, and let its electric scalar potential be $\varphi = \varphi_{\mathrm{body}}$ (Figure 14.16).

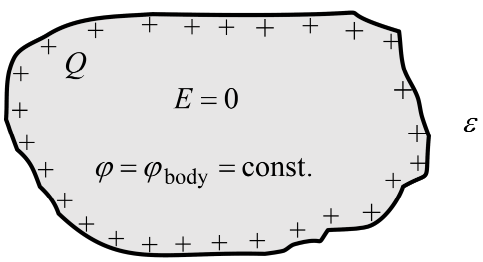


Figure 14.16. Charged isolated conducting body in a perfect LIH dielectric

In an unbounded LIH dielectric, the electric scalar potential of a conducting body is proportional to the total electric charge Q on the body, so the following relation holds:

$$Q = C \cdot \varphi_{\mathrm{body}} \tag{14.51}$$

where C is the *capacitance* of the conducting body. The unit of capacitance is the farad (F), where 1 F = 1 C/V. In LIH dielectrics, the capacitance of a conducting body depends on the geometry and the permittivity of the medium, and it does not depend on the electric charge or the potential of the conducting body.

Let an *isolated solid conducting sphere* of radius R, charged with a total charge Q, be placed in an unbounded perfect LIH dielectric with permittivity ε (Figure 14.9). The electric scalar potential of the sphere is given by expression (14.45), from which it follows that:

$$\varphi_{\text{sphere}} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot R} = \frac{Q}{C} \tag{14.52}$$

and, therefore, the capacitance of an isolated conducting sphere in an unbounded LIH dielectric is:

$$C = 4 \cdot \pi \cdot \varepsilon \cdot R \tag{14.53}$$

which depends on the radius R of the isolated conducting sphere and the permittivity ε, and does not depend on the charge or the potential of the conducting sphere.

14.9. Capacitance of an Electric Capacitor

An *electric capacitor* is a component of an electric circuit used to store electrical energy. It consists of two conducting plates (electrodes) separated by a dielectric. If the dielectric is assumed to be perfect (completely non-conductive), then the capacitor is referred to as a perfect capacitor.

Let an electric capacitor consist of two conducting electrodes, A and B, charged with opposite charges of equal magnitude, with a perfect LIH dielectric of permittivity ε placed between them (Figure 14.17). If the dielectric between the electrodes of an electric capacitor is linear, then the voltage between the electrodes is proportional to the charge on the capacitor plates, and the following relation holds:

$$Q = C \cdot U_{\text{AB}} = C \cdot \left(\varphi_{\text{A}} - \varphi_{\text{B}}\right) \tag{14.54}$$

where C is the *capacitance* of the electric capacitor, which depends on the shape and position of the electrodes, the distance between them, and the permittivity of the surrounding dielectric.

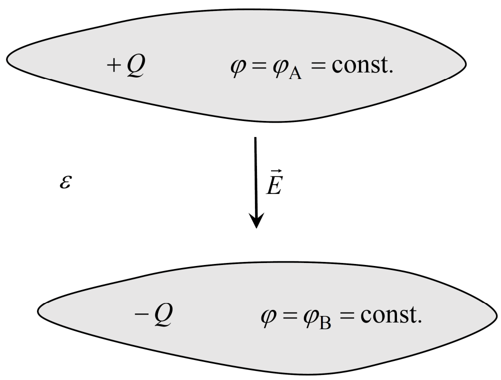


Figure 14.17. Electric capacitor

14.9.1. Capacitance of a Parallel-Plate Electric Capacitor

A parallel-plate electric capacitor consists of two conducting parallel plates of equal area S, separated by a uniform gap of thickness d, filled with a dielectric of permittivity ε (Figure 14.18).

The well-known expression for the capacitance of a parallel-plate electric capacitor can be derived by neglecting edge effects, which implies a homogeneous electric field between the plates. In theory, a parallel-plate capacitor is considered as a segment of two parallel planes charged with opposite charges of constant surface density (Figure 14.14), which allows the electric field intensity to be approximated by the following expression:

$$E=\frac{\sigma}{\varepsilon}=\text{const.} \tag{14.55}$$

whereas the surface electric charge density on the positive plate can be approximated by the expression:

$$\sigma=\frac{Q}{S}=\text{const.} \tag{14.56}$$

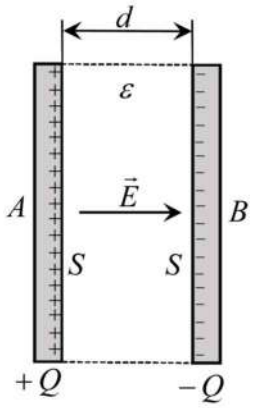


Figure 14.18. Parallel-plate electric capacitor

Furthermore, the voltage between the plates of an electric capacitor can be described by:

$$U=U_{\text{AB}}=\int_A^B \vec{E}\cdot d\vec{\ell}=E\cdot d \tag{14.57}$$

From the previous expressions, it follows that:

$$E=\frac{U}{d}=\frac{Q}{\varepsilon\cdot S} \tag{14.58}$$

and, therefore, the expression for the capacitance of a parallel-plate electric capacitor is given by:

$$C=\frac{Q}{U}=\varepsilon\cdot\frac{S}{d} \tag{14.59}$$

14.9.2. Capacitance of a Cylindrical Electric Capacitor

Let a cylindrical electric capacitor consists of two coaxial conducting cylinders of length ℓ, with a perfect LIH dielectric of permittivity ε placed between them (Figure 14.19).

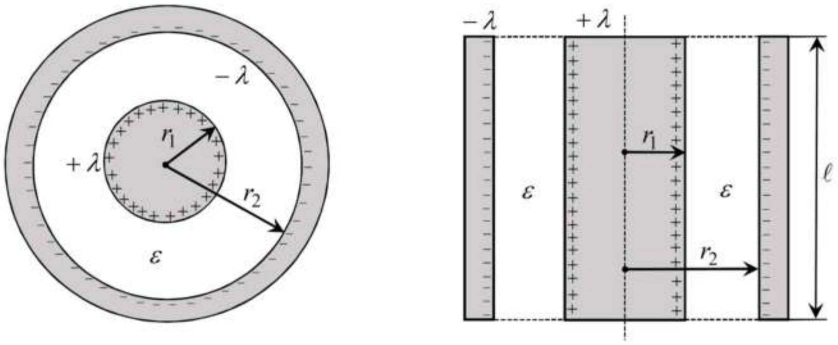


Figure 14.19. Cylindrical electric capacitor

The expression for the capacitance of a cylindrical electric capacitor can be derived by neglecting edge effects, meaning that the capacitor is theoretically modeled as a segment of two coaxial, infinitely long cylinders charged with opposite surface charge densities of equal magnitude. Accordingly, the electric field intensity in the dielectric can be approximated by expression (14.33), which can be written as follows:

$$\vec{E} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon \cdot r} \cdot \vec{e}_{\mathrm{r}} \quad ; \quad r_1 \le r \le r_2 \tag{14.60}$$

where:

$$\lambda = \frac{Q}{\ell} \tag{14.61}$$

Furthermore, the voltage between the cylinders of the electric capacitor can be described by:

$$U = \int_{r_1}^{r_2} E \cdot dr = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{r_2}{r_1} = \frac{Q}{2 \cdot \pi \cdot \varepsilon \cdot \ell} \cdot \ln \frac{r_2}{r_1} \tag{14.62}$$

and, therefore, the expression for the capacitance of the cylindrical electric capacitor is:

$$C = \frac{Q}{U} = \frac{2 \cdot \pi \cdot \varepsilon \cdot \ell}{\ln \frac{r_2}{r_1}} \tag{14.63}$$

14.9.3. Capacitance of a Spherical Electric Capacitor

Let a spherical electric capacitor consist of two concentric conducting spherical electrodes, where the inner electrode is a solid or hollow sphere of radius r_1, whereas the outer electrode is a hollow sphere of radius r_2 (Figure 14.20). A perfect LIH dielectric with permittivity ε is placed between the electrodes. The electrodes are charged with equal and opposite charges, $+Q$ and $-Q$, respectively.

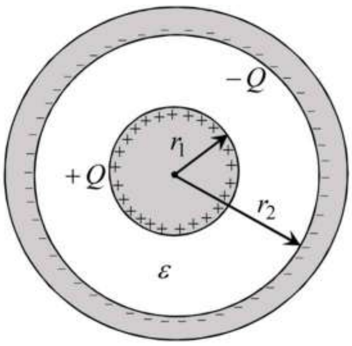


Figure 14.20. Spherical electric capacitor

The electric field intensity in the dielectric is described by expression (14.30), which applies to an isolated solid conducting sphere:

$$\vec{E} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \cdot \vec{r}_0 \quad ; \quad r_1 \le r \le r_2 \tag{14.64}$$

Furthermore, the voltage between the electrodes of an electric capacitor can be described by:

$$U = \int_{r_1}^{r_2} E \cdot dr = \frac{Q}{4 \cdot \pi \cdot \varepsilon} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tag{14.65}$$

from which it follows that the capacitance of the spherical electric capacitor is given by:

$$C = \frac{Q}{U} = \frac{4 \cdot \pi \cdot \varepsilon \cdot r_1 \cdot r_2}{r_2 - r_1} \tag{14.66}$$

In the case where r_2 tends to infinity, the following holds:

$$\lim_{r_2 \to \infty} C = 4 \cdot \pi \cdot \varepsilon \cdot r_1 \tag{14.67}$$

and this is, according to the expression (14.53), the capacitance of an isolated solid conducting sphere of radius r_1 in an unbounded LIH dielectric of permitivity ε.

14.9.4. Calculation of the Capacitance of an Electric Capacitor from Stored Energy

From the definition of electric voltage, it follows that the instantaneous value u of the voltage across an electric capacitor during its charging or discharging can be described by:

$$u = \frac{dW_e}{dq} \tag{14.68}$$

where W_e is the energy stored in the electric field of the capacitor, or energy of an electric capacitor, whereas q is the instantaneous value of the electric capacitor's charge.
If a perfect dielectric is linear and isotropic, then:

$$dW_e = u \cdot dq = \frac{q}{C} \cdot dq \quad ; \quad u = \frac{q}{C} \tag{14.69}$$

and the energy stored in an electric capacitor charged with a charge Q can be expressed as:

$$W_e = \frac{1}{C} \cdot \int_0^Q q \cdot dq = \frac{Q^2}{2 \cdot C} = \frac{Q \cdot U}{2} = \frac{C \cdot U^2}{2} \quad ; \quad Q = C \cdot U \tag{14.70}$$

where U is the voltage across the electric capacitor.

According to expression (12.10), the energy stored in the electric field of a capacitor with a *linear and isotropic* dielectric can be expressed as:

$$W_e = \frac{1}{2} \cdot \int_V E \cdot D \cdot dV = \frac{1}{2} \cdot \int_V \varepsilon \cdot E^2 \cdot dV \tag{14.71}$$

From expressions (14.70) and (14.71), it follows that the capacitance of an electric capacitor with a linear and isotropic dielectric can be calculated from the energy stored in the electric field using the following expression:

$$C = \frac{1}{U^2} \cdot \int_V E \cdot D \cdot dV = \frac{1}{U^2} \cdot \int_V \varepsilon \cdot E^2 \cdot dV = \frac{Q^2}{\int_V \varepsilon \cdot E^2 \cdot dV} \tag{14.72}$$

For a parallel-plate electric capacitor (Figure 14.18), the following holds:

$$C = \frac{1}{U^2} \cdot \int_V \varepsilon \cdot E^2 \cdot dV = \frac{1}{U^2} \cdot \varepsilon \cdot \left(\frac{U}{d}\right)^2 \cdot S \cdot d = \varepsilon \cdot \frac{S}{d} \quad ; \quad E = \frac{U}{d} \tag{14.73}$$

For a cylindrical electric capacitor (Figure 14.19), the following holds:

$$C = \frac{Q^2}{\int_V \varepsilon \cdot E^2 \cdot dV} == \frac{Q^2}{\varepsilon \cdot \int_{r_1}^{r_2} \left(\frac{Q/\ell}{2 \cdot \pi \cdot \varepsilon \cdot r}\right)^2 \cdot 2 \cdot \pi \cdot r \cdot \ell \cdot dr} = \frac{2 \cdot \pi \cdot \varepsilon \cdot \ell}{\ln \frac{r_2}{r_1}} \tag{14.74}$$

For a spherical electric capacitor (Figure 14.20), the following holds:

$$C = \frac{Q^2}{\int\limits_V \varepsilon \cdot E^2 \cdot dV} == \frac{Q^2}{\varepsilon \cdot \int\limits_{r_1}^{r_2} \left(\frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \right)^2 \cdot 4 \cdot \pi \cdot r^2 \cdot dr} = \frac{4 \cdot \pi \cdot \varepsilon \cdot r_1 \cdot r_2}{r_2 - r_1} \tag{14.75}$$

14.9.5. Energy of an Isolated System of Two Electric Capacitors

Let an isolated system (disconnected from the source) consist of two electric capacitors, one of which is charged with a charge Q, whereas the other is uncharged (Figure 14.21).

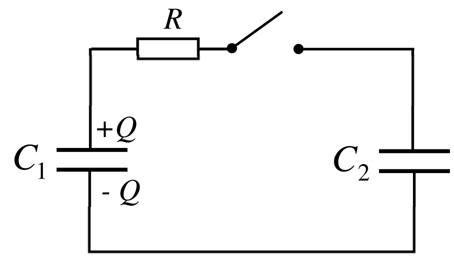


Figure 14.21. An isolated system of electric capacitors with the switch open

In an isolated system, Q = const., and after closing the switch, the electric charge is distributed between both capacitors so that their voltages are equal, as if the capacitors were connected in parallel (Figure 14.22). The Joule losses in resistor R have no effect on the solution.

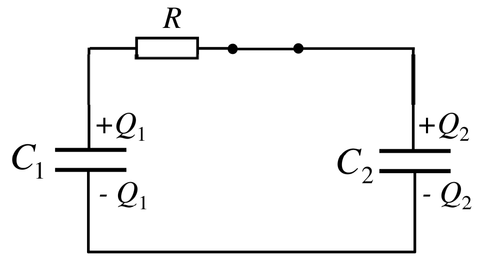


Figure 14.22. An isolated system of electric capacitors with the switch closed

The ratio of the energy of the capacitor system with the switch open to that with the switch closed can be expressed as:

$$\frac{W_{\text{close}}}{W_{\text{open}}} = \frac{\frac{Q^2}{2 \cdot (C_1 + C_2)}}{\frac{Q^2}{2 \cdot C_1}} = \frac{C_1}{C_1 + C_2} \tag{14.76}$$

In the special case where the electric capacitors have equal capacitance, closing the switch reduces the energy of the isolated two-capacitor system by half.

In general, in a complex isolated system, closing a switch results in either a decrease or no change in the total energy of the system. However, in a complex non-isolated system (one connected to an electrical power source), closing a switch may cause the total energy to decrease, increase, or remain the same.

14.10. Electric Scalar Potential and Electric Field of an Electric Dipole

An electric dipole consists of two unlike electric charges of equal magnitude, separated by a distance much smaller than the distance from the charges to the field point ($r >> a$). The dipole is placed in a spherical coordinate system ((r, ϑ, ϕ), and the distribution of the electric scalar potential is independent of the angle ϕ (Figure 14.23). Let the dipole be located in an LIH dielectric with permittivity ε.

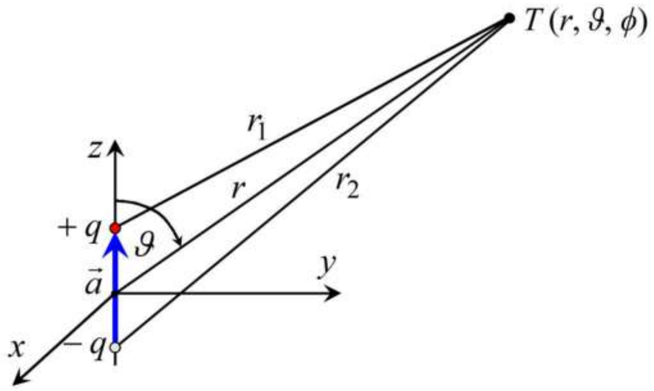


Figure 14.23. Electric dipole in an unbounded perfect LIH dielectric

In an unbounded LIH dielectric, the distribution of the electric scalar potential is given by the following exact expression:

$$\varphi = \frac{q}{4 \cdot \pi \cdot \varepsilon} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) = \frac{q}{4 \cdot \pi \cdot \varepsilon} \cdot \left(\frac{r_2 - r_1}{r_1 \cdot r_2} \right) \tag{14.77}$$

For $r >> a$, the following holds:

$$r_1 \approx r - \frac{a}{2} \cdot \cos \vartheta \quad ; \quad r_2 \approx r + \frac{a}{2} \cdot \cos \vartheta \tag{14.78}$$

and:

$$r_2 - r_1 \approx a \cdot \cos \vartheta \quad ; \quad r_1 \cdot r_2 \approx r^2 - \frac{a^2}{4} \cdot \cos^2 \vartheta \approx r^2 \tag{14.79}$$

from which it follows that:

$$\frac{r_2 - r_1}{r_1 \cdot r_2} \approx \frac{a \cdot \cos \vartheta}{r^2} \tag{14.80}$$

If approximation (14.80) is substituted into expression (14.77), the following expression can be obtained, which approximately describes the distribution of the electric scalar potential of an electric dipole:

$$\varphi = \frac{q \cdot a \cdot \cos \vartheta}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \tag{14.81}$$

Furthermore, the following holds:

$$q \cdot a \cdot \cos \vartheta = \frac{q}{r} \cdot \vec{a} \cdot \vec{r} \tag{14.82}$$

where the vector $\vec{a}$ is directed from the negative electric charge toward the positive electric charge, and its magnitude is equal to the distance between the point charges forming the electric dipole.
Let it be:

$$q \cdot \vec{a} = \vec{p} \tag{14.83}$$

where $\vec{p}$ is the electric dipole moment.

From expressions (14.83) and (14.81), it follows that the distribution of the electric scalar potential of an electric dipole can also be described by the following expression:

$$\varphi = \frac{\vec{p} \cdot \vec{r}}{4 \cdot \pi \cdot \varepsilon \cdot r^3} = \frac{p \cdot \cos \vartheta}{4 \cdot \pi \cdot \varepsilon \cdot r^2} \tag{14.84}$$

From the condition φ = const., it follows that the equation of the equipotential surfaces, as well as the equation of the equipotential lines of the electric dipole, can be written as:

$$\frac{\cos\vartheta}{r^2} = C \tag{14.85}$$

where C is a constant associated with the equipotential surface.
For the electric field intensity in a spherical coordinate system, the following holds:

$$\vec{E} = -\nabla\varphi = -\frac{\partial\varphi}{\partial r}\cdot\vec{e}_\mathrm{r} - \frac{1}{r}\cdot\frac{\partial\varphi}{\partial\vartheta}\cdot\vec{e}_\vartheta - \frac{1}{r\cdot\sin\vartheta}\cdot\frac{\partial\varphi}{\partial\phi}\cdot\vec{e}_\phi \tag{14.86}$$

so it follows that the components of the electric field intensity vector of an electric dipole, whose potential distribution is given by expression (14.84), can be expressed as:

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{p\cdot\cos\vartheta}{2\cdot\pi\cdot\varepsilon\cdot r^3} \tag{14.87}$$

$$E_\vartheta = -\frac{1}{r}\cdot\frac{\partial\varphi}{\partial\vartheta} = \frac{p\cdot\sin\vartheta}{4\cdot\pi\cdot\varepsilon\cdot r^3} \tag{14.88}$$

$$E_\phi = 0 \tag{14.89}$$

The equation of electric field lines, for ϕ = const., follows from the condition:

$$\vec{E}\times d\vec{\ell} = \begin{vmatrix} \vec{e}_\mathrm{r} & \vec{e}_\vartheta & \vec{e}_\phi \\ E_\mathrm{r} & E_\vartheta & 0 \\ dr & r\cdot d\vartheta & 0 \end{vmatrix} = 0 \tag{14.90}$$

from which it follows that:

$$r\cdot E_\mathrm{r}\cdot d\vartheta = E_\vartheta\cdot dr \tag{14.91}$$

and, taking into account expressions (14.87) and (14.88), it follows that:

$$\frac{dr}{r} = \frac{E_r}{E_\vartheta}\cdot d\vartheta = 2\cdot ctg\,\vartheta\cdot d\vartheta \tag{14.92}$$

After integrating both sides of expression (14.92), the following result is obtained:

$$\ln r = 2\cdot\ln(\sin\vartheta) + \ln C = \ln\left(C\cdot\sin^2\vartheta\right) \tag{14.93}$$

from which the final expression for the electric field lines of an electric dipole follows:

$$\frac{r}{\sin^2\vartheta} = C \tag{14.94}$$

where C is the constant associated with the electric field line.

14.11. Energy Stored in an Electrostatic Field

In the general case, the energy stored in an electric field is given by expression (12.2), whereas in a *linear* perfect dielectric, according to expression (12.9), the energy stored in the electric field is given by the following expression:

$$W_\mathrm{e} = \frac{1}{2}\cdot\int_V \vec{E}\cdot\vec{D}\cdot dV \tag{14.95}$$

whereas in the case of a linear and isotropic perfect dielectric, the energy stored in the electric field is given by expression (14.71).

In an *unbounded linear perfect dielectric*, the stored electrostatic energy can also be given by the following expression:

$$W_\mathrm{e} = \frac{1}{2} \cdot \int_V \rho_\mathrm{s} \cdot \varphi \cdot dV + \frac{1}{2} \cdot \sum_{\mathrm{k=1}}^{\mathrm{n}} Q_\mathrm{k} \cdot \varphi_\mathrm{k} \tag{14.96}$$

where Q_k and φ_k are the electric charge and electric scalar potential of the k-th conducting body.

If there is no free electric charge in an unbounded linear medium between conducting bodies, then the stored electrostatic energy is given by the following expression:

$$W_\mathrm{e} = \frac{1}{2} \cdot \sum_{\mathrm{k=1}}^{\mathrm{n}} Q_\mathrm{k} \cdot \varphi_\mathrm{k} \tag{14.97}$$

14.12. Force in the Electrostatic Field of an Isolated System

The force can be computed from the energy of the system or from the electric capacitance of the system. An isolated system is one that does not exchange energy with its surroundings, or a system in which the total electric charge remains constant. Such a system tends toward a state of minimal energy, so the force in an isolated system can be described by the following expression:

$$\vec{F} = -\nabla W_\mathrm{e} \tag{14.98}$$

where W_e is the energy stored in the electrostatic field.

In an isolated system, the force in the direction of the vector $\vec{s}$ can be described by the expression:

$$\vec{F} = -\frac{\partial W_\mathrm{e}}{\partial s} \cdot \vec{s}_0 = -\vec{s}_0 \cdot \left.\frac{\partial W_\mathrm{e}}{\partial s}\right|_{Q=\mathrm{const.}} \tag{14.99}$$

where $\vec{s}_0$ is a unit vector.

In an isolated system, the force in the direction of the vector $\vec{s}$ can also be described by the following expression:

$$\vec{F} = -\vec{s}_0 \cdot \left.\frac{\partial W_\mathrm{e}}{\partial s}\right|_{Q=\mathrm{const.}} = -\vec{s}_0 \cdot \frac{Q^2}{2} \cdot \frac{\partial}{\partial s}\left(\frac{1}{C}\right) = \vec{s}_0 \cdot \frac{1}{2} \cdot \left(\frac{Q}{C}\right)^2 \cdot \frac{\partial C}{\partial s} \tag{14.100}$$

where Q and C are the electric charge and the electric capacitance of the isolated system.

14.12.1. Calculation of the Force on the Plate of a Parallel-Plate Electric Capacitor from Energy

From expression (14.71), it follows that the energy of an isolated parallel-plate electric capacitor (Figure 14.18), which contains a perfect LIH dielectric, can be described by the following expression:

$$W_\mathrm{e} = \frac{1}{2} \cdot E \cdot D \cdot V = \frac{1}{2} \cdot \frac{1}{\varepsilon} \cdot \left(\frac{Q}{S}\right)^2 \cdot S \cdot d \tag{14.101}$$

and, according to expression (14.99), the force on the plate of the electric capacitor can be expressed as:

$$\vec{F} = -\vec{d}_0 \cdot \left.\frac{\partial W_\mathrm{e}}{\partial d}\right|_{Q=\mathrm{const.}} = -\vec{d}_0 \cdot \frac{1}{2} \cdot \frac{1}{\varepsilon} \cdot \left(\frac{Q}{S}\right)^2 \cdot S \tag{14.102}$$

It follows that the force on the plate of an electric capacitor can be expressed as:

$$\vec{F} = \frac{1}{2} \cdot E \cdot D \cdot S \cdot \left(-\vec{d}_0\right) = \frac{W_e}{d} \cdot \left(-\vec{d}_0\right) \tag{14.103}$$

where the direction of the force $-\vec{d}_0$ indicates that the electric force tends to reduce the distance between the capacitor plates, i.e., that the plates of the electric capacitor are attracted to each other.

14.12.2. Calculation of the Force on the Plate of a Parallel-Plate Electric Capacitor from Capacitance

The capacitance of a parallel-plate electric capacitor (Figure 14.18), which contains a perfect LIH dielectric, can be described by expression (14.59), and the following holds:

$$\frac{1}{C} = \frac{d}{\varepsilon \cdot S} \tag{14.104}$$

From expression (14.100), it follows that the force on the plate of an electric capacitor is:

$$\vec{F} = \vec{d}_0 \cdot \frac{Q^2}{2} \cdot \frac{\partial}{\partial d}\left(\frac{1}{C}\right) = -\vec{d}_0 \cdot \frac{Q^2}{2 \cdot \varepsilon \cdot S} = -\vec{d}_0 \cdot \frac{Q^2}{2 \cdot C} \cdot \frac{1}{d} = \frac{W_e}{d} \cdot \left(-\vec{d}_0\right) \tag{14.105}$$

14.12.3. Force on the Electrode of a Cylindrical and Spherical Electric Capacitor

Expressions (14.99) and (14.100) are not suitable for computing the force on the electrodes of a cylindrical and spherical electric capacitor. An important fact is that, in both cases, one electrode is located inside the other. Although parts of the electrodes of these capacitors are attracted under the action of an electric force, the total force on each electrode is zero.

14.13. Force in the Electrostatic Field of a Non-Isolated System

A non-isolated system is one that exchanges energy with its surroundings, i.e., a system in which electric voltages are known and remain constant. Such a system tends toward a state of maximum energy, so the force in a non-isolated system can be described by the following expression:

$$\vec{F} = \nabla W_e \tag{14.106}$$

where W_e is the energy stored in the electrostatic field.

In a non-isolated system, the force in the direction of the vector can be described by the following expression:

$$\vec{F} = \frac{\partial W_e}{\partial s} \cdot \vec{s}_0 = \vec{s}_0 \cdot \left.\frac{\partial W_e}{\partial s}\right|_{U = \text{const.}} \tag{14.107}$$

where $\vec{s}_0$ is a unit vector.

In a non-isolated system, the force in the direction of the vector can also be described by the expression:

$$\vec{F} = \vec{s}_0 \cdot \left.\frac{\partial W_e}{\partial s}\right|_{U = \text{const.}} = \vec{s}_0 \cdot \frac{U^2}{2} \cdot \frac{\partial C}{\partial s} \tag{14.108}$$

where U and C are the electric voltage and electric capacitance of the non-isolated system.

14.13.1. Calculation of the Force on the Plate of a Parallel-Plate Electric Capacitor from Energy

From expression (14.71), it follows that the energy of a non-isolated parallel-plate electric capacitor (Figure 14.18), which contains a perfect LIH dielectric, can be described by the following expression:

$$W_e = \frac{1}{2} \cdot \varepsilon \cdot E^2 \cdot V = \frac{1}{2} \cdot \varepsilon \cdot \left(\frac{U}{d}\right)^2 \cdot S \cdot d = \frac{1}{2} \cdot \varepsilon \cdot \frac{U^2 \cdot S}{d} \tag{14.109}$$

and, according to expression (14.107), the force on the plate of an electric capacitor can be written as:

$$\vec{F} = \vec{d}_0 \cdot \left.\frac{\partial W_e}{\partial d}\right|_{U=\text{const.}} = -\vec{d}_0 \cdot \frac{1}{2} \cdot \varepsilon \cdot \left(\frac{U}{d}\right)^2 \cdot S \tag{14.110}$$

It follows that the force on the plate of an electric capacitor can be expressed as:

$$\vec{F} = \frac{1}{2} \cdot \varepsilon \cdot E^2 \cdot S \cdot \left(-\vec{d}_0\right) = \frac{W_e}{d} \cdot \left(-\vec{d}_0\right) \tag{14.111}$$

where again the direction of the force $-\vec{d}_0$ indicates that the electric force tends to reduce the distance between the capacitor plates, i.e., that the plates of the capacitor attract each other.

14.13.2. Calculation of the Force on the Plate of a Parallel-Plate Electric Capacitor from Capacitance

The capacitance of a parallel-plate electric capacitor (Figure 14.18), which contains a perfect LIH dielectric, can be described by expression (14.59), and the following holds:

$$C = \varepsilon \cdot \frac{S}{d} \tag{14.112}$$

From expression (14.108), it follows that the force on the plate of an electric capacitor is:

$$\vec{F} = \vec{d}_0 \cdot \frac{U^2}{2} \cdot \frac{\partial C}{\partial d} = -\vec{d}_0 \cdot \frac{U^2 \cdot \varepsilon \cdot S}{2 \cdot d^2} = -\vec{d}_0 \cdot \frac{C \cdot U^2}{2} \cdot \frac{1}{d} = \frac{W_e}{d} \cdot \left(-\vec{d}_0\right) \tag{14.113}$$

14.14. Forces and Stresses in the Electrostatic Field

Let the volume V be bounded by a closed surface $S \equiv \partial V$ (Figure 14.24), where $\vec{n}$ is the external unit vector normal to the surface S. The electric force acting on the volume V can be computed by integrating over the closed surface S:

$$\vec{F} = \int_V \rho \cdot \vec{E} \cdot dV = \oint_S \vec{t}_e \cdot dS \tag{14.114}$$

where $\vec{t}_e$ is the *electric stress vector*.

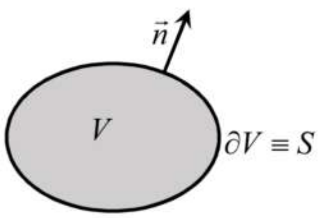


Figure 14.24. Volume bounded by a closed surface S

In an electrostatic field, the electric stress vector in a linear and isotropic perfect dielectric can be described by the following expression:

$$\vec{t}_e = \varepsilon \cdot \left[\vec{E} \cdot \left(\vec{E} \cdot \vec{n}\right) - \frac{1}{2} \cdot E^2 \cdot \vec{n}\right] \tag{14.115}$$

The surface integral described by expression (14.114) can be expressed as a vector sum of three surface integrals of the first kind:

$$\vec{F} = \oint_S \vec{t}_e \cdot dS = \vec{\mathrm{i}} \cdot \oint_S t_{ex} \cdot dS + \vec{\mathrm{j}} \cdot \oint_S t_{ey} \cdot dS + \vec{\mathrm{k}} \cdot \oint_S t_{ez} \cdot dS \tag{14.116}$$

where the electric stress vector in the Cartesian coordinate system can be written as:

$$\vec{t}_e = t_{ex} \cdot \vec{\mathrm{i}} + t_{ey} \cdot \vec{\mathrm{j}} + t_{ez} \cdot \vec{\mathrm{k}} \tag{14.117}$$

The electric field intensity vector $\vec{E}$ bisects the angle between the external unit normal vector $\vec{n}$ and the electric stress vector $\vec{t}_e$ (Figure 14.25):

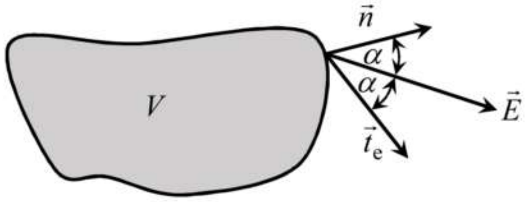


Figure 14.25. Mutual position of three vectors

Based on the electric stress vector at the boundary between two linear and isotropic perfect dielectrics of permittivity ε_1 and ε_2, the following expression for the electrostatic pressure at the boundary of these two dielectrics can be obtained:

$$t_n^e = \frac{|\varepsilon_1 - \varepsilon_2|}{2} \cdot \left(\frac{D_n^2}{\varepsilon_1 \cdot \varepsilon_2} + E_t^2 \right) \tag{14.118}$$

where:

D_n - the normal component of the electric displacement vector,

E_t - the tangential component of the electric field intensity vector.

The electric force at the boundary of two media can be described by a surface integral over the boundary surface between two dielectrics, S_b:

$$\vec{F}_b = \int_{S_b} t_n^e \cdot \vec{n} \cdot dS = \int_{S_b} t_n^e \cdot d\vec{S} \tag{14.119}$$

where $\vec{n}$ is the unit vector normal to the boundary surface of the two dielectrics, directed from the medium of higher permittivity toward the medium of lower permittivity.

The surface integral described by expression (14.119) can be expressed as a vector sum of three surface integrals of the first kind:

$$\vec{F}_b = \int_{S_b} t_n^e \cdot \vec{n} \cdot dS = \vec{i} \cdot \oint_{S_b} t_n^e \cdot n_x \cdot dS + \vec{j} \cdot \oint_{S_b} t_n^e \cdot n_y \cdot dS + \vec{k} \cdot \oint_{S_b} t_n^e \cdot n_z \cdot dS \tag{14.120}$$

where the unit vector normal to the boundary surface of two dielectrics in the Cartesian coordinate system is described by the following expression:

$$\vec{n} = n_x \cdot \vec{i} + n_y \cdot \vec{j} + n_z \cdot \vec{k} = \cos\alpha \cdot \vec{i} + \cos\beta \cdot \vec{j} + \cos\gamma \cdot \vec{k} \tag{14.121}$$

where α, β and γ are the angles between the unit normal vector and the Cartesian coordinate axes.

14.15. Imaging of Electric Charge at the Boundary of Two Dielectrics

Let a point electric charge q be located in front of a dielectric half-space at a distance a. Let the electric charge be located in a half-space of permittivity ε_1, and let the remaining part of the space have permittivity ε_2 (Figure 14.26). For the sake of simplicity, let the electric charge be located on the x-axis of the Cartesian coordinate system (x, y, z).

The expressions for the distribution of the electric scalar potential in both half-spaces can be determined by the method of images. The positions of the image electric charges can be chosen logically, and their magnitudes can be determined by satisfying the boundary conditions (Figures 14.27 and 14.28).

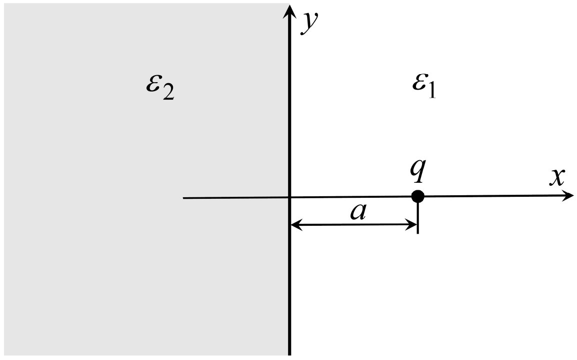


Figure 14.26. Point electric charge in front of the boundary of two dielectric half-spaces

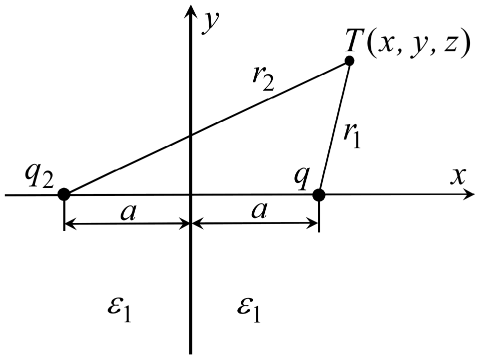


Figure 14.27. Point electric charge and its image when the electric scalar potential is computed in a perfect unbounded dielectric 1

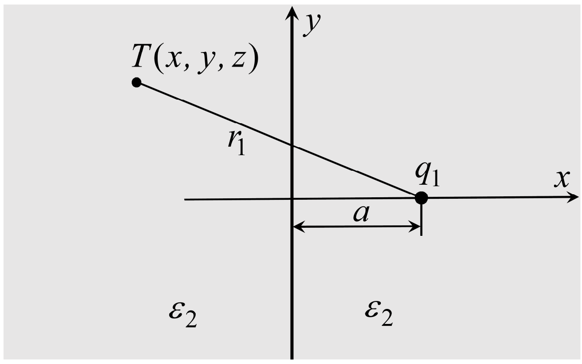


Figure 14.28. Point electric charge image when the electric scalar potential is computed in a perfect unbounded dielectric 2

According to Figure 14.27, the electric scalar potential in a perfect unbounded dielectric 1 is created by a point electric charge q and an image point electric charge q_2, whose position is mirrored with respect to the point electric charge q. According to Figure 14.28, the electric scalar potential in the unbounded perfect dielectric 2 is created by an image point electric charge q_1, which is located at the position of the point electric charge q shown in Figure 14.27.

According to Figure 14.27, the distribution of the electric scalar potential in a perfect dielectric 1 $(x \geq 0)$, can be expressed by the following expression:

$$\varphi_1 = \frac{1}{4 \cdot \pi \cdot \varepsilon_1} \cdot \left(\frac{q}{r_1} + \frac{q_2}{r_2} \right) \tag{14.122}$$

whereas, according to Figure 14.28, the distribution of the electric scalar potential in a perfect dielectric 2 $(x \leq 0)$, can be expressed by the following expression:

$$\varphi_2 = \frac{q_1}{4\cdot\pi\cdot\varepsilon_2\cdot r_1} \tag{14.123}$$

where:

$$r_1 = \sqrt{(x-a)^2 + y^2 + z^2} \quad ; \quad r_2 = \sqrt{(x+a)^2 + y^2 + z^2} \tag{14.124}$$

Expression (14.122) and (14.123) must satisfy the following boundary conditions:

$$\varphi_1\big|_{x=0} = \varphi_2\big|_{x=0} \tag{14.125}$$

$$\varepsilon_1\cdot\frac{\partial\varphi_1}{\partial x}\bigg|_{x=0} = \varepsilon_2\cdot\frac{\partial\varphi_2}{\partial x}\bigg|_{x=0} \tag{14.126}$$

If expressions (14.122) and (14.123) are substituted into the boundary condition (14.125), taking into account expression (14.124), the following expression can be obtained:

$$\frac{1}{\varepsilon_1}\cdot(q+q_2) = \frac{1}{\varepsilon_2}\cdot q_1 \tag{14.127}$$

Similarly, by substituting expressions (14.122) and (14.123) into the boundary condition (14.126), and taking into account expression (14.124), the following expression is obtained:

$$q - q_2 = q_1 \tag{14.128}$$

From the system of two linear equations (14.127) and (14.128), it follows that:

$$q_2 = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2}\cdot q = k_R\cdot q \quad ; \quad k_R = \frac{\varepsilon_1-\varepsilon_2}{\varepsilon_1+\varepsilon_2} \tag{14.129}$$

$$q_1 = \frac{2\cdot\varepsilon_2}{\varepsilon_1+\varepsilon_2}\cdot q = k_T\cdot q \quad ; \quad k_T = \frac{2\cdot\varepsilon_2}{\varepsilon_1+\varepsilon_2} \tag{14.130}$$

where k_R is the reflection factor, whereas k_T is the transmission factor*.

It follows that:

$$k_R + k_T = 1 \tag{14.131}$$

In the special case when $\varepsilon_2 \to \infty$, it is true that:

$$\lim_{\varepsilon_2\to\infty} k_R = -1 \quad \Rightarrow \quad \lim_{\varepsilon_2\to\infty} q_2 = -q \tag{14.132}$$

$$\lim_{\varepsilon_2\to\infty} k_T = 2 \quad \Rightarrow \quad \lim_{\varepsilon_2\to\infty} q_1 = 2\cdot q \quad ; \quad \lim_{\varepsilon_2\to\infty} \varphi_2 = 0 \tag{14.133}$$

from which it follows that a medium with infinite permittivity behaves like a conducting medium.

According to expressions (14.122) i (14.129), the distribution of the electric scalar potential in a perfect dielectric 1 $(x \geq 0)$ can be written as:

$$\varphi_1 = \frac{q}{4\cdot\pi\cdot\varepsilon_1}\cdot\left(\frac{1}{r_1} + \frac{k_R}{r_2}\right) \tag{14.134}$$

whereas, according to expressions (14.123) i (14.130), the distribution of the electric scalar potential in a perfect dielectric 2 $(x \leq 0)$, can be written as:

$$\varphi_2 = \frac{k_T\cdot q}{4\cdot\pi\cdot\varepsilon_2\cdot r_1} \tag{14.135}$$

---

* The reflection factor and transmission factor are static quantities that must be clearly distinguished from the reflection and transmission coefficients, which refer to the behavior of electromagnetic waves.

14.16. Electric Charge Relaxation in an Imperfect Dielectric

Let there be an electric charge of volume density ρ at the observed point of a linear and isotropic imperfect dielectric. Let the imperfect dielectric have permittivity ε and electric conductivity κ. In this case, relaxation of the electric charge occurs. If the medium has higher conductivity, the relaxation of the electric charge is faster.

In this case, the continuity equation can be expressed as:

$$\nabla \cdot \vec{J}_{\mathrm{c}} = \kappa \cdot \nabla \cdot \vec{E} = -\frac{\partial \rho}{\partial t} \tag{14.136}$$

and the corresponding Maxwell's differential equation can be written as:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon} \tag{14.137}$$

where ρ is the volume density of the free electric charge.

From the expressions (14.136) and (14.137), the following differential equation can be derived:

$$\frac{\kappa}{\varepsilon} \cdot \rho = -\frac{\partial \rho}{\partial t} \tag{14.138}$$

which can be written in the following form:

$$\tau \cdot \frac{d\rho}{dt} + \rho = 0 \quad ; \quad \tau = \frac{\varepsilon}{\kappa} \tag{14.139}$$

where τ is the time constant.

After separating the variables, the differential equation (14.139) takes on the following form:

$$\frac{d\rho}{\rho} = -\frac{dt}{\tau} \tag{14.140}$$

and its solution, which can be derived by integrating both sides of the equation, is:

$$\ln \rho = -\frac{t}{\tau} + \ln C \tag{14.141}$$

from which it follows that:

$$\rho = C \cdot e^{-\frac{t}{\tau}} \tag{14.142}$$

In this solution of the differential equation, the initial condition:

$$\rho\big|_{t=0} = \rho_0 \tag{14.143}$$

should be included, and the final solution of the differential equation can be expressed as:

$$\rho = \rho_0 \cdot e^{-\frac{t}{\tau}} \tag{14.144}$$

It follows that the self-discharge of an isolated electric capacitor, which has an imperfect linear and isotropic dielectric with permittivity ε and electrical conductivity κ, occurs, and the time variation of the electric capacitor charge can be described by the following expression:

$$q = Q_0 \cdot e^{-\frac{t}{\tau}} \quad ; \quad \tau = \frac{\varepsilon}{\kappa} \tag{14.145}$$

where q is the instantaneous value, whereas Q_0 is the initial value, of the electric charge on the positive electrode of the electric capacitor.

For a parallel-plate isolated electric capacitor, whose linear and isotropic dielectric is imperfect, the following holds:

$$i = -\frac{dq}{dt} \tag{14.146}$$

where the displacement electric current is the result of the loss of electric charge on the plates of the electric capacitor.

Furthermore, for a parallel-plate electric capacitor, the following holds:

$$i = J \cdot S = \kappa \cdot E \cdot S = \kappa \cdot \frac{\sigma}{\varepsilon} \cdot S = \frac{\kappa}{\varepsilon} \cdot q \tag{14.147}$$

from which the following differential equation arises:

$$\frac{dq}{q} = -\frac{dt}{\tau} \quad ; \quad \tau = \frac{\varepsilon}{\kappa} \tag{14.148}$$

whose solution is described by the expression (14.145), where Q_0 is the initial value of the electric charge on the positive electrode of the parallel-plate electric capacitor.

14.17. Particular and General Solutions of the Laplace Differential Equation

14.17.1. Solutions of the Laplace Equation in the Cartesian Coordinate System

Let the Laplace differential equation in the Cartesian coordinate system (x, y, z) be described by:

$$\Delta\varphi = \frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 0 \tag{14.149}$$

whose particular solution can be obtained after the separation of the variables, so that:

$$\varphi = X(x) \cdot Y(y) \cdot Z(z) \tag{14.150}$$

If expression (14.150) is substituted into expression (14.149), the following differential equation can be obtained:

$$\frac{1}{X} \cdot \frac{d^2X}{dx^2} + \frac{1}{Y} \cdot \frac{d^2Y}{dy^2} + \frac{1}{Z} \cdot \frac{d^2Z}{dz^2} = 0 \tag{14.151}$$

from which the following three ordinary differential equations can be derived:

$$\frac{1}{X} \cdot \frac{d^2X}{dx^2} = \bar{k}_x^2 \quad \Rightarrow \quad \frac{d^2X}{dx^2} - \bar{k}_x^2 \cdot X = 0 \tag{14.152}$$

$$\frac{1}{Y} \cdot \frac{d^2Y}{dy^2} = \bar{k}_y^2 \quad \Rightarrow \quad \frac{d^2Y}{dy^2} - \bar{k}_y^2 \cdot Y = 0 \tag{14.153}$$

$$\frac{1}{Z} \cdot \frac{d^2Z}{dz^2} = \bar{k}_z^2 \quad \Rightarrow \quad \frac{d^2Z}{dz^2} - \bar{k}_z^2 \cdot Z = 0 \tag{14.154}$$

where $\bar{k}_x$, $\bar{k}_y$, and $\bar{k}_z$ are real or imaginary constants for which the following holds:

$$\bar{k}_x^2 + \bar{k}_y^2 + \bar{k}_z^2 = 0 \tag{14.155}$$

For the time being, for the sake of generality, let these constants be complex, and it is important to note that their squares are real constants.

The particular solution of the differential equation (14.152) can be expressed as:

$$X(x)=\begin{cases} C\cdot e^{-k_x \cdot x}+D\cdot e^{k_x \cdot x}=A\cdot \sinh(k_x\cdot x)+B\cdot \cosh(k_x\cdot x) & \text{for } \ k_x^2>0 \\ A\cdot x+B \quad \text{for} \ \ k_x^2=0 \\ A\cdot \sin(\gamma_x\cdot x)+B\cdot \cos(\gamma_x\cdot x) \quad \text{for} \ \ \bar{k}_x^2=-\gamma_x^2<0 \end{cases} \tag{14.156}$$

where in the first and second cases $\bar{k}_x = k_x$ is a real constant, whereas in the third case $\bar{k}_x = \pm j\cdot\gamma_x$ is an imaginary constant and can be replaced by a real constant γ_x in the solution. In the second case, it holds that $\bar{k}_x = k_x = 0$.

The constants A, B, C, and D, given in expression (14.156), are unknown and can be determined from the boundary conditions.

Analogous solutions are also valid for the other two differential equations described by expressions (14.153) and (14.154), i.e., for the functions $Y(y)$ and $Z(z)$.

In the 2D Cartesian coordinate system (x, y), the following holds:

$$\varphi = X(x)\cdot Y(y) \tag{14.157}$$

$$\bar{k}_x^2+\bar{k}_y^2=0 \tag{14.158}$$

and it can be chosen to be:

$$\bar{k}_x^2=k^2 \ ; \ \bar{k}_y^2=-k^2 \quad \Rightarrow \quad k_x=k \ ; \ \bar{k}_y=j\cdot k \tag{14.159}$$

from which it follows that the two corresponding ordinary differential equations can be written as:

$$\frac{d^2X}{dx^2} - k^2\cdot X = 0 \tag{14.160}$$

$$\frac{d^2Y}{dy^2} + k^2\cdot Y = 0 \tag{14.161}$$

If the boundary conditions require it, then it can be assumed that:

$$\bar{k}_x^2=k^2 \ ; \ \bar{k}_y^2=-k^2 \quad \Rightarrow \quad \bar{k}_x=j\cdot k \ ; \ k_y=k \tag{14.162}$$

from which it follows that the two corresponding ordinary differential equations can be written as:

$$\frac{d^2X}{dx^2} + k^2\cdot X = 0 \tag{14.163}$$

$$\frac{d^2Y}{dy^2} - k^2\cdot Y = 0 \tag{14.164}$$

If the boundary conditions can be satisfied by any linear combination of the particular solutions of the Laplace differential equation, then this represents the general solution to the corresponding differential equation, due to the uniqueness of the solution. This solution is known as the solution to the so-called boundary value problem, which is defined by the Laplace differential equation and the boundary conditions. A mathematical or engineering problem described by a differential equation and its boundary conditions is referred to as a *boundary value problem*.

A very commonly used general solution of the Laplace equation in the (x, y) coordinate system is:

$$\varphi(x,y)=\sum_{\mathrm{n}=1}^{\infty} X(x)\cdot\left[C_{\mathrm{n}}\cdot\sin(n\cdot k\cdot y)\ +D_{\mathrm{n}}\cdot\cos(n\cdot k\cdot y)\right] \tag{14.165}$$

where:

$$X(x)=A_{\mathrm{n}}\cdot e^{-n\cdot k\cdot x}+B_{\mathrm{n}}\cdot e^{n\cdot k\cdot x} \tag{14.166}$$

or:

$$X(x)=A_{\mathrm{n}}\cdot\sinh(n\cdot k\cdot x)+B_{\mathrm{n}}\cdot\cosh(n\cdot k\cdot x) \tag{14.167}$$

depending on the boundary conditions that the general solution must satisfy.

It is important to point out that in the general solution of the Laplace equation (14.165) and the corresponding particular solutions (14.166) and (14.167), x and y can be interchanged if the boundary conditions that need to be satisfied require it.

14.17.2. Solutions of the Laplace Equation in a Cylindrical Coordinate System

Let the Laplace differential equation be given in the cylindrical coordinate system (r, ϕ, z):

$$\Delta\varphi=\frac{1}{r}\cdot\frac{\partial}{\partial r}\cdot\left(r\cdot\frac{\partial\varphi}{\partial r}\right)+\frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\phi^2}+\frac{\partial^2\varphi}{\partial z^2}=0 \tag{14.168}$$

whose particular solution can be obtained after the separation of variables, so that:

$$\varphi=R(r)\cdot\Phi(\phi)\cdot Z(z) \tag{14.169}$$

If expression (14.169) is substituted into expression (14.168), the following differential equation can be obtained:

$$\frac{1}{r\cdot R}\cdot\frac{d}{dr}\left(r\cdot\frac{dR}{dr}\right)+\frac{1}{r^2\cdot\Phi}\cdot\frac{d^2\Phi}{d\phi^2}+\frac{1}{Z}\cdot\frac{d^2Z}{dz^2}=0 \tag{14.170}$$

from which the following three ordinary differential equations can be derived:

$$r\cdot\frac{d}{dr}\left(r\cdot\frac{dR}{dr}\right)+\left(\bar{k}^2\cdot r^2-n^2\right)\cdot R=0 \tag{14.171}$$

$$\frac{d^2\Phi}{d\phi^2}+n^2\cdot\Phi=0 \tag{14.172}$$

$$\frac{d^2Z}{dz^2}-\bar{k}^2\cdot Z=0 \tag{14.173}$$

where the complex constant $\bar{k}$ can be either a real or an imaginary constant, whereas n is an integer constant.

The particular solutions of the two ordinary differential equations, described by expressions (14.172) and (14.173), are given by expression (14.156).

If $\bar{k}=k$ is a real constant, then the differential equation (14.171) can be written as:

$$r^2\cdot\frac{d^2R}{dr^2}+r\cdot\frac{dR}{dr}+\left(k^2\cdot r^2-n^2\right)\cdot R=0 \tag{14.174}$$

which is known as the Bessel differential equation, and its particular solution is:

$$R(r) = A \cdot J_n(k \cdot r) + B \cdot N_n(k \cdot r) \qquad (14.175)$$

where:

$J_n(k \cdot r)$ - the Bessel function of the first kind of the n-th order (Figure 14.29),

$N_n(k \cdot r)$ - the Neumann function of the n-th order; the Bessel function of the second kind of the n-th order (Figure 14.29).

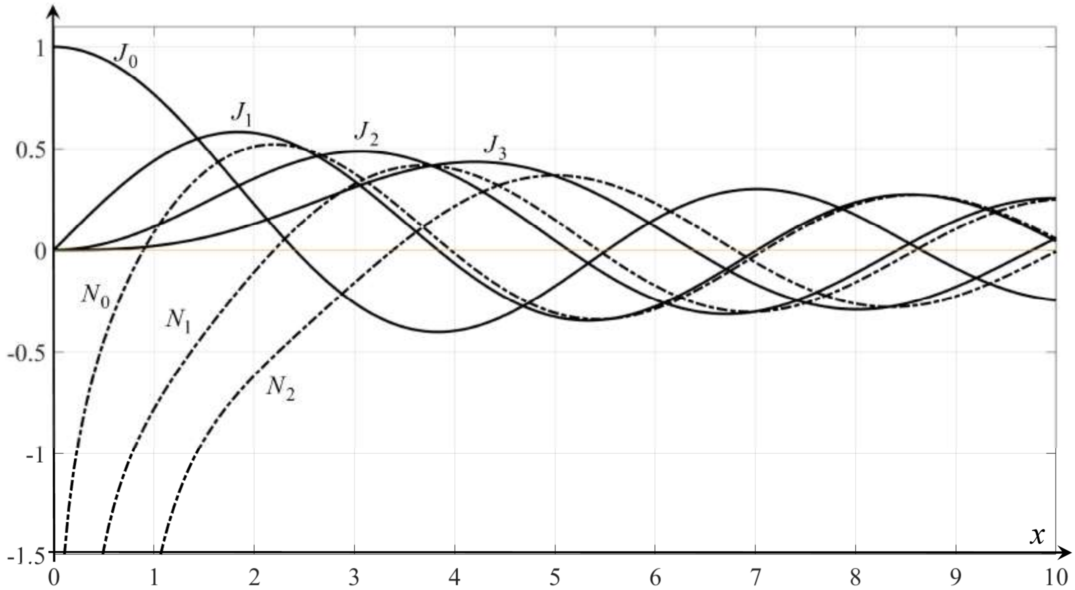


Figure 14.29. Graphic representation of the Bessel functions of the first and second kinds

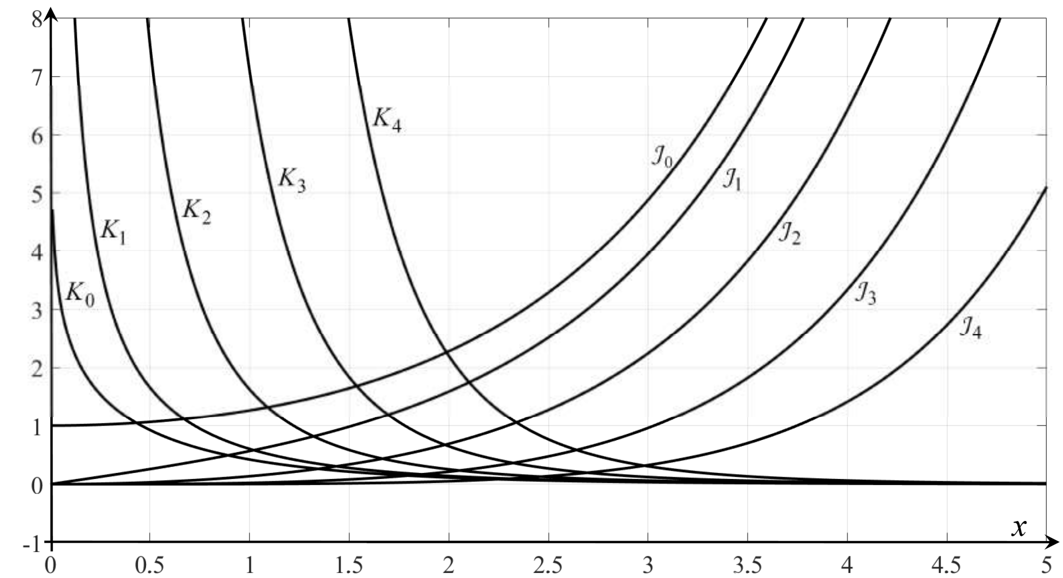


Figure 14.30. Graphic representation of the modified Bessel functions of the first and second kinds

If $\bar{k} = j \cdot \gamma$ x is an imaginary constant and γ is a real constant, then the differential equation (14.168) can be written as:

$$r^2 \cdot \frac{d^2R}{dr^2} + r \cdot \frac{dR}{dr} - \left(\gamma^2 \cdot r^2 + n^2\right) \cdot R = 0 \qquad (14.176)$$

which is known as the modified Bessel differential equation, and its particular solution is:

$$R(r) = A \cdot \mathcal{I}_n(\gamma \cdot r) + B \cdot K_n(\gamma \cdot r) \qquad (14.177)$$

where:

$\mathcal{I}_n(\gamma \cdot r)$ - the modified Bessel function of the first kind of the n-th order (Figure 14.30),

$K_n(\gamma \cdot r)$ - the modified Bessel function of the second kind of the n-th order (Figure 14.30).

If $k = 0,$ then the Bessel differential equation (14.174) reduces to the Euler differential equation:

$$r^2 \cdot \frac{d^2 R}{dr^2} + r \cdot \frac{dR}{dr} - n^2 \cdot R = 0 \tag{14.178}$$

which can be solved by substituting $R = r^t$, resulting in the following equation:

$$r^t \cdot \left(t^2 - n^2\right) = 0 \qquad \Rightarrow \qquad t = \pm n \tag{14.179}$$

If $k = 0$ and $n = 0$, then the Euler differential equation reduces to the differential equation:

$$r \cdot \frac{d}{dr}\left(r \cdot \frac{dR}{dr} \right) = 0 \tag{14.180}$$

All possible solutions of the differential equation (14.171) can be written as:

$$R(r) = \begin{cases} A \cdot J_n(k \cdot r) + B \cdot N_n(k \cdot r) & \text{for } \bar{k}^2 = k^2 > 0 \\ A \cdot \mathcal{I}_n(\gamma \cdot r) + B \cdot K_n(\gamma \cdot r) & \text{for } \bar{k}^2 = -\gamma^2 < 0 \\ A \cdot r^n + B \cdot \dfrac{1}{r^n} & \text{for } \bar{k}^2 = k^2 = 0, \ \ n \neq 0 \\ A \cdot \ln r + B & \text{for } \bar{k}^2 = k^2 = 0 \ \text{ and } \ n = 0 \end{cases} \tag{14.181}$$

A general solution of a differential equation is any linear combination of particular solutions that can satisfy the boundary conditions for a 2D or 3D region.

If the scalar function φ does not depend on z and $n \neq 0$, then the particular solution of the Laplace equation in the polar coordinate system (r, ϕ) is:

$$\varphi(r, \phi) = \left(A \cdot r^n + B \cdot \frac{1}{r^n} \right) \cdot \left[C \cdot \sin(n \cdot \phi) + D \cdot \cos(n \cdot \phi) \right] \tag{14.182}$$

whereas the corresponding general solution of the Laplace equation is:

$$\varphi(r, \phi) = \sum_{n=1}^{\infty} \left(A_n \cdot r^n + B_n \cdot \frac{1}{r^n} \right) \cdot \left[C_n \cdot \sin(n \cdot \phi) + D_n \cdot \cos(n \cdot \phi) \right] \tag{14.183}$$

which must satisfy the boundary conditions.

If the scalar field φ is axisymmetric with respect to the coordinate axis z, then there is no change in the scalar field by the angle ϕ and $n = 0,$ and the general solution of the Laplace equation in the 2D coordinate system (r, z), assuming r cannot be equal to zero in the considered region, is:

$$\varphi(r, z) = \sum_{k=1}^{\infty} (A_k \cdot e^{-k \cdot z} + B_k \cdot e^{k \cdot z}) \cdot \left[C_k \cdot J_0(k \cdot r) + D_k \cdot N_0(k \cdot r) \right] \tag{14.184}$$

whereas in the case when r can be equal to zero in the considered region, the general solution of the Laplace equation in the 2D coordinate system (r, z) is:

$$\varphi(r, z) = \sum_{k=1}^{\infty} (A_k \cdot e^{-k \cdot z} + B_k \cdot e^{k \cdot z}) \cdot J_0(k \cdot r) \tag{14.185}$$

which, in some cases, instead of a sum, can be described by the following integral:

$$\varphi(r, z) = \int_0^\infty \left[A(\lambda) \cdot e^{-\lambda \cdot z} + B(\lambda) \cdot e^{\lambda \cdot z} \right] \cdot J_0(\lambda \cdot r) \cdot d\lambda \tag{14.186}$$

where λ is a continuous parameter.

14.17.3. Solutions of the Laplace Equation in a Spherical Coordinate System

Let the Laplace differential equation be given in the spherical coordinate system (r, ϑ, ϕ):

$$\mathrm{r}^2 \cdot \Delta\varphi = \frac{\partial}{\partial r} \cdot \left(r^2 \cdot \frac{\partial \varphi}{\partial r} \right) + \frac{1}{\sin\vartheta} \cdot \frac{\partial}{\partial \vartheta} \cdot \left(\sin\vartheta \cdot \frac{\partial \varphi}{\partial \vartheta} \right) + \frac{1}{\sin^2\vartheta} \cdot \frac{\partial^2 \varphi}{\partial \phi^2} = 0 \tag{14.187}$$

whose particular solution can be obtained by separation of variables, such that:

$$\varphi = R(r) \cdot \Theta(\vartheta) \cdot \Phi(\phi) \tag{14.188}$$

If expression (14.188) is substituted into expression (14.187), the following three differential equations can be obtained:

$$\frac{d}{dr}\left(r^2 \cdot \frac{dR}{dr} \right) - n \cdot (n+1) \cdot R = 0 \tag{14.189}$$

$$\sin\vartheta \cdot \frac{d}{d\vartheta} \cdot \left(sin\,\vartheta \cdot \frac{d\Theta}{d\vartheta} \right) + \left[n \cdot (n+1) \cdot \sin^2\vartheta - m^2 \right] \cdot \Theta = 0 \tag{14.190}$$

$$\frac{d^2\Phi}{d\phi^2} + m^2 \cdot \Phi = 0 \tag{14.191}$$

where m and n are required to be integers.

Differential equation (14.189) is an Euler differential equation, differential equation (14.190) is a general Legendre differential equation, whereas differential equation (14.191) is an ordinary differential equation whose particular solution is described by expression (14.156). The particular solutions of these three differential equations can be expressed as:

$$R(r) = A \cdot r^{\mathrm{n}} + B \cdot \frac{1}{r^{\mathrm{n+1}}} \tag{14.192}$$

$$\Theta(\vartheta) = C \cdot P_{\mathrm{n}}^{\mathrm{m}}(\cos\vartheta) + D \cdot Q_{\mathrm{n}}^{\mathrm{m}}(\cos\vartheta) \tag{14.193}$$

$$\Phi(\phi) = E \cdot \sin(m \cdot \vartheta) + F \cdot \cos(m \cdot \vartheta) \tag{14.194}$$

where:

$P_{\mathrm{n}}^{\mathrm{m}}(\cos\vartheta)$ - the associated Legendre polynomial of the first kind, of degree n and order m,

$Q_{\mathrm{n}}^{\mathrm{m}}(\cos\vartheta)$ - the associated Legendre polynomial of the second kind, of degree n and order m.

In engineering applications, axisymmetric problems are of particular importance, where there is no variation with respect to the angle ϕ and $m = 0$; in such cases, the Laplace differential equation (14.187) in the 2D coordinate system (r, ϑ) can be written as:

$$\frac{\partial}{\partial r} \cdot \left(r^2 \cdot \frac{\partial \varphi}{\partial r} \right) + \frac{1}{\sin\vartheta} \cdot \frac{\partial}{\partial \vartheta} \cdot \left(\sin\vartheta \cdot \frac{\partial \varphi}{\partial \vartheta} \right) = 0 \tag{14.195}$$

and its particular solution is given by:

$$\varphi(r, \vartheta) = \left(A \cdot r^{\mathrm{n}} + B \cdot \frac{1}{r^{\mathrm{n+1}}} \right) \cdot \left[C \cdot P_{\mathrm{n}}(\cos\vartheta) + D \cdot Q_{\mathrm{n}}(\cos\vartheta) \right] \tag{14.196}$$

where:

$P_n(\cos\vartheta)$ - the Legendre polynomial of the first kind of degree n,

$Q_n(\cos\vartheta)$ - the Legendre polynomial of the second kind of degree n.

Legendre polynomials of the second kind, as well as the associated Legendre polynomials of the second kind, become infinite when $\cos\vartheta = \pm 1$, so these polynomials cannot be used as solutions if the domain includes $\vartheta = 0$ and/or $\vartheta = \pi$. In such cases, according to (14.196), the particular solution of the Laplace equation in the 2D coordinate system (r, ϑ) can be written as:

$$\varphi(r,\vartheta) = \left(A\cdot r^{n} + B\cdot\frac{1}{r^{n+1}} \right)\cdot P_n(\cos\vartheta) \tag{14.197}$$

whereas the corresponding general solution of the Laplace equation is:

$$\varphi(r,\vartheta) = \sum_{n=0}^{\infty}\left(A_n\cdot r^{n} + B_n\cdot\frac{1}{r^{n+1}} \right)\cdot P_n(\cos\vartheta) \tag{14.198}$$

Legendre polynomials of the first and second kind are defined for $x\in[-1,1]$, and for illustration, the Legendre polynomials of the first kind are shown graphically in Figure 14.31.

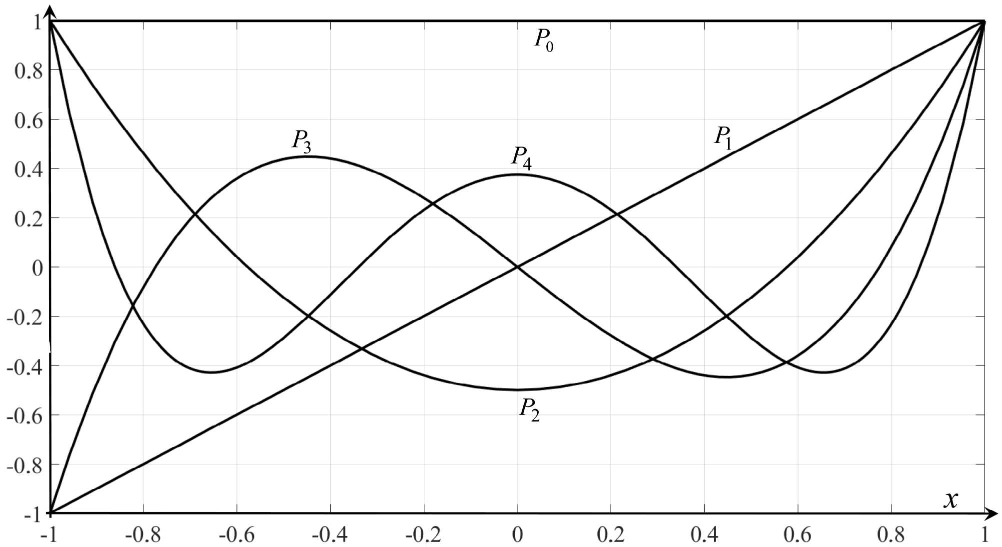


Figure 14.31. Graphical representation of the Legendre polynomials of the first kind

For the Legendre polynomials of the first kind, the following holds true:

$$P_0(x) = 1 \quad ; \quad P_1(x) = x \tag{14.199}$$

and Bonnet's recurrence formula for the Legendre polynomials of the first kind can be written as:

$$P_n(x) = \frac{2\cdot n-1}{n}\cdot x\cdot P_{n-1}(x) - \frac{n-1}{n}\cdot P_{n-2}(x) \quad \text{for } n\geq 2 \tag{14.200}$$

For the Legendre polynomials of the second kind, the following holds true:

$$Q_0(x) = \frac{1}{2}\cdot\ln\frac{1+x}{1-x} \quad ; \quad Q_1(x) = \frac{x}{2}\cdot\ln\frac{1+x}{1-x} - 1 \tag{14.201}$$

and Bonnet's recurrence formula for the Legendre polynomials of the second kind can be written as:

$$Q_n(x) = \frac{2\cdot n-1}{n}\cdot x\cdot Q_{n-1}(x) - \frac{n-1}{n}\cdot Q_{n-2}(x) \quad \text{for } n\geq 2 \tag{14.202}$$

14.18. Dielectric Solid Sphere in a Homogeneous Electric Field

If a dielectric solid sphere is placed in a homogeneous electric field, a mutual interaction occurs between the field and the sphere. Consider a dielectric sphere (medium 1) of radius a placed in a homogeneous electric field, surrounded by an unbounded, perfect linear, isotropic, and homogeneous (LIH) dielectric medium (medium 2). A spherical coordinate system with its origin at the center of the sphere is used (see Figure 14.32).

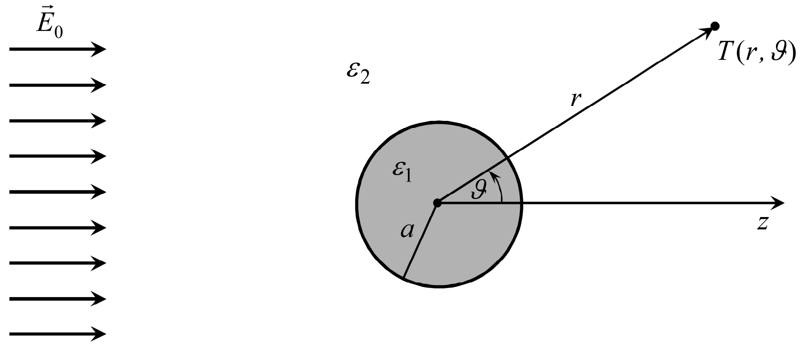


Figure 14.32. Dielectric solid sphere in a homogeneous electric field

The boundary condition at large distances from the sphere requires that the electric field remains homogeneous. To satisfy this condition, the general solution of the Laplace differential equation in medium 2 must be adjusted accordingly. According to expression (14.198), it can be written as:

$$\varphi_2 = \sum_{\mathrm{n}=0}^{\infty} \left(A_{\mathrm{n}} \cdot r^n + B_{\mathrm{n}} \cdot \frac{1}{r^{n+1}} \right) \cdot P_{\mathrm{n}}(\cos\vartheta) \tag{14.203}$$

It is sufficient to adjust the radial component of the electric field intensity at infinity, which, as shown in Figure 14.33, can be described by the following expression:

$$E_{\mathrm{r}} = E_0 \cdot \cos\vartheta = -\left.\frac{\partial \varphi_2}{\partial r}\right|_{r\to\infty} \tag{14.204}$$

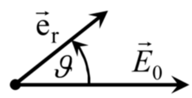


Figure 14.33. Electric field intensity vector far from the sphere

If expression (14.203) is substituted into expression (14.204), the following is obtained:

$$E_0 \cdot \cos\vartheta = -\sum_{\mathrm{n}=0}^{\infty} \left[n \cdot A_{\mathrm{n}} \cdot r^{n-1} - (n+1) \cdot B_{\mathrm{n}} \cdot \frac{1}{r^{n+2}} \right]_{r\to\infty} \cdot P_{\mathrm{n}}(\cos\vartheta) \tag{14.205}$$

Since, according to expression (14.199):

$$P_1(\cos\vartheta) = \cos\vartheta \tag{14.206}$$

it follows that $n = 1$, which means that all coefficients A_{n} and B_{n} are equal to zero, except for A_1 and B_1. Therefore, from expression (14.205), it can be concluded that:

$$E_0 \cdot \cos\vartheta = -\left(A_1 - 2 \cdot B_1 \cdot \frac{1}{r^3} \right)_{r\to\infty} \cdot \cos\vartheta = -A_1 \cdot \cos\vartheta \tag{14.207}$$

implying that:

$$A_1 = E_0 \tag{14.208}$$

whereas B_1 is currently unknown, since $1/r^3$ tends to zero as $r \to \infty$.

Therefore, the distribution of the electric scalar potential in medium 2 is given by the following expression:

$$\varphi_2 = \left(-E_0 \cdot r + B_1 \cdot \frac{1}{r^2} \right) \cdot \cos\vartheta \tag{14.209}$$

Furthermore, to satisfy the boundary conditions between the two media, the electric scalar potential inside the dielectric solid sphere is described by the following expression:

$$\varphi_1 = \left(C_1 \cdot r + D_1 \cdot \frac{1}{r^2} \right) \cdot \cos\vartheta \tag{14.210}$$

From the condition that the electric scalar potential at the center of the dielectric solid sphere must remain finite, it follows that:

$$D_1 = 0 \tag{14.211}$$

and:

$$\varphi_1 = C_1 \cdot r \cdot \cos\vartheta \tag{14.212}$$

Therefore, the distribution of the electric scalar potential in both media is described by expressions (14.209) and (14.212), which contain two unknown coefficients that can be determined from the boundary conditions:

$$\varphi_1\big|_{r=a} = \varphi_2\big|_{r=a} \tag{14.213}$$

$$-\varepsilon_1 \cdot \frac{\partial \varphi_1}{\partial r}\bigg|_{r=a} = -\varepsilon_2 \cdot \frac{\partial \varphi_2}{\partial r}\bigg|_{r=a} \tag{14.214}$$

resulting in the following system of linear equations:

$$\frac{1}{a^2} \cdot B_1 - a \cdot C_1 = E_0 \cdot a \quad ; \quad \frac{2 \cdot \varepsilon_2}{a^3} \cdot B_1 + \varepsilon_1 \cdot C_1 = -\varepsilon_2 \cdot E_0 \tag{14.215}$$

from which it is straightforward to obtain:

$$B_1 = E_0 \cdot a^3 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \quad ; \quad C_1 = -E_0 \cdot \frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \tag{14.216}$$

According to expressions (14.209), (14.212) and (14.216), the distribution of the electric scalar potential is given by the following expressions:

$$\varphi_1 = -\frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot r \cdot \cos\vartheta = -\frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot z \quad ; \quad r \le a \tag{14.217}$$

$$\varphi_2 = -E_0 \cdot r \cdot \cos\vartheta + \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot \left(\frac{a}{r} \right)^3 \cdot E_0 \cdot r \cdot \cos\vartheta \quad ; \quad r \ge a \tag{14.218}$$

from which it follows that the electric scalar potential vanishes at the center of the sphere, that is, on the plane passing through the center where $\vartheta = \pi/2$, corresponding to the plane $z = 0$.

The electric scalar potential of a given homogeneous field, i.e., the distribution of the electric scalar potential in the absence of the dielectric solid sphere, follows from the condition:

$$E_0 = -\frac{\partial \varphi_h}{\partial z} \quad ; \quad \varphi_h\big|_{z=0} = 0 \tag{14.219}$$

from which it follows that:

$$\varphi_h = -E_0 \cdot z = -E_0 \cdot r \cdot \cos\vartheta = -\vec{E}_0 \cdot \vec{r} \tag{14.220}$$

From expressions (14.217) and (14.220), it follows that the electric field inside the sphere is also homogeneous and oriented in the same direction as the electric field far from the sphere, but with a different magnitude. The electric field intensity inside the sphere depends on the ratio of the two permittivities, $\varepsilon_1/\varepsilon_2$.

The distribution of the electric scalar potential in the unbounded medium 2 can also be described by modelling the dielectric solid sphere as an electric dipole:

$$\varphi_2 = -E_0 \cdot z + \frac{\vec{p} \cdot \vec{r}}{4 \cdot \pi \cdot \varepsilon_2 \cdot r^3} = -\vec{E}_0 \cdot \vec{r} + \frac{\vec{p} \cdot \vec{r}}{4 \cdot \pi \cdot \varepsilon_2 \cdot r^3} \tag{14.221}$$

where, according to expression (14.218), the electric dipole moment $\vec{p}$ can be written as:

$$\vec{p} = \vec{E}_0 \cdot 3 \cdot V_{\text{sphere}} \cdot \varepsilon_2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \quad ; \quad V_{\text{sphere}} = \frac{4}{3} \cdot a^3 \cdot \pi \tag{14.222}$$

The electric dipole moment $\vec{p}$ resulting from the bound surface charge is located on the surface of the sphere. The dielectric sphere becomes polarized under the influence of the external electric field (see Figures 14.34 and 14.35).

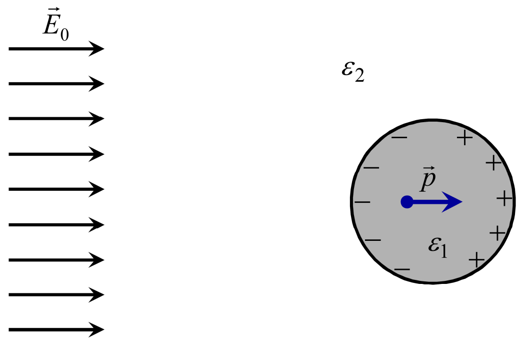


Figure 14.34. Electric dipole moment of a polarized dielectric solid sphere for $\varepsilon_1 > \varepsilon_2$

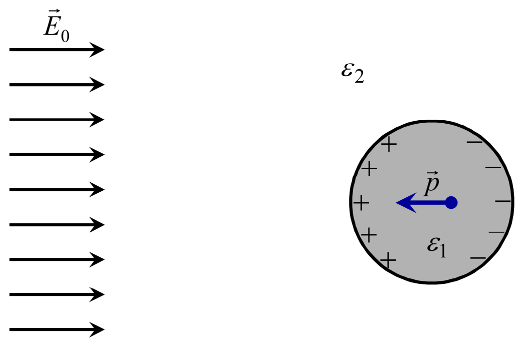


Figure 14.35. Electric dipole moment of a polarized dielectric solid sphere for $\varepsilon_1 < \varepsilon_2$

The electric field tends to penetrate as deeply as possible into the medium with higher permittivity, while avoiding the medium with lower permittivity as much as possible. The electric field lines for $\varepsilon_1 > \varepsilon_2$ are shown in Figure 14.36, whereas the electric field lines for $\varepsilon_1 < \varepsilon_2$ are shown in Figure 14.37.

According to expression (14.222), the electric polarization vector – defined as the vector field representing the volume density of the electric dipole moment in a dielectric – can be written as:

$$\vec{P} = \frac{\vec{p}}{V_{\text{sphere}}} = \vec{E}_0 \cdot 3 \cdot \varepsilon_2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \tag{14.223}$$

whereas the surface density of the bound electric charge is given by the following expression:

$$\sigma_{\text{b}} = \left(\vec{P} \cdot \vec{n}\right)\Big|_{r=a} = \left(\vec{P} \cdot \vec{e}_{\text{r}}\right)\Big|_{r=a} = 3 \cdot E_0 \cdot \varepsilon_2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot \cos\vartheta = \sigma_{\text{bm}} \cdot \cos\vartheta \tag{14.224}$$

where:

$$\sigma_{\text{bm}} = 3 \cdot E_0 \cdot \varepsilon_2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} = \frac{p}{V_{\text{sphere}}} \tag{14.225}$$

which represents the maximum value of the surface density of the bound electric charge for $\varepsilon_1 > \varepsilon_2$, as well as the minimum value for $\varepsilon_1 < \varepsilon_2$.

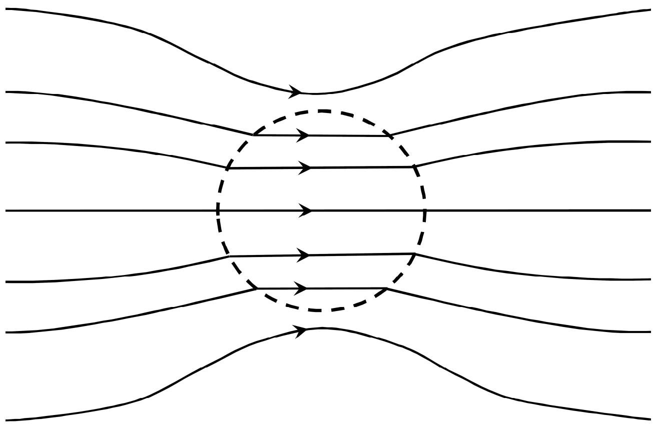


Figure 14.36. Graphical representation of electric field lines for $\varepsilon_1 > \varepsilon_2$

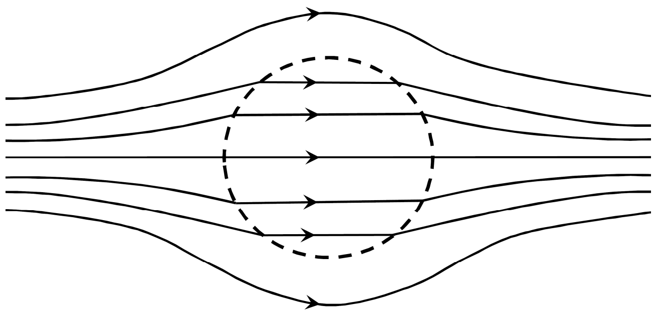


Figure 14.37. Graphical representation of electric field lines for $\varepsilon_1 < \varepsilon_2$

Thus, a surface electric charge sinusoidally distributed over the surface of the sphere creates a homogeneous electric field inside the sphere. It can be assumed that medium 2 extends throughout the entire space, and the influence of the solid dielectric sphere on the distribution of the electric scalar potential in medium 2 is effectively replaced by the surface electric charge (see Figure 14.38).

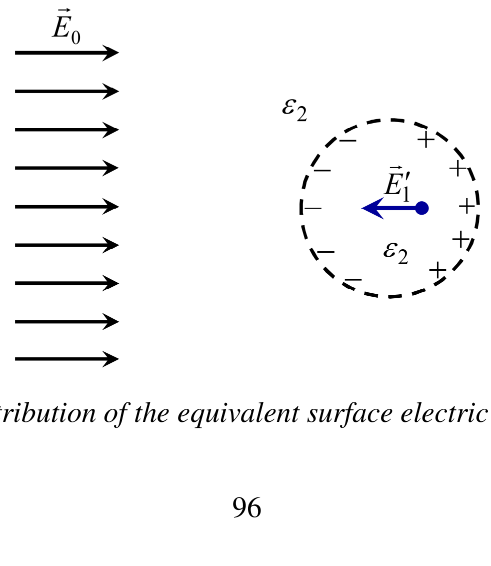


Figure 14.38. Distribution of the equivalent surface electric charge for $\varepsilon_1 > \varepsilon_2$

From expression (14.217), which describes the distribution of the electric scalar potential in medium 1, one can readily obtain the expressions for the non-zero components of the electric field intensity vector in medium 1, which, in the spherical coordinate system, are given by:

$$\vec{E}_1 = E_{1\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{1\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{14.226}$$

$$E_{1\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = \frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot \cos\vartheta \tag{14.227}$$

$$E_{1\vartheta} = -\frac{1}{r \cdot} \frac{\partial \varphi_1}{\partial \vartheta} = -\frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot \sin\vartheta \tag{14.228}$$

whereas in the Cartesian coordinate system, the following holds:

$$\vec{E}_1 = E_{1z} \cdot \vec{\mathrm{k}} = \frac{3 \cdot \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot \vec{\mathrm{k}} \tag{14.229}$$

From expression (14.218), which describes the distribution of the electric scalar potential in medium 2, one can readily obtain the expressions for the non-zero components of the electric field intensity vector in medium 2, which, in the spherical coordinate system, are given by:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{14.230}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\vartheta \cdot \left(1 + 2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot \left(\frac{a}{r} \right)^3 \right) \tag{14.231}$$

$$E_{2\vartheta} = -\frac{1}{r \cdot} \frac{\partial \varphi_2}{\partial \vartheta} = -E_0 \cdot \sin\vartheta \cdot \left(1 - \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot \left(\frac{a}{r} \right)^3 \right) \tag{14.232}$$

The expression for the electric polarization vector (14.223) can also be derived in a different way, not solely from the electric dipole moment of the polarized solid sphere. Since medium 1 is shaped as a solid sphere and is embedded within medium 2, the following relation holds:

$$\vec{D}_1 = \varepsilon_1 \cdot \vec{E}_1 = \varepsilon_2 \cdot \vec{E}_1 + \vec{P} \tag{14.233}$$

from which it follows that the electric polarization vector is given by:

$$\vec{P} = (\varepsilon_1 - \varepsilon_2) \cdot \vec{E}_1 = 3 \cdot \varepsilon_2 \cdot \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + 2 \cdot \varepsilon_2} \cdot E_0 \cdot \vec{\mathrm{k}} \tag{14.234}$$

and, according to expression (14.224), the surface density of the bound electric charge is given by*:

$$\sigma_\mathrm{b} = (\varepsilon_1 - \varepsilon_2) \cdot E_{1\mathrm{r}}\big|_{r=a} \tag{14.235}$$

A distinction should be made between the bound electric charge resulting from polarization and free electric charge. At the boundary between two dielectrics, there is no free electric charge, and boundary condition (14.214) applies, which means that:

$$D_{1\mathrm{r}}\big|_{r=a} = D_{2\mathrm{r}}\big|_{r=a} \tag{14.236}$$

or, written differently:

$$\varepsilon_1 \cdot E_{1\mathrm{r}}\big|_{r=a} = \varepsilon_2 \cdot E_{2\mathrm{r}}\big|_{r=a} \tag{14.237}$$

* Formally, the same expression applies to a dielectric cylinder in a homogeneous electric field.

14.19. Solid Conducting Sphere in a Homogeneous Electric Field

Let a solid conducting sphere (medium 1) of radius *a* be placed in a homogeneous electric field, with an unbounded perfect LIH dielectric (medium 2) surrounding the sphere. A spherical coordinate system with its origin at the center of the sphere is used (see Figure 14.39).

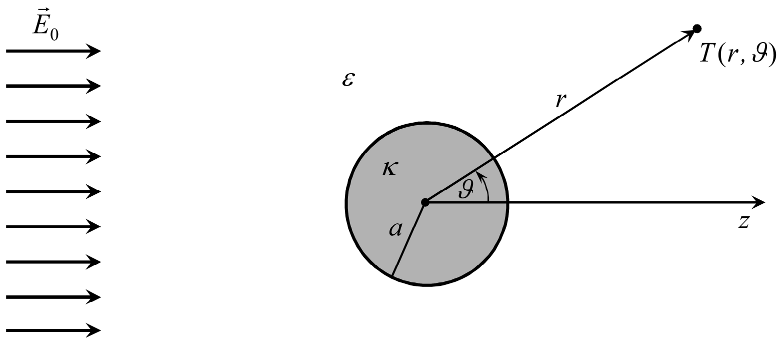


Figure 14.39. Solid conducting sphere in a homogeneous electric field

Expressions for the distribution of the electric scalar potential in both media can be obtained in two ways:

- In the same way as for a solid dielectric sphere in a homogeneous electric field (subchapter 14.18). The procedure begins with the general solution of the Laplace differential equation in a perfect LIH dielectric (medium 2), given by expression (14.203), from which expression (14.209) is derived, and it reads:

$$\varphi_2 = \left(-E_0 \cdot r + B_1 \cdot \frac{1}{r^2} \right) \cdot \cos\vartheta \tag{14.238}$$

  whereas:

$$\varphi_1 = const. = 0 \tag{14.239}$$

  and the unknown constant B_1 is determined from the boundary condition:

$$\varphi_1\big|_{r=a} = \varphi_2\big|_{r=a} = 0 \tag{14.240}$$

- By using the solutions for the distribution of the electric scalar potential given in expressions (14.217) and (14.218), which apply to a solid dielectric sphere (medium 1) in a homogeneous electric field, with the following substitutions:

$$\varepsilon_2 \to \varepsilon \quad ; \quad \varepsilon_1 \to \infty \tag{14.241}$$

Using either of the two described methods, the expressions for the distribution of the electric scalar potential in both media can be easily obtained and are given by:

$$\varphi_1 = 0 \quad ; \quad r \le a \tag{14.242}$$

$$\varphi_2 = -E_0 \cdot r \cdot \cos\vartheta + \left(\frac{a}{r}\right)^3 \cdot E_0 \cdot r \cdot \cos\vartheta \quad ; \quad z = r \cdot \cos\vartheta \quad ; \quad r \ge a \tag{14.243}$$

from which it follows that the electric scalar potential vanishes at the center of the sphere, that is, on the plane passing through the center where $\vartheta = \pi/2$, corresponding to the plane $z = 0$. It is important to note that expression (14.243) does not contain the properties of the medium.

From expression (14.332), which describes the distribution of the electric scalar potential in medium 1, it follows that the electric field intensity vector inside the solid conducting sphere is:

$$\vec{E}_1 = 0 \tag{14.244}$$

From expression (14.333), which describes the distribution of the electric scalar potential in medium 2, one can readily obtain the expressions for the non-zero components of the electric field intensity vector in medium 2, which, in the spherical coordinate system, are given by:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{14.245}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = E_0 \cdot \cos\vartheta \cdot \left(1 + 2 \cdot \left(\frac{a}{r}\right)^3\right) \tag{14.246}$$

$$E_{2\vartheta} = -\frac{1}{r \cdot}\frac{\partial \varphi_2}{\partial \vartheta} = -E_0 \cdot \sin\vartheta \cdot \left(1 - \left(\frac{a}{r}\right)^3\right) \tag{14.247}$$

An electrical influence occurs – that is, at the boundary between the dielectric and the conductor, a free electric charge appears with a surface density given by:

$$\sigma = D_{2\mathrm{r}}\big|_{r=a} - D_{1\mathrm{r}}\big|_{r=a} = D_{2\mathrm{r}}\big|_{r=a} = \varepsilon_2 \cdot E_{2\mathrm{r}}\big|_{r=a} \tag{14.248}$$

By substituting expression (14.246) into expression (14.248), it follows that:

$$\sigma = 3 \cdot \varepsilon \cdot E_0 \cdot \cos\vartheta = \sigma_\mathrm{m} \cdot \cos\vartheta \quad ; \quad \sigma_\mathrm{m} = 3 \cdot \varepsilon \cdot E_0 \tag{14.249}$$

The solid conducting sphere is an equipotential body. This implies that the tangential component of the electric field intensity vector on the surface of the sphere is zero, i.e., the electric field intensity vector in medium 2 is perpendicular to the sphere's surface (see Figure 14.40). However, instead of applying the boundary condition:

$$E_{2\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \vartheta}\bigg|_{r=a} = 0 \tag{14.250}$$

it is preferable – for the sake of simplicity – to use the boundary condition (14.240).

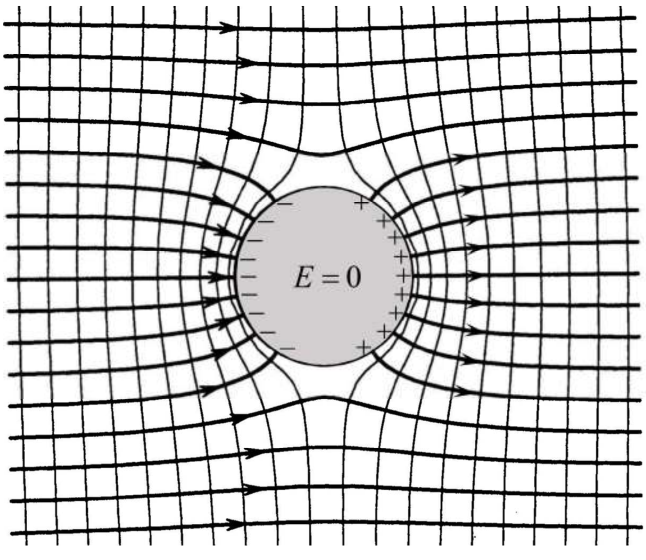


Figure 14.40. Electric field lines and equipotential lines in a dielectric surrounding a conducting sphere

Analogies can be drawn between electrostatic, magnetostatic, and stationary current fields. By analogy with a solid dielectric sphere placed in an electric field, one can derive the solution for a sphere in a homogeneous magnetic field, as well as for a conducting sphere in a homogeneous current field. To do so, it is sufficient to replace the electric field intensity vector with the magnetic field intensity vector and substitute the permittivities with the corresponding permeabilities or electrical conductivities of the media. In place of the electric surface charge at the boundary between two media in electrostatic and stationary current problems, in the magnetostatic case, a linear density of the surface electric current is introduced, given by $K = K_m \cdot \sin\vartheta$.

14.20. Two Infinitely Long Straight Conductors

Let two infinitely long, mutually parallel straight conductors, carrying opposite charges with constant linear density λ (Figure 14.41), be placed in an unbounded, perfect LIH dielectric with permittivity ε.

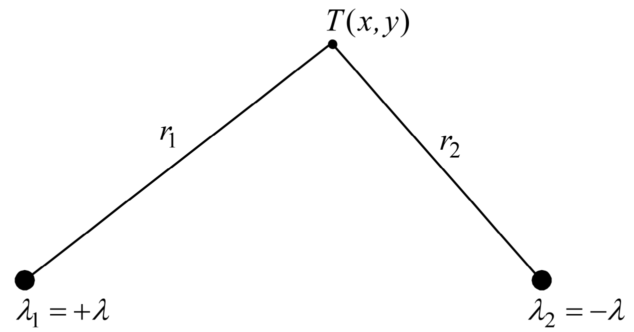


Figure 14.41. Two infinitely long, mutually parallel straight conductors

Based on expression (14.47), which describes the distribution of the electric scalar potential around an isolated, infinitely long straight conductor carrying a constant linear charge density λ, it is straightforward to conclude that the electric scalar potential distribution caused by the charges on two infinitely long, parallel straight conductors shown in Figure 14.41 can be described by the following expression:

$$\varphi = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln\frac{r_2}{r_1} + C \tag{14.251}$$

where r_1 and r_2 are the shortest distances from the field point to the conductor.

It is logical to assume that the electric scalar potential is zero on the symmetrical plane between two infinitely long parallel conductors carrying opposite electric charges of the same linear density. The equation of the symmetrical plane is $r_1 = r_2$, and therefore $C = 0$, which means that the distribution of the electric scalar potential can be described by the following expression:

$$\varphi = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln\frac{r_2}{r_1} \tag{14.252}$$

The equation of the equipotential lines is given by:

$$\frac{r_2}{r_1} = \text{const.} \tag{14.253}$$

The equipotential lines are eccentric circles, meaning the equipotential surfaces are the mantles of eccentric cylinders. To prove this mathematically, let the conductors be placed in the 2D Cartesian coordinate system (x, y), as shown in Figure 14.42.

According to expressions (14.252) and (14.253), for an equipotential line around a positive line electric charge, the following holds:

$$\frac{r_2}{r_1} = k > 1 \tag{14.254}$$

where, according to Figure 14.42, the following applies:

$$r_1 = \sqrt{x^2 + y^2} \quad ; \quad r_2 = \sqrt{(x - 2 \cdot a)^2 + y^2} \tag{14.255}$$

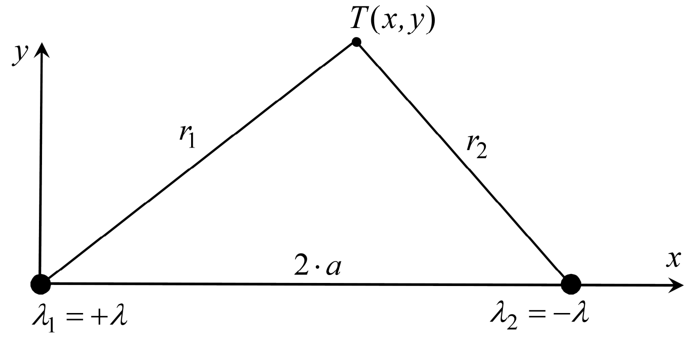


Figure 14.42. Infinitely long, mutually parallel straight conductors in the (x, y) coordinate system

If expressions (14.255) are substituted into expression (14.254) and the resulting expression is then squared, the following expression is obtained:

$$k^2\cdot x^2+k^2\cdot y^2=x^2-4\cdot a\cdot x+4\cdot a^2+y^2 \tag{14.256}$$

from which it follows that:

$$x^2+\frac{4\cdot a\cdot x}{k^2-1}+y^2=\frac{4\cdot a^2}{k^2-1} \tag{14.257}$$

and after completing the square, the following is obtained:

$$\left(x+\frac{2\cdot a}{k^2-1}\right)^2+y^2=\frac{4\cdot a^2}{k^2-1}+\frac{4\cdot a^2}{(k^2-1)^2}=\left(\frac{2\cdot a\cdot k}{k^2-1}\right)^2 \tag{14.258}$$

Expression (14.258) represents the equation of a circle, as shown in Figure 14.43:

$$(x+b)^2+y^2=R^2 \quad ; \quad R=\frac{2\cdot a\cdot k}{k^2-1} \quad ; \quad b=\frac{2\cdot a}{k^2-1}=\frac{R}{k} \tag{14.259}$$

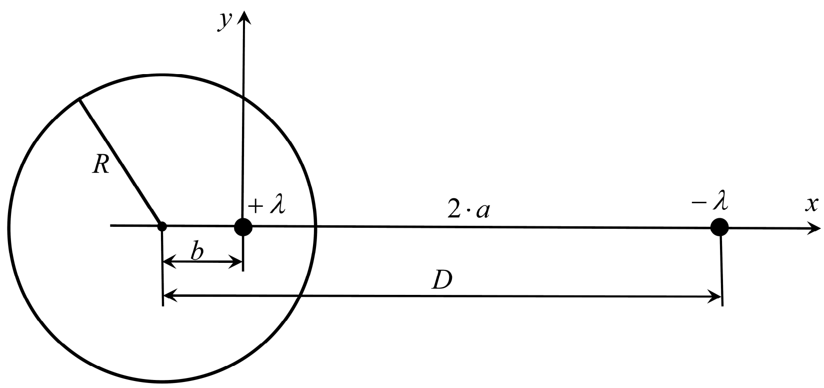


Figure 14.43. The graphical representation of the equipotential circle and the parameters b and D

The following holds:

$$D=2\cdot a+b=2\cdot a+\frac{2\cdot a}{k^2-1}=\frac{2\cdot a\cdot k^2}{k^2-1}=R\cdot k \tag{14.260}$$

from which it follows that:

$$b=\frac{R}{k}=\frac{R^2}{D} \tag{14.261}$$

By substituting:

$$b = s - a \quad ; \quad D = s + a \tag{14.262}$$

from expressions (14.260) and (14.261), it can be derived that:

$$k = \frac{R}{s-a} = \frac{s+a}{R} \quad ; \quad s^2 - a^2 = R^2 \tag{14.263}$$

where s is the shortest distance between the center of the equipotential circle and the plane of symmetry, whereas a is the shortest distance between a single conductor and the plane of symmetry (Figure 14.44). On the plane of symmetry, the electric scalar potential is zero.

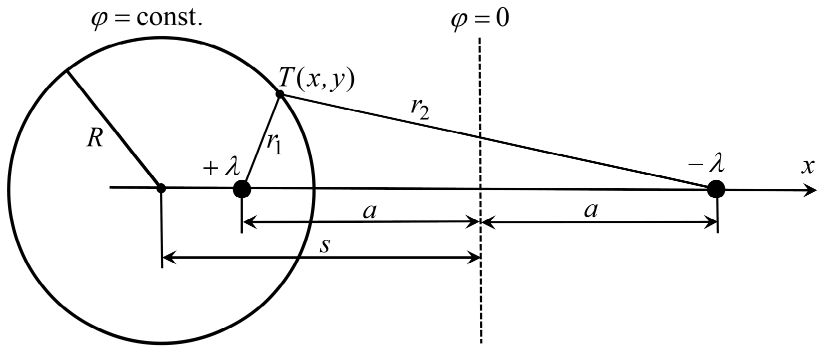


Figure 14.44. The graphical representation of the equipotential circle and the parameters s and a

According to Figure 14.45, $s+a$ represents the shortest distance between the center of the equipotential circle around a positively charged conductor and a negatively charged conductor, whereas $s-a$ represents the shortest distance between the center of the equipotential circle around a positively charged conductor and a positively charged conductor.

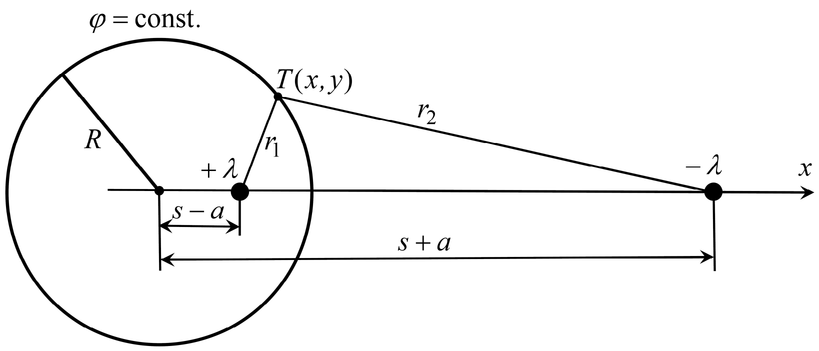


Figure 14.45. Graphical representation of the equipotential circle and the parameters $s+a$ and $s-a$

According to expression (14.263), for $r_2 > r_1$:

$$\frac{r_2}{r_1} = \frac{s+a}{R} = \frac{R}{s-a} = k > 1 \quad ; \quad \varphi > 0 \tag{14.264}$$

$$\varphi = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{r_2}{r_1} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{s+a}{R} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{R}{s-a} \tag{14.265}$$

whereas for $r_2 < r_1$:

$$\frac{r_2}{r_1} = \frac{s-a}{R} = \frac{R}{s+a} = k < 1 \quad ; \quad \varphi < 0 \tag{14.266}$$

$$\varphi = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{r_2}{r_1} = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{s-a}{R} = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{R}{s+a} \tag{14.267}$$

It is important to note that the obtained expressions for the distribution of the electric scalar potential can also be applied in cases where the equipotential surface is filled by a solid, infinitely long cylindrical conductor, or where the cylindrical conductor is placed in front of a conducting plane.

14.21. Capacitance of Eccentric Conducting Cylinders

Let two eccentric conducting cylinders have radii R_1 and R_2, with their centers separated by a distance d (Figure 14.46). Let there be a perfect LIH dielectric with permittivity ε between the cylinders, and let the length of the cylinders be ℓ.

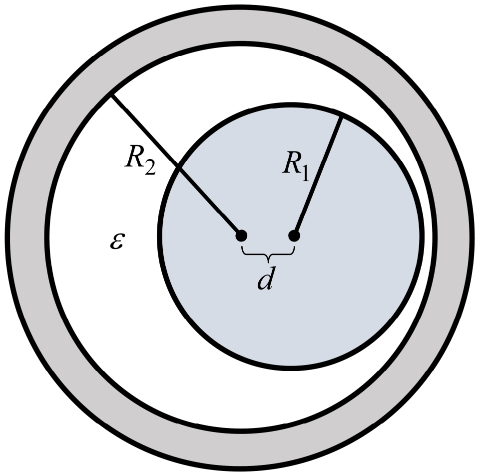


Figure 14.46. Two eccentric conducting cylinders

Neglecting edge effects, the expression for the capacitance of these cylinders can be derived from expression (14.265), which describes the potential distribution of two infinitely long parallel straight conductors charged with opposite charges of the same constant linear charge density λ (Figure 14.44), located in an unbounded perfect LIH dielectric with permittivity ε. Therefore, the two eccentric circles, between which the perfect LIH dielectric is situated, are equipotential circles around the positively charged infinitely long straight conductor (Figure 14.47). It follows that:

$$s_1^2 - a^2 = R_1^2 \quad ; \quad s_2^2 - a^2 = R_2^2 \quad ; \quad s_2 - s_1 = d \tag{14.268}$$

and the unknowns are: s_1, s_2 and a.

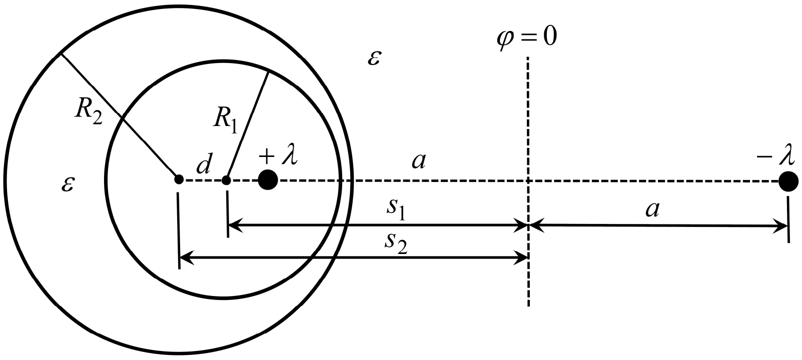


Figure 14.47. Equipotential circles of eccentric conducting cylinders

From the system of equations (14.268), it follows that:

$$s_2^2 - s_1^2 = (s_2 - s_1)\cdot(s_2 + s_1) = d\cdot(s_2 + s_1) = R_2^2 - R_1^2 \tag{14.269}$$

and consequently, the system of two equations with two unknowns can be written as:

$$s_2 - s_1 = d \quad ; \quad s_2 + s_1 = \frac{R_2^2 - R_1^2}{d} \tag{14.270}$$

whose solutions are:

$$s_1 = \frac{R_2^2 - R_1^2}{2\cdot d} - \frac{d}{2} \quad ; \quad s_2 = \frac{R_2^2 - R_1^2}{2\cdot d} + \frac{d}{2} \tag{14.271}$$

whereas, according to the expression (14.268), it holds that:

$$a = \sqrt{s_1^2 - R_1^2} = \sqrt{s_2^2 - R_2^2} \tag{14.272}$$

The electric scalar potentials of both cylinders are positive and, according to expression (14.265), are described by:

$$\varphi_1 = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{s_1 + a}{R_1} \quad ; \quad \varphi_2 = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{s_2 + a}{R_2} \tag{14.273}$$

and the electric voltage between the cylinders is described by the expression:

$$U = \varphi_1 - \varphi_2 = \frac{Q}{2\cdot\pi\cdot\varepsilon\cdot\ell}\cdot\ln\frac{R_2\cdot(s_1 + a)}{R_1\cdot(s_2 + a)} \quad ; \quad \lambda = \frac{Q}{\ell} \tag{14.274}$$

from which it follows that the capacitance of the eccentric conducting cylinders is:

$$C = \frac{Q}{U} = \frac{2\cdot\pi\cdot\varepsilon\cdot\ell}{\ln\dfrac{R_2\cdot(s_1 + a)}{R_1\cdot(s_2 + a)}} \tag{14.275}$$

If the cylinders are coaxial, then the following expression is obtained from expression (14.275):

$$C = \frac{Q}{U} = \frac{2\cdot\pi\cdot\varepsilon\cdot\ell}{\ln\dfrac{R_2}{R_1}} \tag{14.276}$$

because it is:

$$\lim_{d\to 0}\frac{s_1 + a}{s_2 + a} = 1 \tag{14.277}$$

14.22. Capacitance of a Two-Wire Line with Conductors of Different Radii

Let a two-wire line be formed by two isolated, infinitely long straight conductors of different radii R_1 and R_2, with their centers separated by a distance d (Figure 14.48). Let there be an unbounded perfect LIH dielectric with permittivity ε surrounding the cylinders.
In this case, the following holds:

$$s_1^2 - a^2 = R_1^2 \quad ; \quad s_2^2 - a^2 = R_2^2 \quad ; \quad s_1 + s_2 = d \tag{14.278}$$

and the unknowns are: s_1, s_2 and a.

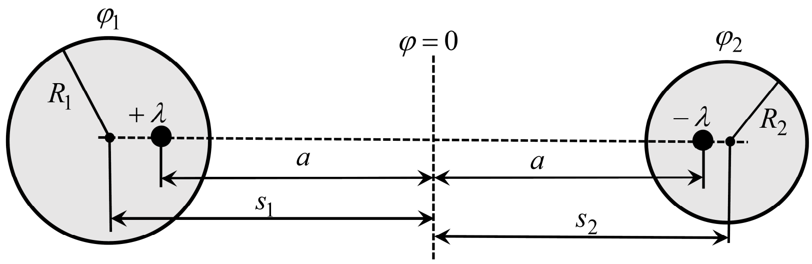


Figure 14.48. Two-wire line with conductors of different radii

From the system of equations (14.278), it follows that:

$$s_2^2 - s_1^2 = (s_2 + s_1)\cdot(s_2 - s_1) = d\cdot(s_2 - s_1) = R_2^2 - R_1^2 \tag{14.279}$$

and consequently, the system of two equations with two unknowns can be written as:

$$s_1 + s_2 = d \quad ; \quad s_2 - s_1 = \frac{R_2^2 - R_1^2}{d} \tag{14.280}$$

whose solutions are:

$$s_1 = \frac{d}{2} - \frac{R_2^2 - R_1^2}{2\cdot d} \quad ; \quad s_2 = \frac{R_2^2 - R_1^2}{2\cdot d} + \frac{d}{2} \tag{14.281}$$

whereas a can be computed from the expression (14.272).
The electric scalar potentials of the conductors, according to (14.265) and (14.267), are given by:

$$\varphi_1 = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{s_1 + a}{R_1} \quad ; \quad \varphi_2 = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{R_2}{s_2 + a} \tag{14.282}$$

and the electric voltage between the conductors is described by the expression:

$$U = \varphi_1 - \varphi_2 = \frac{\lambda}{2\cdot\pi\cdot\varepsilon}\cdot\ln\frac{(s_1 + a)\cdot(s_2 + a)}{R_1\cdot R_2} \tag{14.283}$$

from which it follows that the per-unit-length capacitance of the line is described by the expression:

$$C = \frac{\lambda}{U} = \frac{2\cdot\pi\cdot\varepsilon}{\ln\frac{(s_1 + a)\cdot(s_2 + a)}{R_1\cdot R_2}} \tag{14.284}$$

If both conductors have the same radius R, then $s_1 = s_2 = s$, and the per-unit-length capacitance of such a line is described by the expression:

$$C = \frac{\lambda}{U} = \frac{\pi\cdot\varepsilon}{\ln\frac{s + a}{R}} \tag{14.285}$$

and for $d >> R$, it follows that $a \approx s$, $s + a \approx d$, and the per-unit-length capacitance of such a line is:

$$C = \frac{\lambda}{U} \approx \frac{\pi\cdot\varepsilon}{\ln\frac{d}{R}} = \frac{\pi\cdot\varepsilon}{\ln\frac{2\cdot s}{R}} \tag{14.286}$$

where d is the shortest distance between the axes of the conductors.

14.23. Capacitance of a Cylindrical Conductor in Front of a Conducting Plane

Let an infinitely long straight equipotential conductor of radius R be located in front of a conducting plane, with the electric scalar potential of the plane set to zero. The plane is positioned at a distance s from the center of the conductor (Figure 14.49). The half-space around the conductor is a perfect LIH dielectric with permittivity ε. The solution remains the same if a conducting cylinder is placed in front of a conducting half-space (Figure 14.50), such as the ground, where the electric scalar potential is also zero. In other words, the solution is unaffected by the properties of the medium located in the half-space behind the conducting plane, whose electric scalar potential is, by assumption, zero.

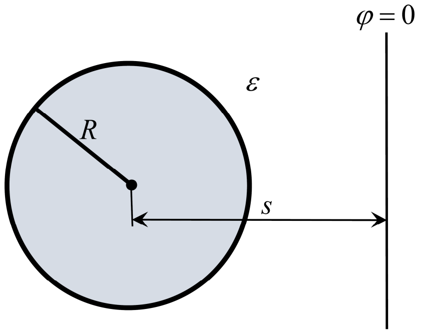


Figure 14.49. Infinitely long cylindrical conductor in front of a conducting plane

Let there be a positive electric charge with linear density $+\lambda$ on the cylindrical conductor. The influence of the conducting plane, whose electric scalar potential is zero, is accounted for using the method of images, which reflects the positive charge into a negative charge with linear density $-\lambda$ (Figure 14.51). This ensures that the electric scalar potential is zero at the conducting plane, thereby satisfying all boundary conditions. The obtained solution is valid only in the half-space where the infinitely long cylindrical equipotential conductor is located.

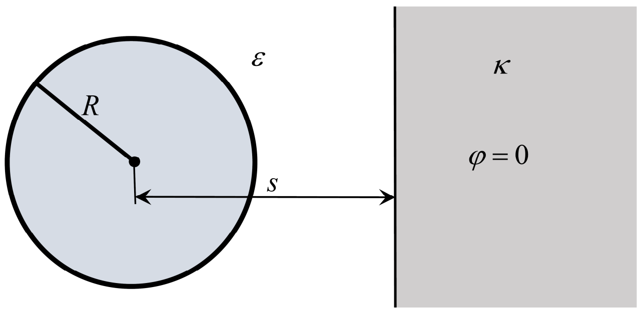


Figure 14.50. Infinitely long cylindrical conductor in front of a conducting half-space

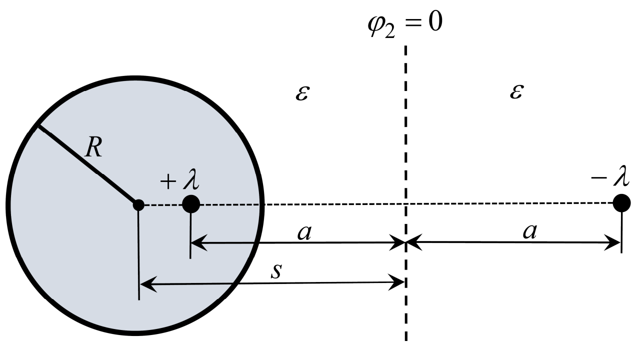


Figure 14.51. Real and mirrored line charge in an unbounded perfect LIH dielectric with *permittivity ε*

The parameters s and R are known, whereas a can be computed from the following expression:

$$s^2 - a^2 = R^2 \quad ; \quad a = \sqrt{s^2 - R^2} \tag{14.287}$$

The electric scalar potential of a conductor, according to expression (14.265), is given by:

$$\varphi_1 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon} \cdot \ln \frac{s+a}{R} \tag{14.288}$$

and the per-unit-length capacitance of the conductor is:

$$C = \frac{\lambda}{\varphi_1} = \frac{2 \cdot \pi \cdot \varepsilon}{\ln \dfrac{s+a}{R}} \tag{14.289}$$

For $s >> R$, it follows that $a \approx s$, $s + a \approx 2 \cdot s$, so the per-unit-length capacitance of such a conductor is:

$$C = \frac{\lambda}{\varphi_1} \approx \frac{2 \cdot \pi \cdot \varepsilon}{\ln \dfrac{2 \cdot s}{R}} \tag{14.290}$$

From expressions (14.285) and (14.289), it follows that for the same values of a, s and R, the per-unit-length capacitance of an infinitely long conductor in front of a conducting plane is double that of the per-unit-length capacitance of a two-wire line with conductors of equal radii R.

14.24. Thin-Wire Conductors and the Average Potential Method

Let a straight segment of a solid cylindrical conductor of radius r_0 and length ℓ be charged with an electric charge Q (Figure 14.52). Let the electric charge be uniformly distributed over the surface of the conductor segment. Let the conductor segment be in an unbounded perfect LIH dielectric of permittivity ε.

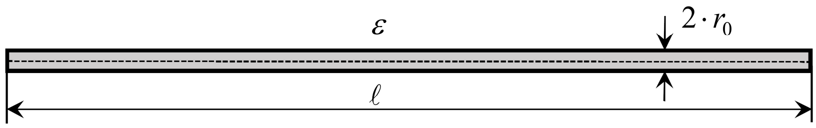


Figure 14.52. Straight segment of a cylindrical conductor

If $r_0 << \ell$, then a thin-wire approximation can be used, which assumes that all the electric charge is concentrated along the axis of the conductor segment. The potential distribution at points outside the conductor segment, including the surface of the conductor segment, must be determined. The potential inside the conductor segment is equal to the potential on its surface. In the numerical model, the electric charge Q is located along the axis of the conductor segment, and its linear charge density λ is described by the following expression:

$$\lambda = \frac{Q}{\ell} = \text{const.} \tag{14.291}$$

For simplicity, the segment is represented in the local 2D coordinate system (u, v), in which the axis of the conductor segment and the field point $T(u, v)$ lie. The position of the segment in the global (x, y, z) coordinate system is arbitrary. According to Figure 14.53, the field point $T(u, v)$ can be located in either the first or second quadrant of the local coordinate system (u, v).

In an unbounded perfect LIH dielectric of permittivity ε, the differential of the electric scalar potential is described by the expression:

$$d\varphi = \frac{\lambda \cdot dt}{4 \cdot \pi \cdot \varepsilon \cdot r} = \frac{Q \cdot dt}{4 \cdot \pi \cdot \varepsilon \cdot \ell \cdot r} = \frac{Q \cdot dt}{4 \cdot \pi \cdot \varepsilon \cdot \ell \cdot \sqrt{(u-t)^2 + v^2}} \tag{14.292}$$

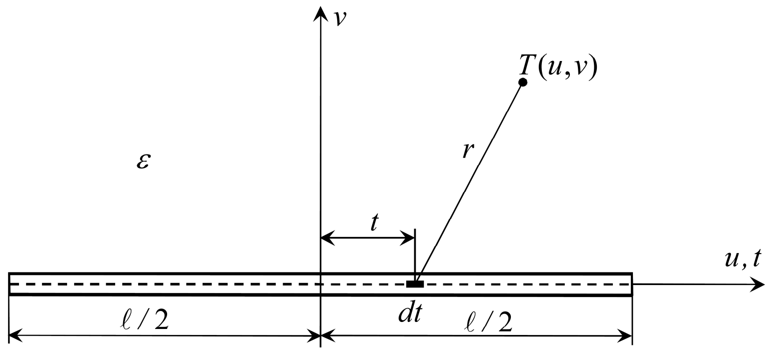


Figure 14.53. Cylindrical conductor segment in the local coordinate system (u, v)

It follows that:

$$\varphi = \frac{Q}{4\cdot\pi\cdot\varepsilon\cdot\ell}\cdot\int_{-\frac{\ell}{2}}^{\frac{\ell}{2}}\frac{dt}{\sqrt{(u-t)^2+v^2}} \tag{14.293}$$

and, after integration, it is obtained that the distribution of the electric scalar potential around the thin-wire segment of the conductor is described by the expression:

$$\varphi = \frac{Q}{4\cdot\pi\cdot\varepsilon\cdot\ell}\cdot\left(\operatorname{arsinh}\frac{u+\frac{\ell}{2}}{v} - \operatorname{arsinh}\frac{u-\frac{\ell}{2}}{v}\right) \tag{14.294}$$

It holds that:

$$\operatorname{arsinh}\frac{w}{v} = \ln\left(\frac{w}{v}+\sqrt{1+\left(\frac{w}{v}\right)^2}\right) = \ln\left(w+\sqrt{v^2+w^2}\right) - \ln v \tag{14.295}$$

and expression (14.294) then takes on the following form:

$$\varphi = \frac{Q}{4\cdot\pi\cdot\varepsilon\cdot\ell}\cdot\ln\frac{\sqrt{v^2+\left(u+\frac{\ell}{2}\right)^2} + u+\frac{\ell}{2}}{\sqrt{v^2+\left(u-\frac{\ell}{2}\right)^2} + u-\frac{\ell}{2}} \tag{14.296}$$

The distribution of the electric scalar potential is axisymmetric with respect to the local axis *u*. It can be easily shown that the equipotential surfaces, described by expressions (14.295) and (14.296), are rotational ellipsoids with foci at the ends of the conductor axis. This means that in the local coordinate system (*u*, *v*), the equipotential lines are ellipses with foci at the ends of the conductor axis (Figure 14.54).

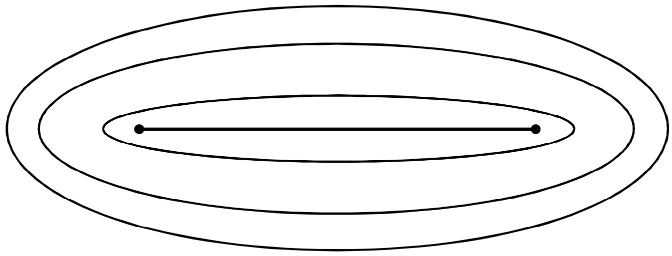


Figure 14.54. Equipotential lines around a cylindrical conductor segment

The distribution of the electric scalar potential along the mantle of a conductor segment can be expressed as:

$$\varphi = \varphi(u, r_0) \; = \; \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot \ell} \cdot \ln \frac{\sqrt{r_0^2 + \left(u + \frac{\ell}{2}\right)^2} \; + \; u + \frac{\ell}{2}}{\sqrt{r_0^2 + \left(u - \frac{\ell}{2}\right)^2} \; + \; u - \frac{\ell}{2}} \tag{14.297}$$

which is shown graphically in Figure 14.55.

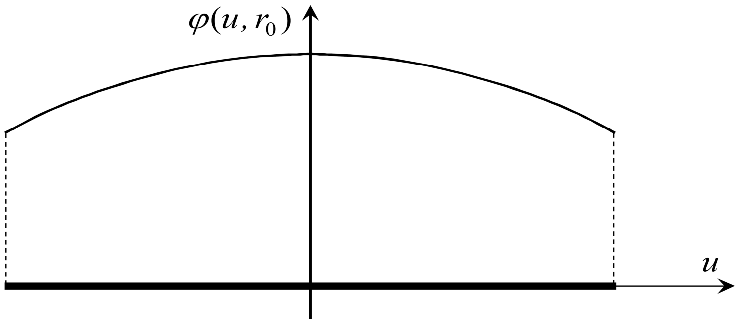


Figure 14.55. Distribution of the electric scalar potential along a conductor segment

Figure 14.55 shows the obtained approximation of the potential distribution along the mantle of the conductor segment and, consequently, along the conductor segment. However, this potential is actually constant. This physically constant potential can be approximated in several ways. One such method is the point collocation method (PCM), in which the electric scalar potential of a chosen point on the conductor segment is taken as the potential for the entire conductor segment. The accuracy of this approximation significantly depends on the choice of the collocation point. Therefore, the point collocation method is considered a poor approximation. A much better approximation is the average potential method (APM), which is a special case of the Galerkin-Bubnov method.

According to the average potential method, the electric scalar potential of a conductor segment is approximated by the average potential along the mantle of the conductor segment (Figure 14.56):

$$\Phi = \Phi_{\text{av}} = \frac{1}{\ell} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \varphi(u, r_0) \cdot du \tag{14.298}$$

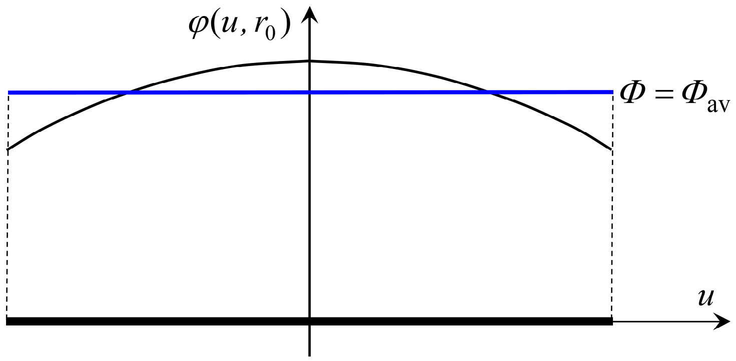


Figure 14.56. Approximation of the electric scalar potential of a conductor segment by the average potential method (APM)

From expressions (14.297) and (14.298), after performing the analytical integration, the following expression is obtained for the electric scalar potential of the conductor segment using the average potential method :

$$\Phi = \Phi_{av} = G(\ell, r_0) \cdot Q \tag{14.299}$$

where $G(\ell, r_0)$ is the *self-potential coefficient* of the conductor segment. It holds that:

$$G(\ell, v) = \frac{1}{2 \cdot \pi \cdot \varepsilon \cdot \ell^2} \cdot \left[\ell \cdot \operatorname{arsinh} \frac{\ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{14.300}$$

or, written differently:

$$G(\ell, v) = \frac{1}{2 \cdot \pi \cdot \varepsilon \cdot \ell^2} \cdot \left[\ell \cdot \ln \frac{\sqrt{\ell^2 + v^2} + \ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{14.301}$$

An equipotential straight thin-wire cylindrical conductor or an equipotential network of conductors, located in a perfect LIH dielectric with permittivity ε, can be divided into n thin-wire segments in a numerical model. Using the average potential method, a system of linear equations can be formed, where the total electric charge of the conductor or network of conductors Q is known, and the unknowns are the electric charges of the segments and the potential of the network of conductors.

Using the method of images, the influence of a conducting plane, such as the ground surface, can also be taken into account. In this case, the potential coefficients include the contributions from both the real and mirrored conductor segments. The matrix of potential coefficients obtained by the average potential method is a symmetric matrix. The system of linear equations can be written as:

$$\begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} & -1 \\ -1 & \cdots & -1 & 0 \end{bmatrix} \begin{Bmatrix} Q_1 \\ \vdots \\ Q_n \\ \Phi_{net} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ -Q \end{Bmatrix} \tag{14.302}$$

where:

$\Phi_{net} = \Phi_1 = \Phi_2 = \cdots = \Phi_n$ - the electric scalar potential of the network of conductors,

Φ_k - the electric scalar potential of the k-th segment of a cylindrical conductor,

Q_k - the electric charge of the k-th segment of a cylindrical conductor,

$\alpha_{i,k} = \alpha_{k,i}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ - the potential coefficients.

According to expressions (14.299) and (14.301), the self-potential coefficients of segments in an unbounded perfect LIH dielectric of permittivity ε are described by the expression:

$$\alpha_{i,i} = G(\ell_i, r_{0i}) = \frac{1}{2 \cdot \pi \cdot \varepsilon \cdot \ell_i^2} \cdot \left[\ell_i \cdot \ln \frac{\sqrt{\ell_i^2 + r_{0i}^2} + \ell_i}{r_{0i}} - \frac{\ell_i^2}{\sqrt{\ell_i^2 + r_{0i}^2} + r_{0i}} \right] \tag{14.303}$$

where:

ℓ_i - the length of the i-th conductor segment,

r_{0i} - the radius of the i-th conductor segment.

If two segments are located in an unbounded perfect LIH dielectric with permittivity ε and are parallel to each other, as shown in Figure 14.57, with a separation distance d, then their mutual potential coefficient can be described by the following expression:

$$\alpha_{\mathrm{i,k}} = G(\ell_{\mathrm{i}}, d) = \frac{1}{2\cdot\pi\cdot\varepsilon\cdot\ell_{\mathrm{i}}^2}\cdot\left[\ell_{\mathrm{i}}\cdot\ln\frac{\sqrt{\ell_{\mathrm{i}}^2+d^2}+\ell_{\mathrm{i}}}{d} - \frac{\ell_{\mathrm{i}}^2}{\sqrt{\ell_{\mathrm{i}}^2+d^2}+d}\right] \tag{14.304}$$

where the lengths of these two segments are equal, i.e., $\ell_{\mathrm{i}} = \ell_{\mathrm{k}}$.

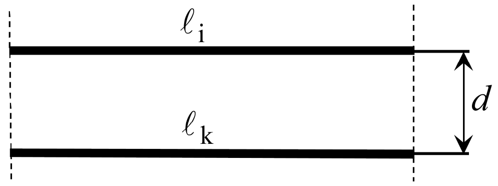


Figure 14.57. Special case of parallelism between two conductor segments

In the general case of parallelism between two conductor segments (Figure 14.58), the k-th conductor segment is observed in the local coordinate system (*u*, *v*) of the i-th conductor segment. In this local coordinate system, the endpoints of the k-th segment are denoted as $T_1(u_1, v_{\mathrm{k}})$ and $T_2(u_2, v_{\mathrm{k}})$.

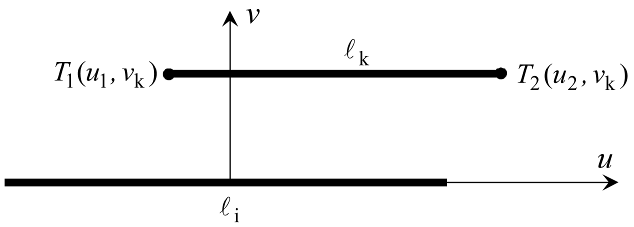


Figure 14.58. General case of parallelism between two conductor segments

If two segments are located in an unbounded perfect LIH dielectric with permittivity ε and are parallel to each other, as shown in Figure 14.58, then their mutual potential coefficient can be described by the following expression:

$$\alpha_{\mathrm{i,k}} = \frac{1}{4\cdot\pi\cdot\varepsilon\cdot\ell_{\mathrm{i}}\cdot\ell_{\mathrm{k}}}\cdot\left(C_1 + C_2 - C_3 - C_4\right) \tag{14.305}$$

where the auxiliary functions C_{m} are defined by the following expressions:

$$C_{\mathrm{m}} = w_{\mathrm{m}}\cdot\ln\left(\sqrt{w_{\mathrm{m}}^2 + v_{\mathrm{k}}^2} + w_{\mathrm{m}}\right) - \sqrt{w_{\mathrm{m}}^2 + v_{\mathrm{k}}^2} \quad ; \quad \mathrm{m} = 1, 2, 3, 4 \tag{14.306}$$

$$w_1 = u_2 + \frac{\ell_{\mathrm{i}}}{2} \quad ; \quad w_2 = u_1 - \frac{\ell_{\mathrm{i}}}{2} \quad ; \quad w_3 = u_1 + \frac{\ell_{\mathrm{i}}}{2} \quad ; \quad w_4 = u_2 - \frac{\ell_{\mathrm{i}}}{2} \tag{14.307}$$

Mutually non-parallel (oblique or perpendicular) conductor segments lie either in the same plane or in two mutually parallel planes. Therefore, if two non-parallel conductor segments do not lie in the same plane, then there exist exactly two parallel planes in which they lie (Figure 14.59). If they do lie in the same plane, this represents a special case in which the two parallel planes coincide. Let the conductor segments be oriented from point *P* to point *K*. This implies that the angle $\alpha \in (0, \pi)$.

The mutual potential coefficient of two non-parallel conductor segments in an unbounded perfect LIH dielectric with permittivity ε is defined by the following expression:

$$\alpha_{\mathrm{i,k}} = \frac{1}{4\cdot\pi\cdot\varepsilon\cdot\ell_{\mathrm{i}}\cdot\ell_{\mathrm{k}}}\cdot\int_{\xi_{\mathrm{P}}}^{\xi_{\mathrm{K}}}\int_{\eta_{\mathrm{P}}}^{\eta_{\mathrm{K}}}\frac{d\xi\cdot d\eta}{r_{\mathrm{i,k}}} \tag{14.308}$$

where:

$$r_{i,k} = \sqrt{\xi^2 + \eta^2 + D^2 - 2 \cdot \xi \cdot \eta \cdot \cos\alpha} \tag{14.309}$$

The parameters $\xi_P, \xi_K, \eta_P, \eta_K, \alpha$, and D can be easily computed from the global coordinates of the starting and ending points of the conductor segments.

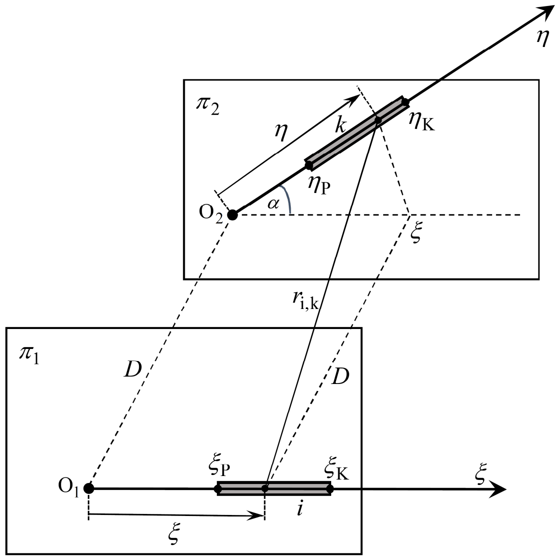


Figure 14.59. General case of non-parallelism between two conductor segments

The solution to the integral (14.308) is known as Tseitlin's formula, which can be written as:

$$\alpha_{i,k} = \frac{1}{4 \cdot \pi \cdot \varepsilon \cdot \ell_i \cdot \ell_k} \cdot \left[A(\xi_P, \eta_P) + A(\xi_K, \eta_K) - A(\xi_P, \eta_K) - A(\xi_K, \eta_P)\right] \tag{14.310}$$

where:

$$\begin{aligned} A(\xi,\eta) = {} & \xi \cdot \ln\left(\eta - \xi \cdot \cos\alpha + r_{i,k}\right) + \eta \cdot \ln\left(\xi - \eta \cdot \cos\alpha + r_{i,k}\right) \\ & + \frac{2 \cdot D}{\sin\alpha} \cdot \arctan\left(\frac{\xi + \eta + r_{i,k}}{D} \cdot \tan\frac{\alpha}{2}\right) \end{aligned} \tag{14.311}$$

If there are multiple equipotential networks of thin-wire conductors, then the system of linear equations (14.302) must be expanded by introducing one additional equation and one additional unknown for each new equipotential network of conductors.

14.25. Partial Capacitances

Let there be *n* conducting charged bodies in the unbounded perfect LIH dielectric. Then, the electric scalar potential of the i-th conducting body can be described by the following equation:

$$\varphi_i = \sum_{k=1}^{n} \alpha_{i,k} \cdot Q_k \quad ; \quad i = 1, 2, ..., n \tag{14.312}$$

where:

φ_i - the electric scalar potential of the i-th conducting body,

Q_k - the electric charge of the k-th conducting body,

$\alpha_{i,k}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ - the potential coefficients.

If the influence of the ground (a conducting body) is taken into account, the electric scalar potential of the ground is assumed to be zero, and the soil surface is approximated as a plane.
In the general case, the system of linear equations (14.312) can be expressed in matrix notation as:

$$\begin{Bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{Bmatrix} = \begin{bmatrix} \alpha_{1,1} & \cdots & \alpha_{1,n} \\ \vdots & \ddots & \vdots \\ \alpha_{n,1} & \cdots & \alpha_{n,n} \end{bmatrix} \cdot \begin{Bmatrix} Q_1 \\ \vdots \\ Q_n \end{Bmatrix} \quad ; \quad \{\varphi\} = [\alpha] \cdot \{Q\} \tag{14.313}$$

where:

$\{\varphi\}$ - the column vector of the electric scalar potentials of the conducting bodies,

$\{Q\}$ - the column vector of the electric charges of the conducting bodies,

$[\alpha]$ - the potential coefficient matrix.

If the electric scalar potentials of conducting bodies are known, then the electric charge of the i-th conducting body is given by:

$$Q_i = \sum_{k=1}^{n} \beta_{i,k} \cdot \varphi_k \quad ; \quad i = 1, 2, ..., n \tag{14.314}$$

where $\beta_{i,k}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ are the so-called *capacitance coefficients.*

The system of linear equations (14.314) can be expressed in matrix notation as:

$$\begin{Bmatrix} Q_1 \\ \vdots \\ Q_n \end{Bmatrix} = \begin{bmatrix} \beta_{1,1} & \cdots & \beta_{1,n} \\ \vdots & \ddots & \vdots \\ \beta_{n,1} & \cdots & \beta_{n,n} \end{bmatrix} \cdot \begin{Bmatrix} \varphi_1 \\ \vdots \\ \varphi_n \end{Bmatrix} \quad ; \quad \{Q\} = [\beta] \cdot \{\varphi\} \tag{14.315}$$

where the capacitance coefficient matrix $[\beta]$ and the potential coefficient matrix $[\alpha]$ are mutually inverse matrices:

$$[\beta] = [\alpha]^{-1} \tag{14.316}$$

The electric charges of conducting bodies can also be expressed as follows:

$$Q_i = \sum_{k=1}^{n} C_{i,k} \cdot U_{i,k} \quad ; \quad i = 1, 2, ..., n \tag{14.317}$$

where:

$C_{i,k}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ - the so-called *partial capacitances*,

$U_{i,k}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ - the electric voltages between the conducting bodies.

The electric voltages between the conducting bodies and the electric scalar potentials of the conducting bodies are related by the following expression:

$$U_{i,k} = \begin{cases} \varphi_i & \text{for } i = k \\ \varphi_i - \varphi_k & \text{for } i \neq k \end{cases} \tag{14.318}$$

The partial capacitance matrix is a symmetric matrix given by:

$$[C]=\begin{bmatrix} C_{1,1} & \cdots & C_{1,n} \\ \vdots & \ddots & \vdots \\ C_{n,1} & \cdots & C_{n,n} \end{bmatrix} \quad ; \quad C_{i,k}=C_{k,i} \tag{14.319}$$

For a system of n conducting bodies, the following holds:

$$C_{i,k}=-\beta_{i,k} \quad \text{for} \quad i \neq k \tag{14.320}$$

$$C_{i,i}=\sum_{k=1}^{n}\beta_{i,k} \tag{14.321}$$

and the following relations hold:

$$\beta_{i,k}=-C_{i,k} \quad \text{for} \quad i \neq k \tag{14.322}$$

$$\beta_{i,i}=\sum_{k=1}^{n}C_{i,k} \tag{14.323}$$

A system of conducting bodies can be graphically represented by a partial capacitance equivalent circuit, in which a partial capacitance $C_{i,k}$ is connected between the i-th and k-th nodes.

14.25.1. Partial Capacitances of a Two-Wire Electric Line

Let a two-wire electric line consist of two infinitely long, straight, thin-wire conductors that are mutually parallel and carry linear charge densities λ_1 and λ_2. The line is located in the air above the ground, which has an electric scalar potential equal to zero (see Figure 14.60). Let the radius of both conductors be $r_0 << d$. Assume that the air is a perfect LIH dielectric.

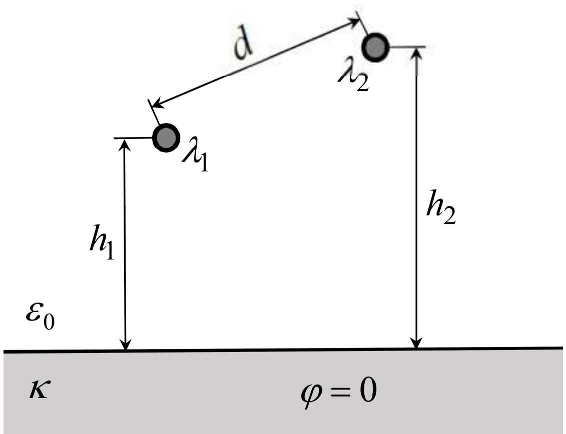


Figure 14.60. Two-wire electric line above the ground

The influence of the ground on the distribution of the electric scalar potential can be accounted for by applying the boundary condition at the ground surface using the method of images. According to this method, two charged conductors in the air (a dielectric half-space) are replaced by four conductors in an unbounded perfect LIH dielectric whose permittivity equals that of air (see Figure 14.61). In addition to the real conductors, there are also imaginary conductors – mirror images of the real ones – each carrying an image charge of equal magnitude and opposite sign to the corresponding real charge.

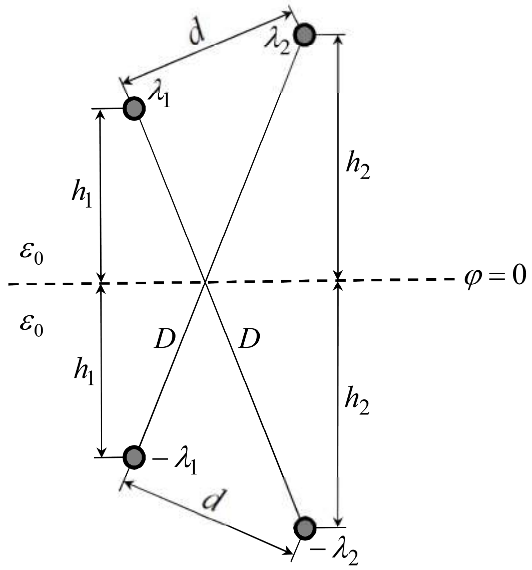


Figure 14.61. A two-wire electric line and its image in unbounded LIH air

From expression (14.252), it follows that the electric scalar potentials of the two conductors are given by the following expressions:

$$\varphi_1 = \frac{\lambda_1}{2\cdot\pi\cdot\varepsilon_0}\cdot\ln\frac{2\cdot h_1}{r_0} + \frac{\lambda_2}{2\cdot\pi\cdot\varepsilon_0}\cdot\ln\frac{D}{d} \tag{14.324}$$

$$\varphi_2 = \frac{\lambda_1}{2\cdot\pi\cdot\varepsilon_0}\cdot\ln\frac{D}{d} + \frac{\lambda_2}{2\cdot\pi\cdot\varepsilon_0}\cdot\ln\frac{2\cdot h_2}{r_0} \tag{14.325}$$

which can be written in matrix notation as:

$$\begin{Bmatrix}\varphi_1\\ \varphi_2\end{Bmatrix} = \begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & \alpha_{2,2}\end{bmatrix}\cdot\begin{Bmatrix}\lambda_1\\ \lambda_2\end{Bmatrix} \quad ; \quad \{\varphi\} = [\alpha]\cdot\{\lambda\} \tag{14.326}$$

where the potential coefficient matrix is given by:

$$[\alpha] = \frac{1}{2\cdot\pi\cdot\varepsilon_0}\cdot\begin{bmatrix}\ln\frac{2\cdot h_1}{r_0} & \ln\frac{D}{d}\\ \ln\frac{D}{d} & \ln\frac{2\cdot h_2}{r_0}\end{bmatrix} \tag{14.327}$$

Furthermore, the following holds:

$$\{\lambda\} = [\beta]\cdot\{\varphi\} = [\alpha]^{-1}\cdot\{\varphi\} \tag{14.328}$$

where the per-unit-length capacitance coefficient matrix is given by:

$$[\beta] = [\alpha]^{-1} = \begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & \beta_{2,2}\end{bmatrix} = \frac{1}{\alpha_{1,1}\cdot\alpha_{2,2} - \alpha_{1,2}\cdot\alpha_{2,1}}\cdot\begin{bmatrix}\alpha_{2,2} & -\alpha_{1,2}\\ -\alpha_{2,1} & \alpha_{1,1}\end{bmatrix} \tag{14.329}$$

It follows that the per-unit-length partial capacitance matrix can be written as:

$$[C]=\begin{bmatrix} \beta_{1,1}+\beta_{1,2} & -\beta_{1,2} \\ -\beta_{2,1} & \beta_{2,1}+\beta_{2,2} \end{bmatrix} \tag{14.330}$$

Per-unit-length partial capacitances describe both capacitive coupling between conductors and capacitive coupling between each conductor and ground (see Figure 14.62).

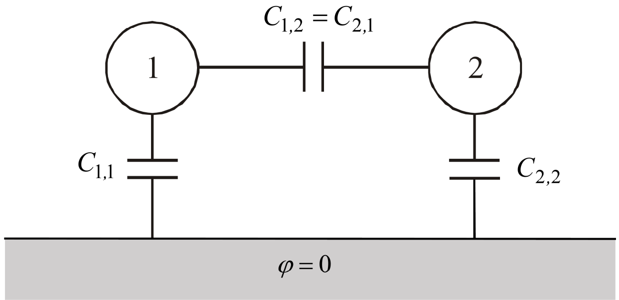


Figure 14.62. Per-unit-length partial capacitances

From expressions (14.317) and (14.319), it follows that:

$$[\beta]=[\alpha]^{-1}=\frac{2\cdot\pi\cdot\varepsilon_0}{\ln\frac{2\cdot h_1}{r_0}\cdot\ln\frac{2\cdot h_2}{r_0}-\ln^2\frac{D}{d}}\cdot\begin{bmatrix} \ln\frac{2\cdot h_2}{r_0} & -\ln\frac{D}{d} \\ -\ln\frac{D}{d} & \ln\frac{2\cdot h_1}{r_0} \end{bmatrix} \tag{14.331}$$

and, according to expression (14.330), it is obtained that:

$$[C]=\frac{2\cdot\pi\cdot\varepsilon_0}{\ln\frac{2\cdot h_1}{r_0}\cdot\ln\frac{2\cdot h_2}{r_0}-\ln^2\frac{D}{d}}\cdot\begin{bmatrix} \ln\frac{2\cdot h_2\cdot d}{r_0\cdot D} & \ln\frac{D}{d} \\ \ln\frac{D}{d} & \ln\frac{2\cdot h_1\cdot d}{r_0\cdot D} \end{bmatrix} \tag{14.332}$$

The effective per-unit-length capacitance of a two-wire line, taking into account the influence of the ground, is the total per-unit-length capacitance between the two conductors:

$$C=C_{1,2}+\frac{C_{1,1}\cdot C_{2,2}}{C_{1,1}+C_{2,2}} \tag{14.333}$$

14.25.2. Partial Capacitances of Mutually Parallel Conductor Segments

Let there be two mutually parallel, straight, cylindrical conductor segments of length ℓ, separated by a distance d (see Figure 14.63). Let them carry charges Q_1 and Q_2, respectively, and be located in air at heights h_1 and h_2 above the ground, which has an electric scalar potential equal to zero (see Figure 14.64). Let the radius of both conductors be $r_0 << d$, and assume that air is a perfect linear, isotropic and homogeneous (LIH) dielectric. The partial capacitances of the conductors are to be determined using the average potential method. The influence of the ground on the distribution of the electric scalar potential can be accounted for by satisfying the boundary condition at the ground surface using the method of images (see Figure 14.65).

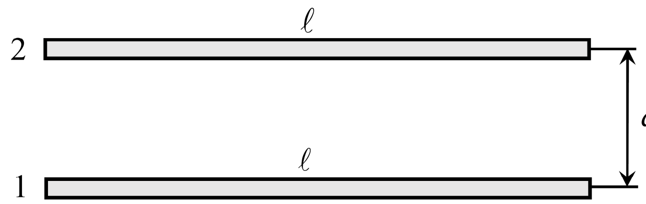


Figure 14.63. Two mutually parallel cylindrical conductor segments

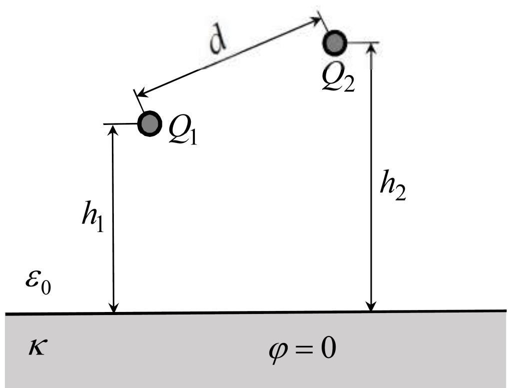


Figure 14.64. Two mutually parallel cylindrical conductor segments above the ground

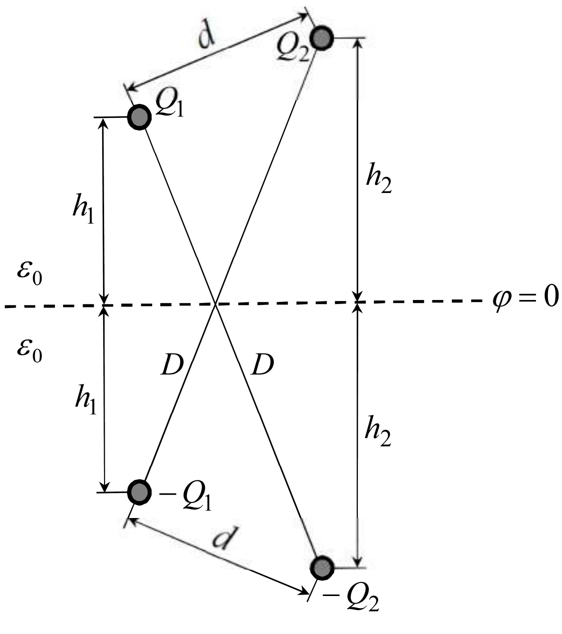


Figure 14.65. Two conductor segments and their images in unbounded LIH air

According to the average potential method, the linear electric charge density along the axis of a thin-wire conductor segment is approximated as constant, whereas the electric scalar potential of the segment is approximated by the average potential over its surface. This method is described in detail in subchapter 14.24.

The following holds:

$$\begin{Bmatrix} \Phi_1 \\ \Phi_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} \cdot \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \quad ; \quad \lambda_1 = \frac{Q_1}{\ell} \quad ; \quad \lambda_2 = \frac{Q_2}{\ell} \tag{14.334}$$

where Φ_1 and Φ_2 are the (average) electric scalar potentials of the conductor segments, whereas Q_1 and Q_2 are the electric charges of the conductor segments.

Based on expression (14.303) and Figure 14.61, the potential coefficient matrix – determined using the average potential method while taking into account the influence of the ground – is given by the following expression:

$$[\alpha]=\begin{bmatrix}\alpha_{1,1} & \alpha_{1,2}\\ \alpha_{2,1} & \alpha_{2,2}\end{bmatrix}=\begin{bmatrix}G(\ell,r_0)-G(\ell,2\cdot h_1) & G(\ell,d)-G(\ell,D)\\ G(\ell,d)-G(\ell,D) & G(\ell,r_0)-G(\ell,2\cdot h_2)\end{bmatrix} \tag{14.335}$$

where, according to expression (14.301), in the case where the perfect LIH dielectric is air:

$$G(\ell,v)=\frac{1}{2\cdot\pi\cdot\varepsilon_0\cdot\ell^2}\cdot\left[\ell\cdot\ln\frac{\sqrt{\ell^2+v^2}+\ell}{v}-\frac{\ell^2}{\sqrt{\ell^2+v^2}\;+\;v}\right] \tag{14.336}$$

The capacitance coefficient matrix can be obtained by inverting the potential coefficient matrix described by expression (14.335):

$$[\beta]=[\alpha]^{-1}=\begin{bmatrix}\beta_{1,1} & \beta_{1,2}\\ \beta_{2,1} & \beta_{2,2}\end{bmatrix}=\frac{1}{\alpha_{1,1}\cdot\alpha_{2,2}-\alpha_{1,2}\cdot\alpha_{2,1}}\cdot\begin{bmatrix}\alpha_{2,2} & -\alpha_{1,2}\\ -\alpha_{2,1} & \alpha_{1,1}\end{bmatrix} \tag{14.337}$$

from which it follows that the of partial capacitance matrix is given by the following expression:

$$[C]=\begin{bmatrix}C_{1,1} & C_{1,2}\\ C_{2,1} & C_{2,2}\end{bmatrix}=\begin{bmatrix}\beta_{1,1}+\beta_{1,2} & -\beta_{1,2}\\ -\beta_{2,1} & \beta_{2,1}+\beta_{2,2}\end{bmatrix} \tag{14.338}$$

Since the previous expressions are complex, it is not practical to derive analytical expressions for the elements of the capacitance coefficient matrix or the elements of the partial capacitance matrix.

14.26. Force at the Boundary Between Two Dielectrics and the Force on an Electric Charge

Let a point electric charge q be located at a distance d from the boundary of a dielectric half-space. Assume the charge is situated in the half-space with permittivity ε_1, whereas the remaining half-space has permittivity ε_2 (see Figure 14.66). Let $\varepsilon_1>\varepsilon_2$.

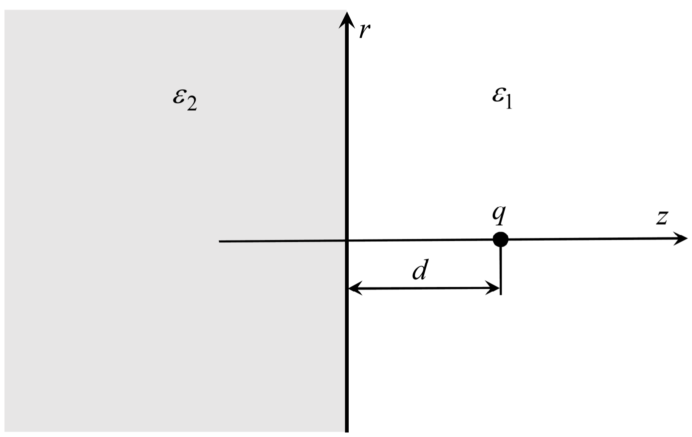


Figure 14.66. Point electric charge in a cylindrical coordinate system

Using the method of images, it is straightforward to derive that at the boundary between two dielectrics (z = 0), the electric field intensity vector in medium 2 can be described by the following expression:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\mathrm{z}} \cdot \vec{\mathrm{e}}_\mathrm{z} = \frac{k_\mathrm{T} \cdot q}{4 \cdot \pi \cdot \varepsilon_2} \cdot \left[\frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} - d \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(d^2 + r^2\right)^3}} \right] \tag{14.339}$$

where, according to expression (14.130), the transmission factor is is given by:

$$k_\mathrm{T} = \frac{2 \cdot \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tag{14.340}$$

It follows that at the boundary between two media (z = 0), for $\varepsilon_1 > \varepsilon_2$, the tangential component of the electric field intensity vector and the normal component of the electric displacement vector in both dielectrics are described by the following expressions:

$$E_\mathrm{t} = E_{2\mathrm{r}} = \frac{q}{2 \cdot \pi \cdot (\varepsilon_1 + \varepsilon_2)} \cdot \frac{r}{\sqrt{\left(d^2 + r^2\right)^3}} \tag{14.341}$$

$$D_\mathrm{n} = D_{2\mathrm{n}} = \varepsilon_2 \cdot E_{2\mathrm{z}} = \frac{\varepsilon_2 \cdot q}{2 \cdot \pi \cdot (\varepsilon_1 + \varepsilon_2)} \cdot \frac{-d}{\sqrt{\left(d^2 + r^2\right)^3}} \tag{14.342}$$

According to expression (14.118), the electrostatic pressure at the boundary between these two dielectrics can be described by the following expression:

$$t_\mathrm{n}^\mathrm{e} = \frac{\varepsilon_1 - \varepsilon_2}{2} \cdot \left(\frac{D_\mathrm{n}^2}{\varepsilon_1 \cdot \varepsilon_2} + E_\mathrm{t}^2 \right) \tag{14.343}$$

and, by substituting expressions (14.341) and (14.342) into expression (14.343), it follows that:

$$t_\mathrm{n}^\mathrm{e} = \frac{q^2}{8 \cdot \pi^2} \cdot \frac{k_\mathrm{R}}{(\varepsilon_1 + \varepsilon_2)} \cdot \left(\frac{\varepsilon_2}{\varepsilon_1} \cdot \frac{d^2}{\left(d^2 + r^2\right)^3} + \frac{r^2}{\left(d^2 + r^2\right)^3} \right) \tag{14.344}$$

where, according to expression (14.129):

$$k_\mathrm{R} = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} \tag{14.345}$$

is the reflection factor.

The force at the boundary between two dielectrics can be obtained by integrating the electrostatic pressure over the boundary surface $z = 0$, and the force is directed from the medium with higher permittivity to the medium with lower permittivity:

$$\vec{F}_\mathrm{b} = (-\vec{e}_\mathrm{z}) \cdot \int_{S_\mathrm{b}} t_\mathrm{n}^\mathrm{e} \cdot dS = (-\vec{e}_\mathrm{z}) \cdot \int_0^\infty t_\mathrm{n}^\mathrm{e} \cdot 2 \cdot \pi \cdot r \cdot dr \tag{14.346}$$

and it follows that:

$$\vec{F}_\mathrm{b} = (-\vec{e}_\mathrm{z}) \cdot \frac{k_\mathrm{R} \cdot q^2}{4 \cdot \pi \cdot (\varepsilon_1 + \varepsilon_2)} \left(\frac{\varepsilon_2}{\varepsilon_1} \cdot \int_0^\infty \frac{d^2 \cdot r \cdot dr}{\left(d^2 + r^2\right)^3} + \int_0^\infty \frac{r^3 \cdot dr}{\left(d^2 + r^2\right)^3} \right) \tag{14.347}$$

The following holds:

$$\int_0^\infty \frac{d^2 \cdot r \cdot dr}{\left(d^2 + r^2\right)^3} = \int_0^\infty \frac{r^3 \cdot dr}{\left(d^2 + r^2\right)^3} = \frac{1}{(2 \cdot d)^2} \tag{14.348}$$

from which it follows that the final expression for the force at the boundary between two dielectrics is given by:

$$\vec{F}_{\mathrm{b}} = (-\vec{e}_{\mathrm{z}}) \cdot \frac{q \cdot (k_{\mathrm{R}} \cdot q)}{4 \cdot \pi \cdot \varepsilon_1 \cdot (2 \cdot d)^2} \tag{14.349}$$

Since, according to expression (14.345), the reflection factor is positive for $\varepsilon_1 > \varepsilon_2$, the real point electric charge q and its image charge have the same polarity (see Figure 14.67). This means they repel each other, and the force on the point electric charge q can be written as:

$$\vec{F}_{\mathrm{q}} = \vec{e}_{\mathrm{z}} \cdot \frac{q \cdot (k_{\mathrm{R}} \cdot q)}{4 \cdot \pi \cdot \varepsilon_1 \cdot (2 \cdot d)^2} = -\vec{F}_{\mathrm{b}} \tag{14.350}$$

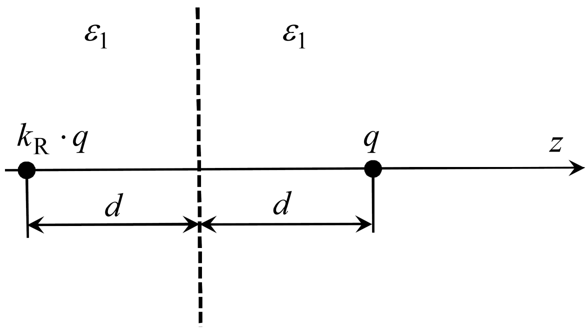


Figure 14.67. Real and mirrored point electric charge

Conclusion: The force at the boundary between two dielectrics arises from the tendency of the medium with higher permittivity to displace the medium with lower permittivity. The force acting on the electric charge is equal in magnitude to the force at the dielectric boundary but directed oppositely. If the charge is located in the medium with higher permittivity, it tends to move deeper into that medium. According to the law of action and reaction, the medium with lower permittivity and the electric charge in the higher-permittivity medium repel each other. Conversely, if the charge is in the medium with lower permittivity, it is attracted toward the medium with higher permittivity. Therefore, the force on the electric charge that generates the electrostatic field is always directed opposite to the force at the boundary between the two dielectrics. This behavior is described by expression (14.350).

14.27. Solved Examples

Example 14.1. Determine the linear charge density of an infinitely long cylindrical capacitor based on the solution of the Laplace differential equation in a cylindrical coordinate system. A dielectric of relative permittivity $\varepsilon_\mathrm{r} = 10$ is present between the electrodes of the capacitor. There is an electric scalar potential $\varphi = \varphi_\mathrm{a} = 100\ \mathrm{V}$ on the inner cylinder and an electric scalar potential $\varphi = \varphi_\mathrm{b} = 0\ \mathrm{V}$ on the outer cylinder. Let $a = 1\ \mathrm{cm}, b = 2\ \mathrm{cm}$. Assume that the z-axis of the cylindrical coordinate system coincides with the axis of the cylindrical capacitor. The linear charge density per cylinder is constant.

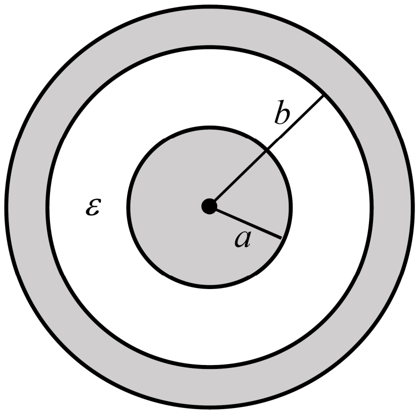


Figure 14.68. Infinitely long cylindrical capacitor

Solution:

In the cylindrical coordinate system (r, ϕ, z), the Laplace differential equation of the electric scalar potential is described by the expression (14.168), which can be written for the axisymmetric problem, i.e., for $\varphi = \varphi(r)$, as follows:

$$\Delta\varphi = \frac{1}{r} \cdot \frac{d}{dr} \cdot \left(r \cdot \frac{d\varphi}{dr} \right) = 0 \tag{14.351}$$

from which it follows that:

$$r \cdot \frac{d\varphi}{dr} = C_1 \qquad \Rightarrow \qquad \frac{d\varphi}{dr} = \frac{C_1}{r} \tag{14.352}$$

and:

$$\varphi = C_1 \cdot \int \frac{dr}{r} = C_1 \cdot \ln r + C_2 \tag{14.353}$$

where C_1 and C_2 are unknown constants.

From the solution (14.353) of the Laplace differential equation (14.351) and given boundary conditions:

$$\varphi\big|_{r=a} = \varphi_\mathrm{a} = 100\ \mathrm{V} \quad ; \quad \varphi\big|_{r=b} = \varphi_\mathrm{b} = 0\ \mathrm{V} \tag{14.354}$$

we obtain the following system of linear equations:

$$C_1 \cdot \ln a + C_2 = \varphi_\mathrm{a} \tag{14.355}$$

$$C_1 \cdot \ln b + C_2 = \varphi_\mathrm{b} \tag{14.356}$$

whose solutions are:

$$C_1 = -\frac{\varphi_a - \varphi_b}{\ln\frac{b}{a}} \quad ; \quad C_2 = \varphi_b + \frac{\varphi_a - \varphi_b}{\ln\frac{b}{a}} \cdot \ln b \tag{14.357}$$

In this particular case, the electric field intensity in the dielectric of a cylindrical capacitor is defined by the following expression:

$$\vec{E} = -\nabla\varphi = -\frac{d\varphi}{dr} \cdot \vec{e}_r \tag{14.358}$$

and it is obtained that according to expressions (14.353) and (14.358):

$$\vec{E} = -\frac{C_1}{r} \cdot \vec{e}_r \tag{14.359}$$

Using Gauss's law, it is easy to obtain the expression (14.33) for the electric field intensity in the dielectric of a cylindrical capacitor, which can be written as:

$$\vec{E} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon \cdot r} \cdot \vec{e}_r \tag{14.360}$$

From expressions (14.359) and (14.360), it follows that the linear charge density of a cylindrical capacitor is:

$$\lambda = -2 \cdot \pi \cdot \varepsilon \cdot C_1 = \frac{2 \cdot \pi \cdot \varepsilon}{\ln\frac{\mathrm{b}}{\mathrm{a}}} \cdot (\varphi_a - \varphi_b) \tag{14.361}$$

and, for the given data, it can be obtained that:

$$\lambda = \frac{2000 \cdot \pi \cdot \varepsilon_0}{\ln 2} = 8.0260736 \times 10^{-8}\ \text{C/m} = 80.260736\ \text{nC/m} \tag{14.362}$$

Since the inner cylinder is at a higher electric scalar potential than the outer cylinder, there is a positive charge of linear density λ on the inner cylinder, and a negative charge of linear density $-\lambda$ on the outer cylinder, as shown in Figure 14.19.

Example 14.2. In the air, an electric charge is distributed uniformly throughout a sphere, forming a ball with constant volume charge density $\rho_0 = 1\ \text{nC/m}^3$. Determine the distribution of the electric scalar potential at all points in spherical coordinates using Gauss's law. Let the radius of the sphere be $r_0 = 1$ m.

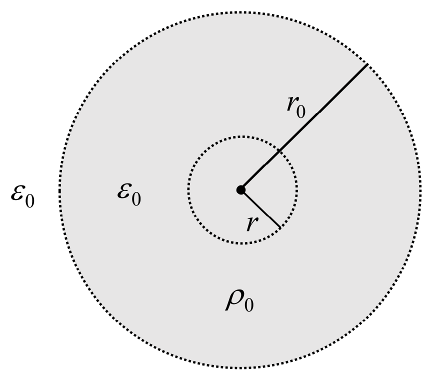


Figure 14.69. Uniformly distributed charge within the sphere

Solution:

Inside the sphere (region 1), the following holds:

$$\oint_S \vec{D}_1 \cdot d\vec{S} = \int_V \rho_0 \cdot dV \tag{14.363}$$

where the closed surface S is a sphere of radius $r \le r_0$ (Figure 14.69).
From expression (14.363), it follows that:

$$D_1 \cdot S = D_1 \cdot 4 \cdot r^2 \cdot \pi = \rho_0 \cdot V = \rho_0 \cdot \frac{4}{3} \cdot r^3 \cdot \pi \tag{14.364}$$

thus it is:

$$D_1 = \frac{\rho_0}{3} \cdot r \quad ; \quad E_1 = \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot r \tag{14.365}$$

Outside the sphere (region 2), the following holds:

$$\oint_S \vec{D}_2 \cdot d\vec{S} = \int_V \rho_0 \cdot dV \tag{14.366}$$

where the closed surface S is a sphere of radius $r > r_0$ (Figure 14.69).
From expression (14.366), it follows that:

$$D_2 \cdot S = D_2 \cdot 4 \cdot r^2 \cdot \pi = \rho_0 \cdot V_0 = \rho_0 \cdot \frac{4}{3} \cdot r_0^3 \cdot \pi \tag{14.367}$$

thus it is:

$$D_2 = \frac{\rho_0}{3} \cdot \frac{r_0^3}{r^2} \quad ; \quad E_2 = \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r_0^3}{r^2} \tag{14.368}$$

The electric field intensity is defined by the following expression::

$$\vec{E} = -\nabla \varphi = -\frac{d\varphi}{dr} \cdot \vec{e}_\mathrm{r} \tag{14.369}$$

from which it follows that the distribution of the electric scalar potential is described by expressions:

$$\varphi_1 = -\int E_1 \cdot dr = -\int \frac{\rho_0 \cdot r}{3 \cdot \varepsilon_0} \cdot dr = -\frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r^2}{2} + C_1 \tag{14.370}$$

$$\varphi_2 = -\int E_2 \cdot dr = -\int \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r_0^3}{r^2} \cdot dr = \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r_0^3}{r} + C_2 \tag{14.371}$$

It is a usual assumption that the reference point, where the electric scalar potential is equal to zero, is at infinity, from which it follows that the constant $C_2 = 0$ and:

$$\varphi_2 = \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r_0^3}{r} \quad \text{for} \quad r > r_0 \tag{14.372}$$

From the boundary condition:

$$\varphi_1 \big|_{r=r_0} = \varphi_2 \big|_{r=r_0} \tag{14.373}$$

it is easy to obtain that it is:

$$C_1 = \frac{\rho_0 \cdot r_0^2}{3 \cdot \varepsilon_0} \cdot \left(1 + \frac{1}{2}\right) = \frac{\rho_0 \cdot r_0^2}{2 \cdot \varepsilon_0} \tag{14.374}$$

and, according to expressions (14.370) and (14.374), the distribution of the electric scalar potential in region 1 is described by the following expression:

$$\varphi_1 = \frac{\rho_0}{2 \cdot \varepsilon_0} \cdot \left(r_0^2 - \frac{r^2}{3}\right) \quad \text{for} \quad r \le r_0 \tag{14.375}$$

For the given data, it can be obtained that:

$$\varphi_1 = 56.4704533 \cdot \left(1 - \frac{r^2}{3}\right) \ \text{V} \quad \text{for} \quad r \le 1 \ \text{m} \tag{14.376}$$

$$\varphi_2 = \frac{37.64696889}{r} \ \text{V} \quad \text{for} \quad r > 1 \ \text{m} \tag{14.377}$$

Example 14.3. In the air, an electric charge is uniformly distributed within a sphere and has a constant volume charge density $\rho_0 = 1 \ \text{nC/m}^3$. Determine the distribution of the electric scalar potential at all points in spherical coordinates using the solutions of Poisson's differential equation. Let the radius of the sphere be $r_0 = 1$ m.

Solution:

This example is textually almost identical to the previous example, so Figure 14.69 can be included; the two examples differ only in the method of solution.

In the spherical coordinate system (r, ϑ, ϕ), the Laplacian of the electric scalar potential is described by expression (14.87), which, for a centrally symmetric problem, i.e., for $\varphi = \varphi(r)$, can be written as:

$$\Delta \varphi = \frac{1}{r^2} \cdot \frac{\partial}{\partial r}\left(r^2 \cdot \frac{\partial \varphi}{\partial r}\right) \tag{14.378}$$

and the corresponding Poisson's differential equation is given by:

$$\Delta \varphi = \frac{1}{r^2} \cdot \frac{d}{dr} \cdot \left(r^2 \cdot \frac{d\varphi}{dr}\right) = -\frac{\rho}{\varepsilon} \tag{14.379}$$

In this particular case, the following holds:

$$\Delta \varphi_1 = \frac{1}{r^2} \cdot \frac{d}{dr} \cdot \left(r^2 \cdot \frac{d\varphi_1}{dr}\right) = -\frac{\rho_0}{\varepsilon_0} \quad \text{for} \quad r \le r_0 \tag{14.380}$$

$$\Delta \varphi_2 = \frac{1}{r^2} \cdot \frac{d}{dr} \cdot \left(r^2 \cdot \frac{d\varphi_2}{dr}\right) = 0 \quad \text{for} \quad r > r_0 \tag{14.381}$$

After double integration of the two previous expressions, the following expressions are obtained:

$$\varphi_1 = -\frac{1}{3} \cdot \frac{\rho_0}{\varepsilon_0} \cdot \frac{r^2}{2} - \frac{C_1}{r} + C_2 \quad \text{for} \quad r \le r_0 \tag{14.382}$$

$$\varphi_2 = -\frac{C_3}{r} + C_4 \quad \text{for} \quad r > r_0 \tag{14.383}$$

From the usual assumption that the reference point is at infinity, it follows that the constant $C_4 = 0.$ Furthermore, for $r = 0,$ the electric scalar potential must be finite, from which it follows that the constant $C_1 = 0.$ After including these constants, the expressions (14.382) and (14.383) take on a new form:

$$\varphi_1 = -\frac{1}{3} \cdot \frac{\rho_0}{\varepsilon_0} \cdot \frac{r^2}{2} + C_2 \quad ; \qquad \varphi_2 = -\frac{C_3}{r} \tag{14.384}$$

Expressions (14.384) should satisfy the boundary conditions:

$$\left.\varphi_1\right|_{r=r_0} = \left.\varphi_2\right|_{r=r_0} \quad ; \quad \varepsilon_0 \cdot \left.\frac{d\varphi_1}{dr}\right|_{r=r_0} = \varepsilon_0 \cdot \left.\frac{d\varphi_2}{dr}\right|_{r=r_0} \tag{14.385}$$

Based on the satisfaction of the boundary conditions, it is easy to obtain that:

$$C_2 = \frac{\rho_0 \cdot r_0^2}{2 \cdot \varepsilon_0} \quad ; \quad C_3 = -\frac{\rho_0 \cdot r_0^3}{3 \cdot \varepsilon_0} \tag{14.386}$$

and, according to expressions (14.384) and (14.386), the distribution of the electric scalar potential can be described by the following expressions:

$$\varphi_1 = \frac{\rho_0}{2 \cdot \varepsilon_0} \cdot \left(r_0^2 - \frac{r^2}{3} \right) \quad \text{for} \quad r \le r_0 \tag{14.387}$$

$$\varphi_2 = \frac{\rho_0}{3 \cdot \varepsilon_0} \cdot \frac{r_0^3}{r} \quad \text{for} \quad r > r_0 \tag{14.388}$$

For the given data, it can be obtained that:

$$\varphi_1 = 56.4704533 \cdot \left(1 - \frac{r^2}{3} \right) \text{ V} \quad \text{for} \quad r \le 1 \text{ m} \tag{14.389}$$

$$\varphi_2 = \frac{37.64696889}{r} \text{ V} \quad \text{for} \quad r > 1 \text{ m} \tag{14.390}$$

Example 14.4. Determine the equation of the equipotential surface of zero electric scalar potential for two unlike point electric charges q_1 and q_2, separated by a distance *d*. Assume that $|q_1| > |q_2|$. The electric charges are in an unbounded LIH dielectric of permittivity ε.

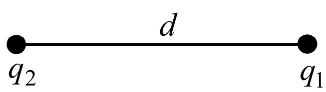


Figure 14.70. Two point charges in an unbounded LIH dielectric

Solution:

The distribution of the electric scalar potential is described by the following expression:

$$\varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \left(\frac{q_1}{r_1} + \frac{q_2}{r_2} \right) \tag{14.391}$$

where r_1 denotes the distance of the charge q_1 from the field point, whereas r_2 denotes the distance of the charge q_2 from the field point.

If the charges are of the same polarity, the equipotential surface corresponding to zero electric scalar potential is located at infinity. Therefore, it is physically meaningful to consider only the case in which the charges are of opposite polarity, i.e., when:

$$\operatorname{sign} q_1 = -\operatorname{sign} q_2 \tag{14.392}$$

Then $\varphi = 0$ for $r_1, r_2 \to \infty$ and in the case when:

$$\frac{q_1}{r_1} + \frac{q_2}{r_2} = 0 \tag{14.393}$$

from which it follows that:

$$\frac{r_1}{r_2} = -\frac{q_1}{q_2} = k > 1 \tag{14.394}$$

due to the adopted assumption $|q_1| > |q_2|$.

For the sake of simplicity, we assume that the charges are positioned within the two-dimensional Cartesian coordinate system (x, y) shown in Figure 14.71. This assumption is justified by the fact that the equipotential surface corresponding to zero electric scalar potential is axisymmetric with respect to the x-axis.

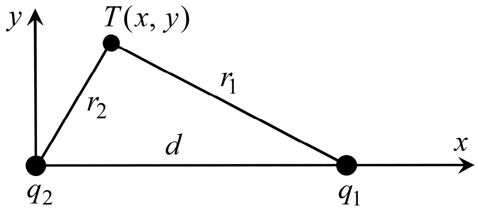


Figure 14.71. Two point charges in a 2D coordinate system (x, y)

From Figure 14.71, it can be seen that:

$$\frac{r_1}{r_2} = \frac{\sqrt{(d-x)^2 + y^2}}{\sqrt{x^2 + y^2}} = k \tag{14.395}$$

The following expression is obtained as a direct consequence of squaring equation (14.395):

$$x^2 + \frac{2 \cdot d}{k^2 - 1} \cdot x + y^2 = \frac{d^2}{k^2 - 1} \tag{14.396}$$

which takes the following form after completing the square:

$$\left(x + \frac{d}{k^2 - 1} \right)^2 + y^2 = \frac{d^2 \cdot k^2}{\left(k^2 - 1\right)^2} \tag{14.397}$$

and that is the equation of a circle $(x - p)^2 + y^2 = R^2$, whose center lies on the x-axis, whereas R is the radius of the circle. The zero-potential surface is therefore a sphere with its center on the x-axis.

It follows from expression (14.397) that:

$$b = -p = \frac{d}{k^2 - 1} \quad ; \quad R = \frac{d \cdot k}{k^2 - 1} = b \cdot k \tag{14.398}$$

where the parameters b and R are explicitly depicted in Figure 14.72.

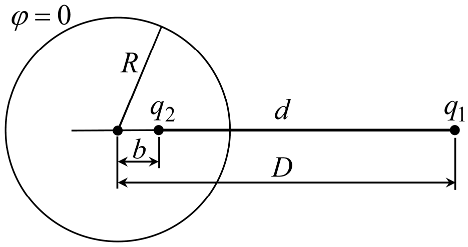


Figure 14.72. Zero-potential line in a 2D coordinate system (x, y)

As shown in Figure 14.72, the parameter b represents the leftward displacement of the center of the zero-potential circle relative to the location of the charge q_2, whereas D denotes the distance between the charge q_1 and the center of the zero-potential circle. The following relation holds:

$$D = d + b = d + \frac{d}{k^2 - 1} = \frac{d \cdot k^2}{k^2 - 1} = R \cdot k \tag{14.399}$$

implying that:

$$k = \frac{D}{R} = -\frac{q_1}{q_2} \tag{14.400}$$

It follows from expressions (14.398) and (14.400) that:

$$b = \frac{R}{k} = \frac{R^2}{D} \tag{14.401}$$

Without recourse to analytical derivation, the method of images yields the following expressions:

$$q_2 = \; = -\,q_1 \cdot \frac{R}{D} \quad ; \quad b = \frac{R^2}{D} \tag{14.402}$$

If $q_1 = -\,q_2$, then the equation of the zero-potential surface becomes $r_1 = r_2$, which corresponds to the plane of symmetry between the two electric charges.

Example 14.5. Determine the magnitude of the electric field intensity at the point $T(x, y, z)$, produced by a positive point electric charge q located on the bisector of the first quadrant in the plane $z = 0$, assuming that the planes $x = 0$ and $y = 0$ are perfectly conducting.

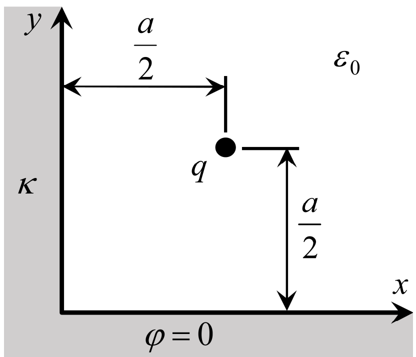


Figure 14.73. Positive point electric charge in front of two conducting half-planes

Solution:

This problem can be solved using the method of images. The real point electric charge, together with three image charges (as shown in Figure 14.74), all located in the plane $z = 0$, ensures that the following boundary conditions are satisfied:

$$\varphi\big|_{x=0} = \varphi\big|_{y=0} = 0 \tag{14.403}$$

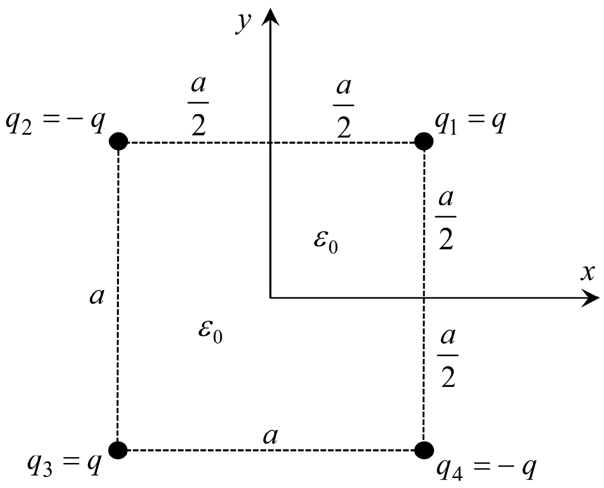


Figure 14.74. Real electric charge and corresponding image charges

According to the method of images, the entire space is declared to be a perfect LIH dielectric (air), and the obtained solution is valid only within the region of space where the conditions $x \geq 0$ and $y \geq 0$ are satisfied. The electric field intensity vector generated by the i-th point electric charge is described by the expression:

$$\vec{E}_\mathrm{i} = \frac{q_i}{4 \cdot \pi \cdot \varepsilon_0} \cdot \frac{r_\mathrm{ix} \cdot \vec{\mathrm{i}} + r_\mathrm{iy} \cdot \vec{\mathrm{j}} + r_\mathrm{iz} \cdot \vec{\mathrm{k}}}{r_\mathrm{i}^3} \tag{14.404}$$

where:

$$q_1 = q_3 = q \quad ; \quad q_2 = q_4 = -q \tag{14.405}$$

whereas:

$$r_\mathrm{1x} = x - \frac{a}{2} \quad ; \quad r_\mathrm{1y} = y - \frac{a}{2} \quad ; \quad r_\mathrm{1z} = z \quad ; \quad r_1 = \sqrt{r_\mathrm{1x}^2 + r_\mathrm{1y}^2 + r_\mathrm{1z}^2} \tag{14.406}$$

$$r_\mathrm{2x} = x + \frac{a}{2} \quad ; \quad r_\mathrm{2y} = y - \frac{a}{2} \quad ; \quad r_\mathrm{2z} = z \quad ; \quad r_2 = \sqrt{r_\mathrm{2x}^2 + r_\mathrm{2y}^2 + r_\mathrm{2z}^2} \tag{14.407}$$

$$r_\mathrm{3x} = x + \frac{a}{2} \quad ; \quad r_\mathrm{3y} = y + \frac{a}{2} \quad ; \quad r_\mathrm{3z} = z \quad ; \quad r_3 = \sqrt{r_\mathrm{3x}^2 + r_\mathrm{3y}^2 + r_\mathrm{3z}^2} \tag{14.408}$$

$$r_\mathrm{4x} = x - \frac{a}{2} \quad ; \quad r_\mathrm{4y} = y + \frac{a}{2} \quad ; \quad r_\mathrm{4z} = z \quad ; \quad r_4 = \sqrt{r_\mathrm{4x}^2 + r_\mathrm{4y}^2 + r_\mathrm{4z}^2} \tag{14.409}$$

The total electric field intensity vector is described by the following expression:

$$\vec{E} = \sum_{\mathrm{i}=1}^{4} \vec{E}_\mathrm{i} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \sum_{\mathrm{i}=1}^{4} \left(q_\mathrm{i} \cdot \frac{r_\mathrm{ix} \cdot \vec{\mathrm{i}} + r_\mathrm{iy} \cdot \vec{\mathrm{j}} + r_\mathrm{iz} \cdot \vec{\mathrm{k}}}{r_\mathrm{i}^3} \right) = \left\{ E_\mathrm{x}, E_\mathrm{y}, E_\mathrm{z} \right\} \tag{14.410}$$

and its magnitude is given by the expression:

$$E = \sqrt{E_\mathrm{x}^2 + E_\mathrm{y}^2 + E_\mathrm{z}^2} \tag{14.411}$$

Example 14.6. Determine the distribution of the electric scalar potential generated by a point charge q located in the air above a conducting plane featuring a hemispherical bulge of radius R. The charge is positioned at a distance D from the plane $y = 0$, and both the point electric charge and the center of the hemispherical bulge lie in the plane z = 0.

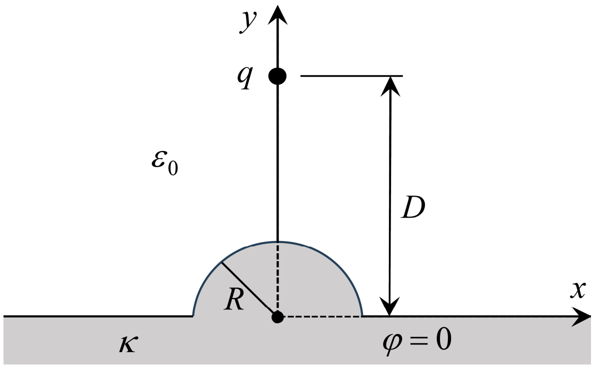


Figure 14.75. Point charge positioned above a conducting plane featuring a hemispherical bulge

Solution:

This problem can be solved using the method of images. The real point electric charge, together with three image charges (as shown in Figure 14.76), all located in the plane z = 0, ensures that the boundary conditions are satisfied.

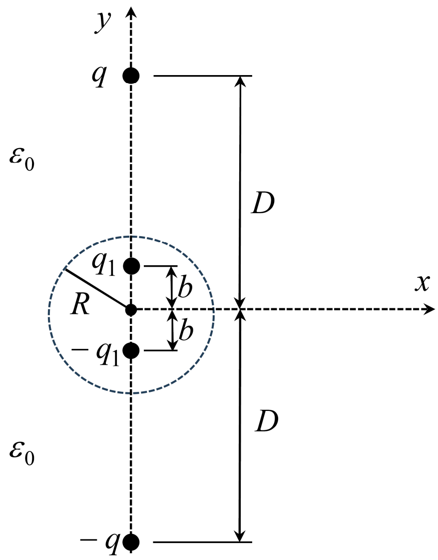


Figure 14.76. Real electric charge and corresponding image charges

According to expression (14.402), the following relation holds:

$$q_1 = -q \cdot \frac{R}{D} \quad ; \quad b = \frac{R^2}{D} \tag{14.412}$$

Therefore, the distribution of the electric scalar potential in the air can be described by the following expression:

$$\varphi = \frac{q}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{1}{r} - \frac{1}{r_\mathrm{s}} - \frac{R}{r_1 \cdot D} + \frac{R}{r_{1\mathrm{s}} \cdot D} \right) \tag{14.413}$$

where:

$$r = \sqrt{x^2 + (y - D)^2 + z^2} \quad ; \quad r_s = \sqrt{x^2 + (y + D)^2 + z^2} \tag{14.414}$$

$$r_1 = \sqrt{x^2 + (y - b)^2 + z^2} \quad ; \quad r_{1s} = \sqrt{x^2 + (y + b)^2 + z^2} \tag{14.415}$$

where r is the distance between the electric charge q and the field point $T(x, y, z)$; r_s is the distance between the electric charge $-q$ and the field point $T(x, y, z)$; r_1 is the distance between the electric charge q_1 and the field point $T(x, y, z)$; and r_{1s} is the distance between the electric charge $-q_1$ and the field point $T(x, y, z)$.

Example 14.7. A point electric charge q = 10 nC is located at a distance D = 0.1 m from the center of a grounded solid conducting sphere with radius R = 0.05 m. Determine the electric scalar potential of the sphere after the grounding is removed and the point charge q is taken to infinity. Assume that the influence of the grounding conductor on the distribution of the electric scalar potential can be neglected.

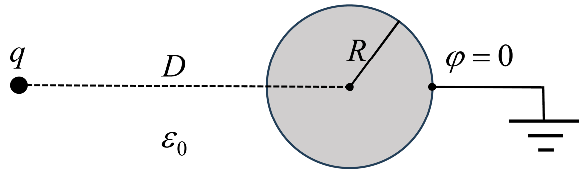


Figure 14.77. Point electric charge and grounded solid conducting sphere

Solution:

In order for the surface of the solid conducting sphere to be at zero electric scalar potential, a total induced electric charge must exist on the surface of the sphere:

$$q_1 = -q \cdot \frac{R}{D} \tag{14.416}$$

After the grounding is interrupted, the induced electric charge q_1 remains on the surface of the sphere. If the point charge q is then moved to infinity, the remaining charge q_1 becomes uniformly distributed over the surface of the sphere. The electric scalar potential of the sphere is then given by:

$$\varphi_{\text{sphere}} = \frac{q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot R} = -\frac{q}{4 \cdot \pi \cdot \varepsilon_0 \cdot D} = -898.7551788 \text{ V} \tag{14.417}$$

From expression (14.417), it follows that the electric scalar potential of the solid sphere, after the grounding is removed and the charge q is taken to infinity, is independent of the sphere's radius R.

Example 14.8. A point electric charge q is located at the origin of the Cartesian coordinate system at a distance D from the center of a charged solid conducting sphere of radius R. Let the electric scalar potential of the conducting sphere be φ_{sphere}. Determine the magnitude of the electric scalar potential and the electric field intensity at the point $T(x, y, z)$.

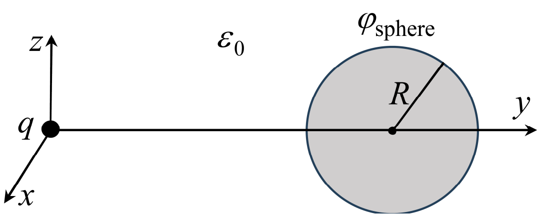


Figure 14.78. Point electric charge and a charged solid conducting sphere

Solution:

The problem can be solved by the method of images. The real electric charge q, the image electric charge q_1, and the equivalent electric charge q_2, all located along the y-axis (as shown in Figure 14.79), are arranged to satisfy the boundary conditions.

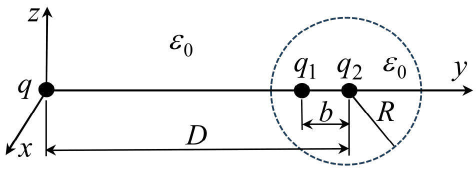


Figure 14.79. Real, image, and equivalent electric charges in an unbounded LIH dielectric (air)

Electric charges q and q_1 ensure that the electric scalar potential remains constant (φ = 0) on the surface of the solid conducting sphere, where:

$$q_1 = -q \cdot \frac{R}{D} \tag{14.418}$$

In order to raise the conducting sphere to a potential φ_{sphere}, an equivalent electric charge q_2 must be placed at the center of the sphere. The following relation holds:

$$\varphi_{\text{sphere}} = \frac{q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot R} \quad \Rightarrow \quad q_2 = 4 \cdot \pi \cdot \varepsilon_0 \cdot R \cdot \varphi_{\text{sphere}} \tag{14.419}$$

It follows that the distribution of the electric scalar potential in the air is described by the following expression:

$$\varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{q}{r} + \frac{q_1}{r_1} + \frac{q_2}{r_2} \right) \tag{14.420}$$

where:

$$r = \sqrt{x^2 + y^2 + z^2} \quad ; \quad r_1 = \sqrt{x^2 + (y - D + b)^2 + z^2} \tag{14.421}$$

$$r_2 = \sqrt{x^2 + (y - D)^2 + z^2} \tag{14.422}$$

The electric field intensity vector is described by the expression:

$$\vec{E} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{q \cdot \vec{r}}{r^3} + \frac{q_1 \cdot \vec{r}_1}{r_1^3} + \frac{q_2 \cdot \vec{r}_2}{r_2^3} \right) = \{E_{\text{x}}, E_{\text{y}}, E_{\text{z}}\} \tag{14.423}$$

where:

$$\vec{r} = \{x, y, z\} \quad ; \quad \vec{r}_1 = \{x, y - D + b, z\} \quad ; \quad \vec{r}_2 = \{x, y - D, z\} \tag{14.424}$$

whereas the magnitude of the electric field intensity is described by the expression:

$$E = \sqrt{E_{\text{x}}^2 + E_{\text{y}}^2 + E_{\text{z}}^2} \tag{14.425}$$

Example 14.9. A point electric charge q = 0.5 nC is located at a distance D = 2 m from the center of a grounded solid conducting sphere of radius R = 0.2 m (Figure 14.77). Determine the distribution of the surface charge density over the surface of the sphere.

Solution:

In order for the surface of the solid conducting sphere to be at zero electric scalar potential, a total induced electric charge is present on the grounded surface of the sphere:

$$q_1 = -q \cdot \frac{R}{D} \tag{14.426}$$

which can be replaced by an image point charge q_1, located at a distance:

$$b = \frac{R^2}{D} \tag{14.427}$$

from the center of the sphere (Figure 14.80).

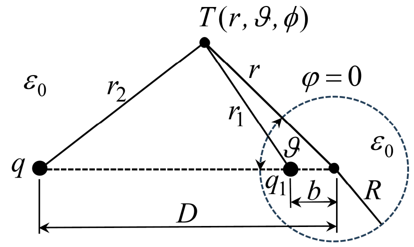


Figure 14.80. Real and image electric charges in an unbounded LIH dielectric (air)

The distribution of the electric scalar potential in the air, which includes the surface of the sphere, is described by the following expression:

$$\varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{q}{r_2} + \frac{q_1}{r_1} \right) = \frac{q}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{1}{r_2} - \frac{R}{D \cdot r_1} \right) \tag{14.428}$$

where r_1 and r_2, according to Figure 14.80, are given by the following expressions:

$$r_1 = \sqrt{r^2 + b^2 - 2 \cdot r \cdot b \cdot \cos\vartheta} \quad ; \quad r_2 = \sqrt{r^2 + D^2 - 2 \cdot r \cdot D \cdot \cos\vartheta} \tag{14.429}$$

where r and ϑ are the coordinates of the field point T in the spherical coordinate system, whose origin coincides with the center of the sphere.

The axisymmetric surface charge density over the surface of the solid sphere is described by the following expression:

$$\sigma = -\varepsilon_0 \cdot \left. \frac{\partial \varphi}{\partial r} \right|_{r=R} = \frac{q}{4 \cdot \pi} \cdot \left(\frac{1}{r_2^2} \cdot \frac{\partial r_2}{\partial r} - \frac{R}{D \cdot r_1^2} \cdot \frac{\partial r_1}{\partial r} \right)_{r=R} \tag{14.430}$$

where:

$$\frac{\partial r_1}{\partial r} = \frac{r - b \cdot \cos\vartheta}{r_1} = \frac{D \cdot r - R^2 \cdot \cos\vartheta}{r_1 \cdot D} \tag{14.431}$$

$$\frac{\partial r_2}{\partial r} = \frac{R - D \cdot \cos\vartheta}{r_2} \tag{14.432}$$

from which it follows that:

$$\sigma = \frac{q}{4 \cdot \pi} \cdot \frac{R - D \cdot \cos\vartheta}{\sqrt{\left(R^2 + D^2 - 2 \cdot R \cdot D \cdot \cos\vartheta\right)^3}} - \frac{q}{4 \cdot \pi} \cdot \frac{R \cdot \left(R \cdot D - R^2 \cdot \cos\vartheta\right)}{D^2 \cdot \sqrt{\left(R^2 + \frac{R^4}{D^2} - \frac{2 \cdot R^3}{D} \cdot \cos\vartheta\right)^3}} \tag{14.433}$$

The following relation holds:

$$R^2 + \frac{R^4}{D^2} - \frac{2 \cdot R^3}{D} \cdot \cos\vartheta = \frac{R^2}{D^2} \cdot \left(R^2 + D^2 - 2 \cdot R \cdot D \cdot \cos\vartheta\right) \tag{14.434}$$

and expression (14.433) takes the following form:

$$\sigma = \frac{q}{4 \cdot \pi} \cdot \frac{R - D \cdot \cos\vartheta - \dfrac{D^2}{R} + D \cdot \cos\vartheta}{\sqrt{\left(R^2 + D^2 - 2 \cdot R \cdot D \cdot \cos\vartheta\right)^3}} \tag{14.435}$$

from which it follows that:

$$\sigma = \sigma(\vartheta) = -\frac{q}{4 \cdot \pi} \cdot \frac{1}{\sqrt{\left(R^2 + D^2 - 2 \cdot R \cdot D \cdot \cos\vartheta\right)^3}} \cdot \frac{D^2 - R^2}{R} \tag{14.436}$$

If a new parameter ℓ is introduced to represent the distance between the charge q and a point on the surface of the sphere (Figure 14.81), this distance is given by the expression:

$$\ell = r_2\big|_{r=R} = \sqrt{R^2 + D^2 - 2 \cdot R \cdot D \cdot \cos\vartheta} \tag{14.437}$$

then the expression (14.436) takes the following form:

$$\sigma = \sigma(\ell) = -\frac{q}{4 \cdot \pi} \cdot \frac{1}{\ell^3} \cdot \frac{D^2 - R^2}{R} \tag{14.438}$$

After incorporating the given data, it follows that for ℓ (in meters):

$$\sigma = \sigma(\ell) = -\frac{78.78169683}{\ell^3} \ \frac{\mathrm{nC}}{\mathrm{m}^2} \tag{14.439}$$

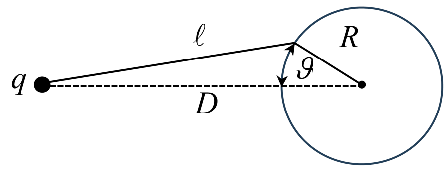


Figure 14.81. Graphical illustration of the parameter ℓ

Example 14.10. Consider a circular thin-wire loop of radius a, carrying a uniform line charge with linear charge density λ. The loop lies in the plane $z = 0$, and the surrounding medium is air. Determine the expression for the spatial distribution of the electric scalar potential in the cylindrical coordinate system (r, ϕ, z).

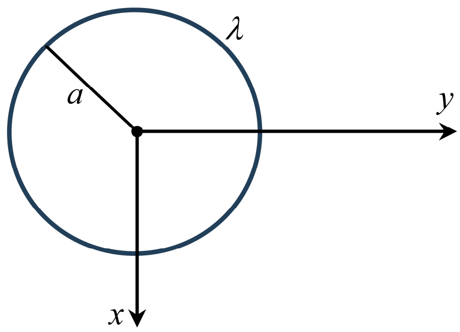


Figure 14.82. 2D representation of a uniformly charged circular thin-wire loop

Solution:

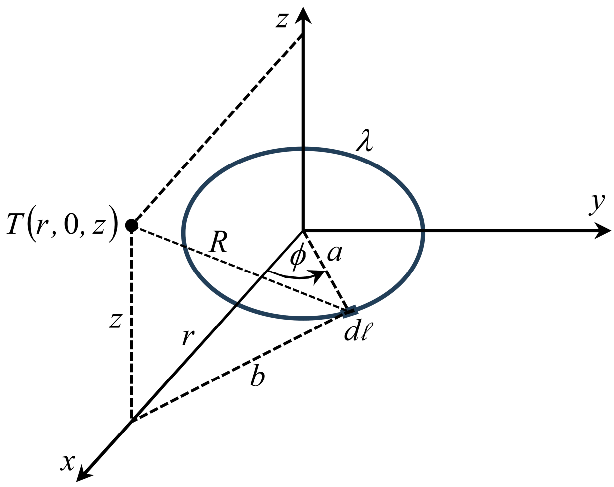


Figure 14.83. 3D representation of a uniformly charged circular thin-wire loop

The distribution of the electric scalar potential does not depend on the angle ϕ, as the system is axisymmetric with respect to the z-axis. Therefore, it is sufficient to derive an expression for the electric scalar potential at $\phi = 0$ (Figure 14.83).

An infinitesimal amount of electric charge $\lambda \cdot d\ell$ generates an infinitesimal contribution to the electric scalar potential at the field point $T(r, 0, z)$:

$$d\varphi = \frac{\lambda \cdot d\ell}{4 \cdot \pi \cdot \varepsilon_0 \cdot R} = \frac{\lambda \cdot a \cdot d\phi}{4 \cdot \pi \cdot \varepsilon_0 \cdot R} \tag{14.440}$$

According to Figure 14.83, the following relation holds:

$$R^2 = b^2 + z^2 = r^2 + a^2 - 2 \cdot a \cdot r \cdot \cos\phi + z^2 \tag{14.441}$$

from which it follows that:

$$\varphi = \frac{\lambda \cdot a}{2 \cdot \pi \cdot \varepsilon_0} \cdot \int_0^{\pi} \frac{d\phi}{\sqrt{r^2 + a^2 + z^2 - 2 \cdot a \cdot r \cdot \cos\phi}} \tag{14.442}$$

The integral in expression (14.442) does not have an analytical solution and can be reduced to the evaluation of the so-called elliptic integrals:

$$F(\phi_0, k) = \int_0^{\phi_0} \frac{d\phi}{\sqrt{1 - k^2 \cdot \sin^2\phi}} \tag{14.443}$$

$$E(\phi_0, k) = \int_0^{\phi_0} \sqrt{1 - k^2 \cdot \sin^2\phi} \cdot d\phi \tag{14.444}$$

Expression (14.443) corresponds to an elliptic integral of the first kind, whereas expression (14.444) represents an elliptic integral of the second kind.

If the limits of elliptic integrals range from 0 to $\pi/2$, such elliptic integrals are referred to as complete elliptic integrals:

$$F(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - k^2 \cdot \sin^2\phi}} \tag{14.445}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1 - k^2 \cdot \sin^2\phi} \cdot d\phi \tag{14.446}$$

Expression (14.445) represents a complete elliptic integral of the first kind, whereas expression (14.446) corresponds to a complete elliptic integral of the second kind.

In order to reduce the evaluation of the integral in expression (14.442) to a complete elliptic integral of the first kind, the following substitution is introduced:

$$\phi = \pi - 2 \cdot \beta \quad ; \quad d\phi = -2 \cdot d\beta \tag{14.447}$$

By applying the substitution given in expression (14.447) to expression (14.442), we obtain the following expression:

$$\varphi = -\frac{\lambda \cdot a}{\pi \cdot \varepsilon_0} \cdot \int_{\pi/2}^{0} \frac{d\beta}{\sqrt{r^2 + a^2 + z^2 - 2 \cdot a \cdot r \cdot \cos(\pi - 2 \cdot \beta)}} \tag{14.448}$$

The following relation holds:

$$\cos(\pi - 2 \cdot \beta) = -\cos(2 \cdot \beta) = -1 + 2 \cdot \sin^2\beta \tag{14.449}$$

and expression (14.448) takes the following form:

$$\varphi = \frac{\lambda \cdot a}{\pi \cdot \varepsilon_0} \cdot \frac{1}{\sqrt{(r+a)^2 + z^2}} \cdot \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - \frac{4 \cdot a \cdot r}{(a+r)^2 + z^2} \cdot \sin^2\beta}} \tag{14.450}$$

With the substitution:

$$k^2 = \frac{4 \cdot a \cdot r}{(a+r)^2 + z^2} \tag{14.451}$$

expression (14.450) takes the following form:

$$\varphi = \frac{\lambda \cdot a}{\pi \cdot \varepsilon_0} \cdot \frac{k}{\sqrt{4 \cdot a \cdot r}} \cdot \int_0^{\pi/2} \frac{d\beta}{\sqrt{1 - k^2 \cdot \sin^2\beta}} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \sqrt{\frac{a}{r}} \cdot k \cdot F(\mathrm{k}) \tag{14.452}$$

The values of the complete elliptic integral of the first kind, $F(k)$, can be found in mathematical tables for the corresponding parameter k, which depends on the field point T. Alternatively, the complete elliptic integral in expression (14.445) can be evaluated numerically. It is also possible to calculate the original integral in expression (14.442) numerically.

Example 14.11. Consider an isolated thin-wire cylindrical conductor of radius r_0 and length ℓ, carrying a total electric charge Q. Assume that the charge is uniformly distributed over the surface (mantle) of the conductor. Using the thin-wire approximation and assuming the conductor is placed in unbounded air, determine expressions for the electric scalar potential and the electric capacitance of the conductor using the following two methods: a) the point collocation method (PCM), where the collocation point is located at the central point of the conductor mantle, b) the average potential method (APM).

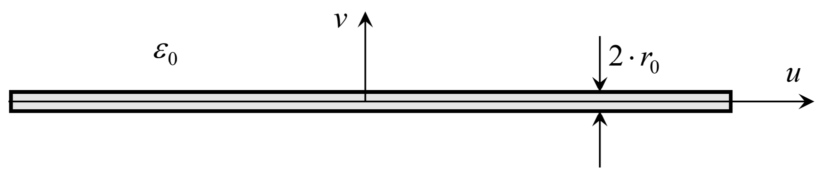


Figure 14.84. Thin-wire cylindrical conductor in the local coordinate system (u, v)

Solution:

According to expression (14.296), the electric scalar potential distribution is given by:

$$\varphi = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \ell} \cdot \ln \frac{\sqrt{v^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{v^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{14.453}$$

whereas, according to expression (14.297), the potential distribution along the conductor is given by:

$$\varphi = \varphi(u, r_0) = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \ell} \cdot \ln \frac{\sqrt{r_0^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{r_0^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{14.454}$$

a) Solution using the point collocation method

The problem states that the collocation point is located at the midpoint of the surface (mantle) of the thin-wire conductor (Figure 14.85), i.e., the collocation point is $T(0, r_0)$.

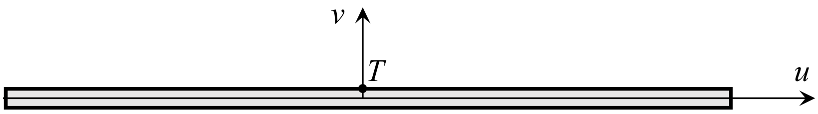


Figure 14.85. Selected collocation point $T(0, r_0)$

According to the point collocation method, the electric scalar potential of the conductor is approximated by the potential of the collocation point:

$$\Phi = \Phi_{\mathrm{PCM}} = \varphi(0, r_0) = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot \ell} \cdot \ln \frac{\sqrt{r_0^2 + \left(\frac{\ell}{2}\right)^2} + \frac{\ell}{2}}{\sqrt{r_0^2 + \left(\frac{\ell}{2}\right)^2} - \frac{\ell}{2}} \tag{14.455}$$

For numerical reasons, it is useful to reformulate expression (14.455) by rationalizing the denominator. After rationalization, it follows that:

$$\Phi_{\mathrm{PCM}} = \varphi(0, r_0) = \frac{Q}{2 \cdot \pi \cdot \varepsilon_0 \cdot \ell} \cdot \ln \frac{\sqrt{r_0^2 + \left(\frac{\ell}{2}\right)^2} + \frac{\ell}{2}}{r_0} \tag{14.456}$$

The electric capacitance of the conductor, according to the point collocation method, is given by the following expression:

$$C_{\mathrm{PCM}} = \frac{Q}{\Phi_{\mathrm{PCM}}} = \frac{2 \cdot \pi \cdot \varepsilon_0 \cdot \ell}{\ln \frac{\sqrt{r_0^2 + \left(\frac{\ell}{2}\right)^2} + \frac{\ell}{2}}{r_0}} \tag{14.457}$$

Since $r_0 << \ell,$ an additional approximation can be introduced, so that:

$$\Phi_{\mathrm{PCM}} \approx \frac{Q}{2\cdot\pi\cdot\varepsilon_0\cdot\ell}\cdot\ln\frac{\ell}{r_0} \tag{14.458}$$

$$C_{\mathrm{PCM}} \approx \frac{2\cdot\pi\cdot\varepsilon_0\cdot\ell}{\ln\dfrac{\ell}{r_0}} \tag{14.459}$$

The physically constant electric scalar potential of the conductor is approximated by the electric scalar potential at the collocation point (Figure 14.86). The solution strongly depends on the choice of the collocation point. Therefore, from a numerical perspective, the point collocation method is relatively poor.

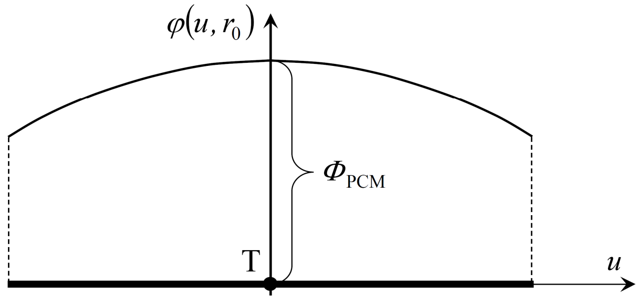


Figure 14.86. Approximation of the electric scalar potential of the conductor using the PCM

b) Solution using the average potential method

According to expressions (14.299) and (14.300), the electric scalar potential of the conductor, approximated using the average potential method, is given by the following expression:

$$\Phi = \Phi_{\mathrm{av}} = \Phi_{\mathrm{APM}} = G(\ell, r_0)\cdot Q \tag{14.460}$$

where:

$$G(\ell, r_0) = \frac{1}{2\cdot\pi\cdot\varepsilon_0\cdot\ell^2}\cdot\left[\ell\cdot\ln\frac{\sqrt{\ell^2+r_0^2}+\ell}{r_0} - \frac{\ell^2}{\sqrt{\ell^2+r_0^2}+r_0}\right] \tag{14.461}$$

The electric capacitance of the conductor, as determined by the average potential method, is given by:

$$C_{\mathrm{APM}} = \frac{Q}{\Phi_{\mathrm{APM}}} = \frac{1}{G(\ell, r_0)} = \frac{2\cdot\pi\cdot\varepsilon_0\cdot\ell^2}{\ell\cdot\ln\dfrac{\sqrt{\ell^2+r_0^2}+\ell}{r_0} - \dfrac{\ell^2}{\sqrt{\ell^2+r_0^2}+r_0}} \tag{14.462}$$

Since $r_0 << \ell,$ an additional approximation can be introduced, so that:

$$\Phi_{\mathrm{APM}} \approx \frac{Q}{2\cdot\pi\cdot\varepsilon_0\cdot\ell}\cdot\left[\ln\frac{2\cdot\ell}{r_0} - 1\right] \tag{14.463}$$

$$C_{\mathrm{APM}} \approx \frac{2\cdot\pi\cdot\varepsilon_0\cdot\ell}{\ln\dfrac{2\cdot\ell}{r_0} - 1} \tag{14.464}$$

According to the average potential method, the physically constant electric scalar potential of the conductor is approximated by the average value of the electric scalar potential along the conductor (Figure 14.87). From a numerical standpoint, this method is significantly more accurate than the point collocation method.

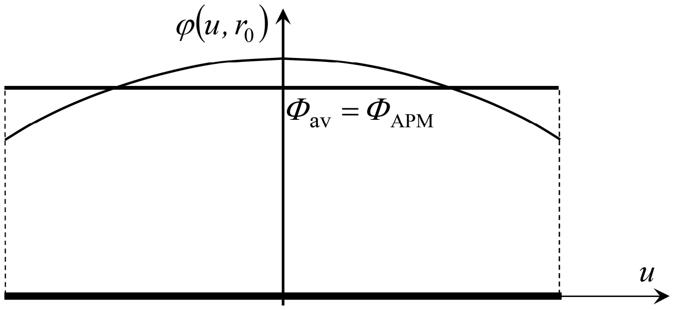


Figure 14.87. Approximation of the electric scalar potential of the conductor using the APM

In reality, the conductor is an equipotential body, whereas the linear charge density varies along its axis. A satisfactory approximate solution can be obtained numerically by dividing the conductor into several thin-wire segments, each with a constant λ over its length. The unknowns are Q_i; $i = 1, 2, ...n$ and the electric scalar potential Φ of the conductor, which are determined by solving a system of linear equations. The sum of all Q_i corresponds to the total electric charge Q.

Example 14.12. A point charge q is located in an LIH half-space with relative permittivity $\varepsilon_{r1} = 3$ (LIH medium 1), at a distance d = 0.5 m from the boundary plane between a perfect LIH dielectric and air (LIH medium 2). Calculate the angles that the electric field line at point A (where h = 0.4 m) makes with the normal to the boundary plane.

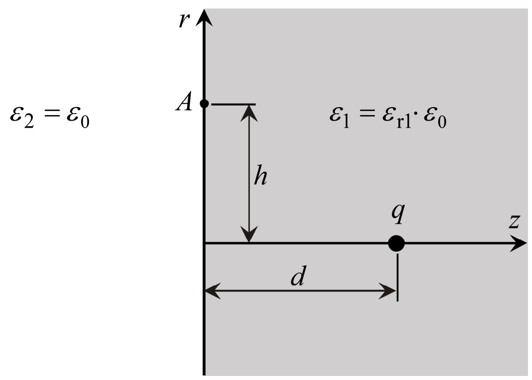


Figure 14.88. Point charge in a half-space filled with a perfect LIH dielectric

Solution:

The problem is most easily solved using the method of images. Since the electric field distribution is axisymmetric, a cylindrical coordinate system will be used (Figure 14.88).

The electric field vector at point A in medium 1 is calculated according to Figure 14.89, whereas the electric field vector at point A in medium 2 is calculated according to Figure 14.90.

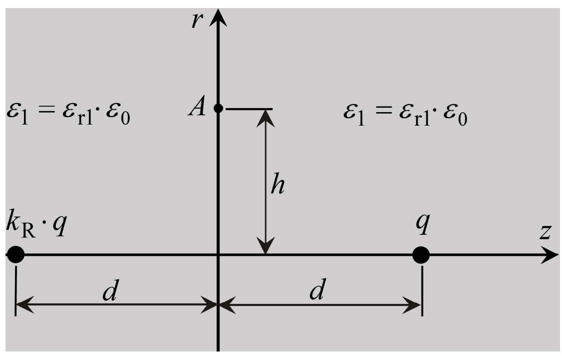


Figure 14.89. Calculation of the electric field intensity in medium 1

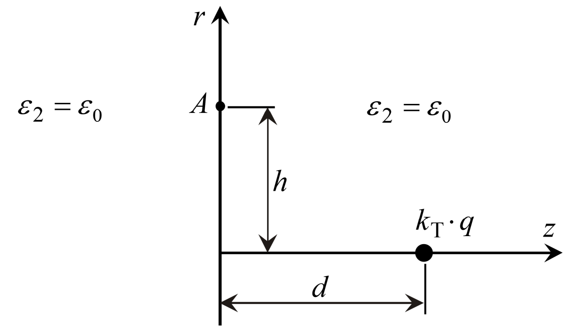


Figure 14.90. Calculation of the electric field intensity in medium 2

According to Figure 14.89, it is easy to conclude that the electric field intensity vector at point A in medium 1 is given by the following expression:

$$\vec{E}_1 = E_{r1} \cdot \vec{e}_r + E_{z1} \cdot \vec{e}_z = \frac{q}{4 \cdot \pi \cdot \varepsilon_1} \cdot \left[\frac{h \cdot (1 + k_R) \cdot \vec{e}_r - d \cdot (1 - k_R) \cdot \vec{e}_z}{\sqrt{(d^2 + h^2)^3}} \right] \tag{14.465}$$

whereas, according to Figure 14.90, the electric field intensity vector at point A in medium 2 is given by the following expression:

$$\vec{E}_2 = E_{r2} \cdot \vec{e}_r + E_{z2} \cdot \vec{e}_z = \frac{q \cdot k_T}{4 \cdot \pi \cdot \varepsilon_2} \cdot \left[\frac{h \cdot \vec{e}_r - d \cdot \vec{e}_z}{\sqrt{(d^2 + h^2)^3}} \right] \tag{14.466}$$

because in an unbounded perfect LIH dielectric with permittivity ε, the electric field intensity vector at a point located at a distance r from an isolated point charge q is given by the following expression:

$$\vec{E} = \frac{q}{4 \cdot \pi \cdot \varepsilon} \cdot \frac{\vec{r}}{r^3} \tag{14.467}$$

Thus, expressions (14.465) and (14.466) are derived based on Figures 14.89 and 14.90, as well as expression (14.467), whereas the superposition principle was also used in the expression (14.465).
According to expression (14.129), the reflection factor is:

$$k_R = \frac{\varepsilon_1 - \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = \frac{3 - 1}{3 + 1} = \frac{1}{2} \tag{14.468}$$

whereas, according to expression (14.130), the transmission factor is:

$$k_T = \frac{2 \cdot \varepsilon_2}{\varepsilon_1 + \varepsilon_2} = \frac{2}{1 + 3} = \frac{1}{2} \tag{14.469}$$

and after substituting these factors into expressions (14.465) and (14.466), the resulting expressions are:

$$\vec{E}_1 = E_{r1} \cdot \vec{e}_r + E_{z1} \cdot \vec{e}_z = \frac{q}{8 \cdot \pi \cdot \varepsilon_1} \cdot \left[\frac{3 \cdot h \cdot \vec{e}_r - d \cdot \vec{e}_z}{\sqrt{(d^2 + h^2)^3}} \right] \tag{14.470}$$

$$\vec{E}_2 = E_{r2} \cdot \vec{e}_r + E_{z2} \cdot \vec{e}_z = \frac{q}{8 \cdot \pi \cdot \varepsilon_2} \cdot \left[\frac{h \cdot \vec{e}_r - d \cdot \vec{e}_z}{\sqrt{(d^2 + h^2)^3}} \right] \tag{14.471}$$

According to Figure 14.91, the electric field intensity vector at point A, in each medium, has two components, and the field line (i.e., electric field vector) forms an angle α with the normal to the boundary plane, which is given by the following expressions:

$$\tan \alpha = \left| E_r / E_z \right| \tag{14.472}$$

$$\tan\alpha_1 = \left| E_{r1} / E_{z1} \right| = \frac{3 \cdot h}{d} = 2.4 \quad ; \quad \tan\alpha_2 = \left| E_{r2} / E_{z2} \right| = \frac{h}{d} = 0.8 \tag{14.473}$$

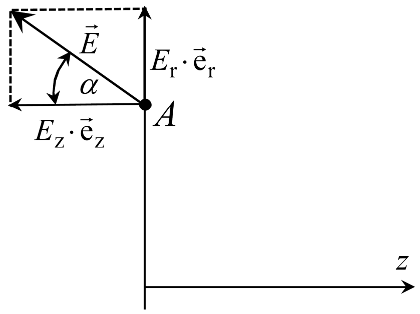


Figure 14.91. Components of the electric field intensity vector at point A

It follows from expression (14.473) that:

$$\alpha_1 = 67.380135^{\circ} \quad ; \quad \alpha_2 = 38.659808^{\circ} \tag{14.474}$$

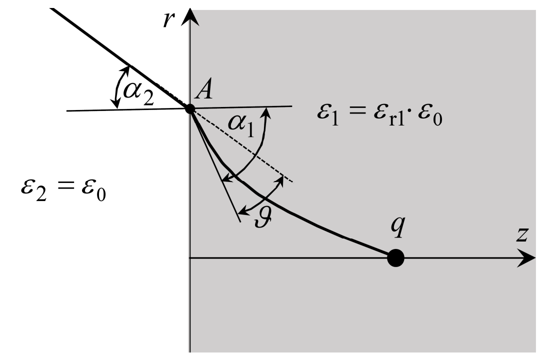


Figure 14.92. Angle at which the field line refracts at point A

The field line passing through point A tends to remain in the medium with higher permittivity (medium 1), and therefore takes the shape shown in Figure 14.92. From Figure 14.92 and expression (14.474), it follows that the field line refracts at point *A* at an angle given by:

$$\vartheta = \alpha_1 - \alpha_2 = 28.7203278^{\circ} \tag{14.475}$$

Example 14.13. The insulator of a single-core high-voltage 10 kV cable consists of three hollow cylindrical layers with relative permittivities $\varepsilon_{r1} = 4$, $\varepsilon_{r2} = 3$, and $\varepsilon_{r3} = 2$. Their outer radii are $r_1 = 0.02$ m, $r_2 = 0.03$ m, and $r_3 = 0.04$ m, whereas the radius of the conductor is $r_0 = 0.01$ m. Determine the distribution of the electric field intensity along the axis r in the cylindrical coordinate system. Assume that the cable is buried in the soil and that the conductor is at an electric scalar potential $\varphi_c = 10$ kV.

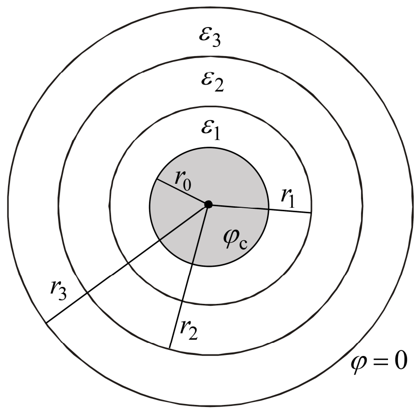


Figure 14.93. Conductor and its three-layer insulator

Solution:

The three-layer insulator is, in fact, the dielectric of a three-layer cylindrical capacitor. Neglecting edge effects – i.e., assuming the cylindrical capacitor is infinitely long – the distribution of the electric field intensity in the i-th layer of the multilayer dielectric is described by the following expression:

$$E_{\mathrm{i}} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_{\mathrm{i}} \cdot r} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_{\mathrm{ri}} \cdot r} \quad ; \quad \mathrm{i} = 1, 2, 3 \tag{14.476}$$

where λ is the linear charge density located on the surface of the conductor.
Furthermore, the voltage of the conductor with respect to the surrounding soil is given by:

$$U = \sum_{\mathrm{i}=1}^{3} \int_{r_{\mathrm{i}-1}}^{r_{\mathrm{i}}} E_{\mathrm{i}} \cdot dr = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \sum_{\mathrm{i}=1}^{3} \frac{1}{\varepsilon_{\mathrm{ri}}} \cdot \int_{r_{\mathrm{i}-1}}^{r_{\mathrm{i}}} \frac{dr}{r} \quad ; \quad U = \varphi_{\mathrm{c}} \tag{14.477}$$

from which, after integration, the following expression for the conductor voltage is obtained:

$$U = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \left(\frac{1}{\varepsilon_{\mathrm{r1}}} \cdot \ln \frac{r_1}{r_0} + \frac{1}{\varepsilon_{\mathrm{r2}}} \cdot \ln \frac{r_2}{r_1} + \frac{1}{\varepsilon_{\mathrm{r3}}} \cdot \ln \frac{r_3}{r_2} \right) = \frac{\lambda}{C} = \frac{\lambda \cdot K}{2 \cdot \pi \cdot \varepsilon_0} \tag{14.478}$$

where:

$$C = \frac{2 \cdot \pi \cdot \varepsilon_0}{\frac{1}{\varepsilon_{\mathrm{r1}}} \cdot \ln \frac{r_1}{r_0} + \frac{1}{\varepsilon_{\mathrm{r2}}} \cdot \ln \frac{r_2}{r_1} + \frac{1}{\varepsilon_{\mathrm{r3}}} \cdot \ln \frac{r_3}{r_2}} = \frac{2 \cdot \pi \cdot \varepsilon_0}{K} \tag{14.479}$$

is the per-unit-length capacitance of the single-core cable, whereas an auxiliary parameter *K* is:

$$K = \frac{1}{\varepsilon_{\mathrm{r1}}} \cdot \ln \frac{r_1}{r_0} + \frac{1}{\varepsilon_{\mathrm{r2}}} \cdot \ln \frac{r_2}{r_1} + \frac{1}{\varepsilon_{\mathrm{r3}}} \cdot \ln \frac{r_3}{r_2} \tag{14.480}$$

From expression (14.478), it follows that the linear charge density can be expressed as:

$$\lambda = \frac{2 \cdot \pi \cdot \varepsilon_0 \cdot U}{K} \tag{14.481}$$

and from expression (14.476), it follows that the distribution of the electric field intensity in the i-th layer of the multilayer insulator (dielectric) is described by the expression:

$$E_{\mathrm{i}} = \frac{U}{\varepsilon_{\mathrm{ri}} \cdot r \cdot K} \quad ; \quad \mathrm{i} = 1, 2, 3 \tag{14.482}$$

If the given data are substituted, it follows that the auxiliary parameter is:

$$K = \frac{1}{4} \cdot \ln \frac{2}{1} + \frac{1}{3} \cdot \ln \frac{3}{2} + \frac{1}{2} \cdot \ln \frac{4}{3} = 0.45228287 \tag{14.483}$$

whereas:

$$E_{\mathrm{i}} = \frac{2.2210057 \times 10^4}{\varepsilon_{\mathrm{ri}} \cdot r} \quad ; \quad \mathrm{i} = 1, 2, 3 \tag{14.484}$$

The electric field intensity at the boundary points of the insulator is:

$$E\big|_{r=r_0} = E_1\big|_{r=r_0} = 5.5275143 \times 10^5 \ \text{V/m} \tag{14.485}$$

$$E\big|_{r=r_3} = E_3\big|_{r=r_3} = 2.7637571 \times 10^5 \ \text{V/m} \tag{14.486}$$

If the entire perfect dielectric were homogeneous, that is, if all layers had the same permittivity, then the following expression would hold:

$$E = \frac{U}{r \cdot \ln \frac{r_3}{r_0}} \tag{14.487}$$

and in that case, the electric field intensity at the boundary points of the LIH perfect dielectric would be:

$$E\Big|_{r=r_0} = 7.2134752 \times 10^5 \text{ V/m} \tag{14.488}$$

$$E\Big|_{r=r_3} = 1.8033688 \times 10^5 \text{ V/m} \tag{14.489}$$

This means that, in this case, the electric field intensity at the conductor surface is higher, indicating a greater probability of dielectric breakdown.

Example 14.14. Using the concept of electric stress, calculate the force with which the two halves of an isolated solid conducting sphere of radius R = 0.1 m, carrying a total electric charge Q = 1μC, repel each other. Assume the sphere is in the air. Solve the problem by integrating over the flat dividing plane that separates the two hemispheres.

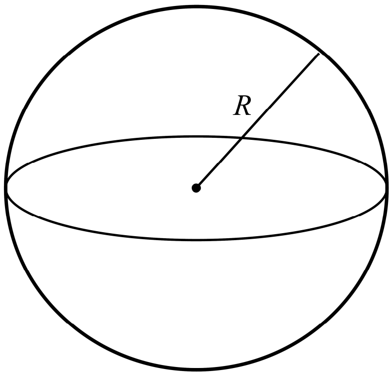


Figure 14.94. Two arbitrarily chosen hemispheres of a solid conducting sphere

Solution:

The electric charge Q is uniformly distributed over the surface of the solid sphere; the electric field intensity inside the sphere is zero, whereas the electric field intensity outside the sphere is given by the expression:

$$E = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot r^2} \tag{14.490}$$

where r denotes the distance from the center of the sphere to the field point.

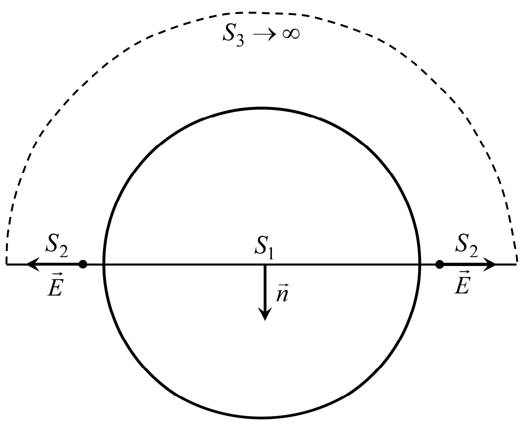


Figure 14.95. Integration surface S enclosing one hemisphere and closed at infinity

The integration surface $S = S_1 \cup S_2 \cup S_3$ around the hemisphere can be closed so that it coincides with the boundary surface of the hemisphere under consideration. However, for the purpose of force calculation, it is simpler to close the surface around the hemisphere by dividing the solid sphere with a plane, whereas the remaining part of the surface *S* is closed at infinity (Figure 14.95), where the electric field vanishes.

On surfaces S_1 and S_3, the electric field intensity is zero, and therefore the electric stress vector is also zero on those surfaces.
It follows that the force on the 'upper' hemisphere is given by the expression:

$$\vec{F} = \oint_S \vec{t}_e \cdot dS = \int_{S_2} \vec{t}_e \cdot dS = \vec{n} \cdot \int_{S_2} t_e \cdot dS \tag{14.491}$$

where $\vec{n}$ is the unit normal vector of the surface (Figure 14.95).
According to expression (14.115), the electric stress vector on the surface S_2 is given by the expression:

$$\vec{t}_e = \varepsilon_0 \cdot \left[\vec{E} \cdot \left(\vec{E} \cdot \vec{n} \right) - \frac{E^2}{2} \cdot \vec{n} \right] = -\frac{\varepsilon_0}{2} \cdot E^2 \cdot \vec{n} = -\frac{Q^2}{32 \cdot \pi^2 \cdot \varepsilon_0 \cdot r^4} \cdot \vec{n} \tag{14.492}$$

and thus the force on the 'upper' hemisphere is:

$$\vec{F} = \left(-\vec{n} \right) \cdot \frac{Q^2}{32 \cdot \pi^2 \cdot \varepsilon_0} \cdot \int_R^\infty \frac{2 \cdot \pi \cdot r \cdot dr}{r^4} = \vec{n} \cdot \frac{Q^2}{16 \cdot \pi \cdot \varepsilon_0} \cdot \frac{1}{2 \cdot r^2} \Bigg|_R^\infty \tag{14.493}$$

from which it follows that:

$$\vec{F} = \left(-\vec{n} \right) \cdot \frac{Q^2}{32 \cdot \pi \cdot \varepsilon_0 \cdot R^2} = \left(-\vec{n} \right) \cdot 0.1123444 \text{ N} \tag{14.494}$$

Example 14.15. Using the concept of electric stress, calculate the force with which the two halves of an isolated solid conducting sphere of radius R = 0.1 m, carrying a total electric charge *Q* = 1μC, repel each other. Assume the sphere is in the air. Solve the problem by integrating over the surface of one hemisphere. Two arbitrarily chosen conducting hemispheres are shown in Figure 14.94.

Solution:

In this problem, unlike the integration surface in the previous one, the integration surface $S = S_1 \cup S_2$ is the outer surface of the considered hemisphere on which the repulsive force acts (Figure 14.96).

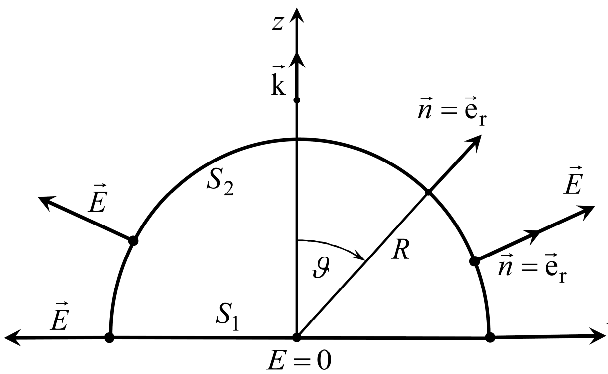


Figure 14.96. Integration surface around the conducting hemisphere

On surface S_1, the electric field intensity is zero, and therefore the electric stress vector is also zero on that surface. It follows that the force on the 'upper' hemisphere is described by the expression:

$$\vec{F} = \oint_S \vec{t}_e \cdot dS = \int_{S_2} \vec{t}_e \cdot dS = \int_{S_2} t_e \cdot \vec{e}_r \cdot dS \tag{14.495}$$

where $\vec{e}_r$ is the unit normal vector of the surface S_2 (Figure 14.96).

The electric charge Q is uniformly distributed over the surface of the solid sphere. The electric field intensity on the surface of the solid sphere (in the air) is given by the expression:

$$E = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot R^2} \tag{14.496}$$

where R is the radius of the sphere.

According to expression (14.115), in this case for $r = R$, i.e., on surface S_2, the electric stress vector has a constant magnitude and is given by the expression:

$$\vec{t}_e = \varepsilon_0 \cdot \left[\vec{E} \cdot \left(\vec{E} \cdot \vec{n} \right) - \frac{E^2}{2} \cdot \vec{n} \right] = \frac{\varepsilon_0}{2} \cdot E^2 \cdot \vec{n} = \frac{Q^2}{32 \cdot \pi^2 \cdot \varepsilon_0 \cdot R^4} \cdot \vec{e}_r \tag{14.497}$$

whereas the electric stress vector on surface S_1 is equal to zero.

The force on the 'upper' hemisphere acts in the direction of the z-axis, that is, in the direction of the unit vector $\vec{k}$. This means that it is sufficient to integrate only the z-component of the electric stress vector:

$$\vec{F} = \vec{k} \cdot \int_{S_2} t_{ez} \cdot dS = \vec{k} \cdot t_e \cdot \int_{S_2} \cos\vartheta \cdot dS \tag{14.498}$$

It follows that:

$$\vec{F} = \vec{k} \cdot \frac{Q^2}{32 \cdot \pi^2 \cdot \varepsilon_0 \cdot R^4} \cdot \int_0^{2\cdot\pi} d\varphi \cdot \int_0^{\pi/2} \cos\vartheta \cdot R^2 \cdot \sin\vartheta \cdot d\vartheta \tag{14.499}$$

and therefore:

$$\vec{F} = \vec{k} \cdot \frac{Q^2}{16 \cdot \pi \cdot \varepsilon_0 \cdot R^2} \cdot \int_0^{\pi/2} \sin\vartheta \cdot \cos\vartheta \cdot d\vartheta = \vec{k} \cdot \frac{Q^2}{16 \cdot \pi \cdot \varepsilon_0 \cdot R^2} \cdot \left. \frac{\sin^2\vartheta}{2} \right|_0^{\pi/2} \tag{14.500}$$

$$\vec{F} = \frac{Q^2}{32 \cdot \pi \cdot \varepsilon_0 \cdot R^2} \cdot \vec{k} = 0.1123444 \cdot \vec{k} \text{ N} \tag{14.501}$$

Example 14.16. The space between two concentric spheres is filled with a dielectric ($\varepsilon_r = 3$) within which a positive electric charge is located. The volume electric charge density is spherically symmetric and described by the expression $\rho = 10^{-5} \cdot r^2$ C/m^3. Assume that there is no electric charge in the remaining space (air). Using the concept of electric stress, determine the force with which the two halves of the dielectric repel each other. Let $r_1 = 0.05$ m, $r_2 = 0.1$ m.

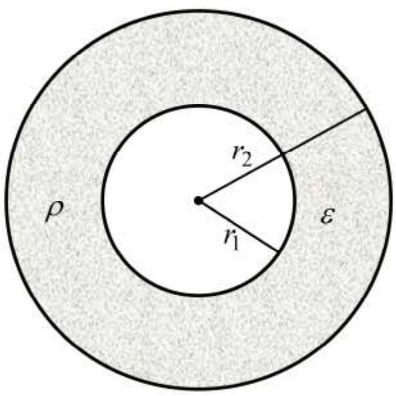


Figure 14.97. Electric charge distribution between two concentric spherical surfaces

Solution:

This example is very similar to the previous one and is therefore not solved in detail.
In the case of $r_1 \le r \le r_2$, it holds that:

$$\oint_S \vec{D}_1 \cdot d\vec{S} = \varepsilon_0 \cdot \varepsilon_r \cdot E_1 \cdot 4 \cdot \pi \cdot r^2 = \int_{r_1}^{r} \rho \cdot 4 \cdot r^2 \cdot \pi \cdot dr \tag{14.502}$$

from which, after integration, it follows that:

$$E_1 = 2 \times 10^{-6} \cdot \frac{1}{\varepsilon_0 \cdot \varepsilon_r} \cdot \left(r^3 - \frac{r_1^5}{r^2} \right) \tag{14.503}$$

and thus, on the plane that divides the two halves of the dielectric (similar to Figure 14.95):

$$\vec{t}_{\mathrm{e}1} = -\frac{1}{2} \cdot \varepsilon_0 \cdot \varepsilon_r \cdot E_1^2 \cdot \vec{n} = t_{\mathrm{e}1} \cdot \vec{n} \tag{14.504}$$

$$\vec{F}_1 = \vec{n} \cdot \int_{S_1} t_{\mathrm{e}1} \cdot dS = \vec{n} \cdot \int_{r_1}^{r_2} t_{\mathrm{e}1} \cdot 2 \cdot \pi \cdot r \cdot dr \tag{14.505}$$

and after integration, it is obtained that:

$$\vec{F}_1 = \frac{4 \cdot \pi \times 10^{-12}}{\varepsilon_0 \cdot \varepsilon_\mathrm{r}} \cdot \left(\frac{r_2^8 - r_1^8}{8} - \frac{2 \cdot r_1^5 \cdot \left(r_2^3 - r_1^3\right)}{3} + \frac{r_1^{10} \cdot \left(r_2^2 - r_1^2\right)}{2 \cdot r_1^2 \cdot r_2^2} \right) \cdot \left(-\vec{n}\right) \tag{14.506}$$

In the case of $r > r_2$, it holds that:

$$\oint_S \vec{D}_2 \cdot d\vec{S} = \varepsilon_0 \cdot E_2 \cdot 4 \cdot \pi \cdot r^2 = \int_{r_1}^{r_2} \rho \cdot 4 \cdot r^2 \cdot \pi \cdot dr \tag{14.507}$$

and integrating yields:

$$E_2 = 2 \times 10^{-6} \cdot \frac{1}{\varepsilon_0} \cdot \frac{r_2^5 - r_1^5}{r^2} \tag{14.508}$$

On the plane dividing the two halves of the dielectric (as in Figure 14.95), it holds that:

$$\vec{t}_{\mathrm{e}2} = -\frac{1}{2} \cdot \varepsilon_0 \cdot E_2^2 \cdot \vec{n} = t_{\mathrm{e}2} \cdot \vec{n} \tag{14.509}$$

$$\vec{F}_2 = \vec{n} \cdot \int_{S_2} t_{\mathrm{e}2} \cdot dS = \vec{n} \cdot \int_{r_2}^{\infty} t_{\mathrm{e}2} \cdot 2 \cdot \pi \cdot r \cdot dr \tag{14.510}$$

and integrating yields:

$$\vec{F}_2 = \frac{4 \cdot \pi \times 10^{-12}}{\varepsilon_0} \cdot \frac{\left(r_2^5 - r_1^5\right)^2}{2 \cdot r_2^2} \cdot \left(-\vec{n}\right) \tag{14.511}$$

The force with which the two halves of the dielectric repel each other is:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 \tag{14.512}$$

and the final result is:

$$\vec{F} = \left(-\vec{n}\right) \cdot 7.1696 \times 10^{-9}\ \mathrm{N} \tag{14.513}$$

Example 14.17. A parallel-plate capacitor has a plate separation of d = 0.05 m, an individual plate area of S = 0.04 m², and is charged with a charge of Q = 1.5 μC. The dielectric between the plates has a relative permittivity of 90. Neglecting edge effects, calculate the force acting on the positive plate of the capacitor using the electric field stress.

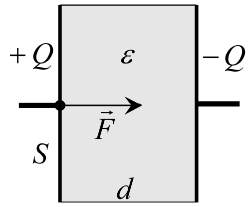


Figure 14.98. Force on the positive plate of a parallel-plate capacitor

Solution:

An integration surface should be closed around the positive plate of the parallel-plate capacitor, as shown in Figure 14.99.

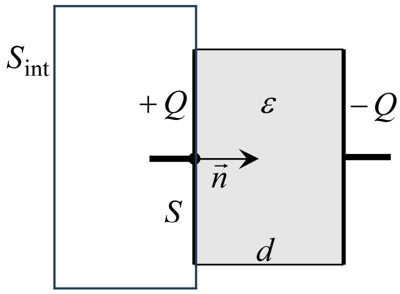


Figure 14.99. Integration surface

The electric field intensity is nonzero only within the dielectric of the parallel-plate capacitor and is given by the following expression:

$$\vec{E} = \frac{Q}{C \cdot d} \cdot \vec{n} = \frac{Q}{\varepsilon \cdot S} \cdot \vec{n} \tag{14.514}$$

where $\vec{n}$ is the unit normal vector of the positive electrode of the capacitor (Figure 14.99), and S is the surface area of the capacitor plate.

According to expression (14.115), the electric stress vector on the positive plate of the capacitor (within the dielectric) is given by the following expression:

$$\vec{t}_{\mathrm{e}} = \varepsilon \cdot \left[\vec{E} \cdot \left(\vec{E} \cdot \vec{n} \right) - \frac{E^2}{2} \cdot \vec{n} \right] = \frac{\varepsilon}{2} \cdot E^2 \cdot \vec{n} = \frac{Q^2}{2 \cdot \varepsilon \cdot S^2} \cdot \vec{n} \tag{14.515}$$

and the force on the positive electrode is:

$$\vec{F} = \oint_{S_{\mathrm{int}}} \vec{t}_{\mathrm{e}} \cdot dS = t_{\mathrm{e}} \cdot S \cdot \vec{n} = \frac{Q^2}{2 \cdot \varepsilon \cdot S} \cdot \vec{n} = 0.035294033 \cdot \vec{n} \ \mathrm{N} \tag{14.516}$$

Example 14.18. A dielectric slab of thickness d partially enters between the rectangular plates of a parallel-plate capacitor connected to an ideal voltage source, which keeps the capacitor voltage U constant. Let the plate area of the capacitor be S. Determine the force on the dielectric slab as a function of x, using two methods: a) from electrostatic energy, b) using electric stress (force at the boundary between two dielectrics). Neglect edge effects. Given: U = 12 V, S = 0.25 m^2, d = 0.02 m, b = 0.5 m, $\varepsilon_{\mathrm{r}} = 20$.

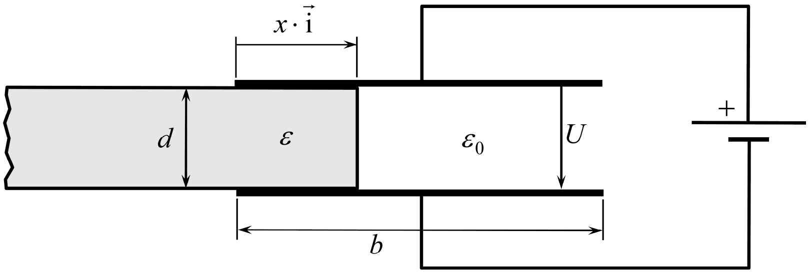


Figure 14.100. Force on the dielectric slab – non-isolated system

Solution:

a) Calculation of force from electrostatic energy

In this particular case, the force in the non-isolated system is given by the following expression:

$$\vec{F} = \vec{\mathrm{i}} \cdot \left.\frac{\partial W_\mathrm{e}}{\partial x}\right|_{U=\text{const.}} = \vec{\mathrm{i}} \cdot \frac{U^2}{2} \cdot \frac{\partial C}{\partial x} \tag{14.517}$$

where W_e is the energy stored in the electrostatic field.

Neglecting edge effects, the capacitance of the capacitor is given by the following expression:

$$C = \varepsilon_0 \cdot \varepsilon_r \cdot \frac{S}{d} \cdot \frac{x}{b} + \varepsilon_0 \cdot \frac{S}{d} \cdot \frac{b-x}{b} \tag{14.518}$$

where S is the surface area of the capacitor plate.

It follows that the force on the dielectric slab is given by the following expression:

$$\vec{F} = \vec{\mathrm{i}} \cdot \frac{U^2}{2} \cdot \frac{\partial C}{\partial x} = \frac{U^2}{2} \cdot \varepsilon_0 \cdot \frac{S}{d \cdot b} \cdot (\varepsilon_\mathrm{r} - 1) \cdot \vec{\mathrm{i}} = 3.0281322 \times 10^{-7} \cdot \vec{\mathrm{i}} \ \ \mathrm{N} \tag{14.519}$$

The force acting on the dielectric slab tends to pull it farther into the space between the plates of the capacitor. If two different dielectrics are present in an electric field, the dielectric with the higher permittivity tends to displace the one with the lower permittivity.

From expression (14.519), it follows that the force on the dielectric slab does not depend on x. This results from the introduced simplifications, which lead to a homogeneous electric field between the capacitor plates in the non-isolated system.

b) Calculation of force using electric stress – force at the boundary between two dielectrics

According to expression (14.118), the pressure at the boundary between the two dielectrics is given by the following expression:

$$t_\mathrm{n}^\mathrm{e} = \frac{\varepsilon_0 \cdot (\varepsilon_\mathrm{r} - 1)}{2} \cdot \left(\frac{D_\mathrm{n}^2}{\varepsilon_\mathrm{r} \cdot \varepsilon_0^2} + E_\mathrm{t}^2 \right) \tag{14.520}$$

where:

$$D_\mathrm{n} = 0 \quad ; \quad E_\mathrm{t} = \frac{U}{d} \tag{14.521}$$

and therefore:

$$t_\mathrm{n}^\mathrm{e} = \frac{\varepsilon_0 \cdot (\varepsilon_\mathrm{r} - 1) \cdot U^2}{2 \cdot d^2} \tag{14.522}$$

The force at the boundary between the two dielectrics is equal to the integral of the pressure over the boundary:

$$\vec{F} = \int_{S_b} t_n^e \cdot \vec{n} \cdot dS = t_n^e \cdot \frac{S \cdot d}{b} \cdot \vec{i} \tag{14.523}$$

and therefore:

$$\vec{F} = \frac{\varepsilon_0 \cdot (\varepsilon_r - 1) \cdot U^2 \cdot S}{2 \cdot d \cdot b} \cdot \vec{i} = 3.0281322 \times 10^{-7} \cdot \vec{i} \ \text{N} \tag{14.524}$$

Example 14.19. A dielectric slab of thickness d partially enters between the rectangular plates of a parallel-plate capacitor connected to an ideal voltage source, which was previously charged with an electric charge Q and then disconnected from the source. Let the plate area of the capacitor be S. Determine the force on the dielectric slab as a function of x, using two methods: a) from electrostatic energy, b) using electric stress (force at the boundary between two dielectrics). Neglect edge effects. Given: $Q = 12$ μC, $S = 0.25$ m^2, $d = 0.02$ m, $b = 0.5$ m, $\varepsilon_r = 20$.

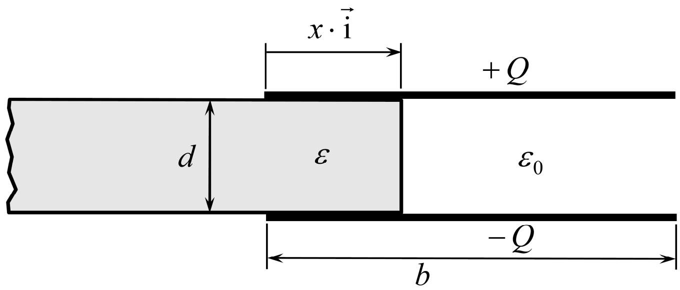


Figure 14.101. Force on the dielectric slab – isolated system

Solution:

a) Calculation of force from electrostatic energy

In this particular case, the force in the isolated system is given by the following expression:

$$\vec{F} = -\vec{i} \cdot \left.\frac{\partial W_e}{\partial x}\right|_{Q=\text{const.}} = -\vec{i} \cdot \frac{Q^2}{2} \cdot \frac{\partial (1/C)}{\partial x} = \vec{i} \cdot \frac{Q^2}{2 \cdot C^2} \cdot \frac{\partial C}{\partial x} \tag{14.525}$$

where W_e is the energy stored in the electrostatic field.

Neglecting edge effects, the capacitance of the capacitor is given by the following expression:

$$C = \varepsilon_0 \cdot \varepsilon_r \cdot \frac{S}{d} \cdot \frac{x}{b} + \varepsilon_0 \cdot \frac{S}{d} \cdot \frac{b-x}{b} = \varepsilon_0 \cdot \frac{S}{d \cdot b} \cdot [(\varepsilon_r - 1) \cdot x + b] \tag{14.526}$$

It follows that the force on the dielectric slab is given by the following expression:

$$\vec{F} = \vec{i} \cdot \frac{Q^2}{2 \cdot C^2} \cdot \frac{\partial C}{\partial x} = \frac{Q^2}{2 \cdot C^2} \cdot \varepsilon_0 \cdot \frac{S}{d \cdot b} \cdot (\varepsilon_r - 1) \cdot \vec{i} \tag{14.527}$$

thus, after substituting expression (14.526) into expression (14.527), we obtain:

$$\vec{F} = \frac{Q^2}{2} \cdot \frac{d \cdot b}{\varepsilon_0 \cdot S} \cdot \frac{\varepsilon_r - 1}{[(\varepsilon_r - 1) \cdot x + b]^2} \cdot \vec{i} = \frac{6.1801264}{(19 \cdot x + 0.5)^2} \cdot \vec{i} \ \text{N} \tag{14.528}$$

The force acting on the dielectric slab tends to pull it deeper between the plates of the capacitor. If two different dielectrics are present in an electric field, the one with the higher permittivity tends to displace the one with the lower permittivity.

From expression (14.528), it follows that the force on the dielectric slab decreases with increasing parameter x, because the capacitance of the capacitor increases, leading to a decrease in the voltage across the capacitor in an isolated system. As the voltage of the capacitor decreases, the intensity of the electric field between the plates also decreases.

b) Calculation of force using electric stress – force at the boundary between two dielectrics

According to expression (14.118), the pressure at the boundary between the two dielectrics is given by the following expression:

$$t_{\mathrm{n}}^{\mathrm{e}} = \frac{\varepsilon_0 \cdot (\varepsilon_{\mathrm{r}} - 1)}{2} \cdot \left(\frac{D_{\mathrm{n}}^2}{\varepsilon_{\mathrm{r}} \cdot \varepsilon_0^2} + E_{\mathrm{t}}^2 \right) \tag{14.529}$$

where:

$$D_{\mathrm{n}} = 0 \quad ; \qquad E_{\mathrm{t}} = \frac{Q}{C \cdot d} \tag{14.530}$$

and therefore:

$$t_{\mathrm{n}}^{\mathrm{e}} = \frac{\varepsilon_0 \cdot (\varepsilon_{\mathrm{r}} - 1) \cdot Q^2}{2 \cdot d^2 \cdot C^2} \tag{14.531}$$

where the capacitance of the capacitor is described by expression (14.526).

The force at the boundary between the two dielectrics is equal to the integral of the pressure over the boundary:

$$\vec{F} = \int_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{e}} \cdot \vec{n} \cdot dS = t_{\mathrm{n}}^{\mathrm{e}} \cdot \frac{S \cdot d}{b} \cdot \vec{\mathrm{i}} \tag{14.532}$$

and therefore:

$$\vec{F} = \frac{\varepsilon_0 \cdot (\varepsilon_{\mathrm{r}} - 1) \cdot Q^2 \cdot S}{2 \cdot d \cdot b \cdot C^2} \cdot \vec{\mathrm{i}} \tag{14.533}$$

After substituting expression (14.526) into expression (14.533), we obtain:

$$\vec{F} = \frac{Q^2}{2} \cdot \frac{d \cdot b}{\varepsilon_0 \cdot S} \cdot \frac{\varepsilon_{\mathrm{r}} - 1}{[(\varepsilon_{\mathrm{r}} - 1) \cdot x + b]^2} \cdot \vec{\mathrm{i}} = \frac{6.1801264}{(19 \cdot \mathrm{x} + 0.5)^2} \cdot \vec{\mathrm{i}} \ \mathrm{N} \tag{14.534}$$

Example 14.20. A thin, isolated, conducting circular plate is located in the air and is charged with an electric charge of Q =100 pC. From the requirement that the surface of the plate is an equipotential surface, it follows that the surface charge density (on both sides of the plate) is described by the expression $\sigma = k \cdot Q / \sqrt{a^2 - r^2}$, where a = 0.1 m is the radius of the plate, and r is the distance of a point on the plate from the center of the plate. Calculate: a) the constant k, b) the distribution of the electric scalar potential along the z-axis perpendicular to the plate, and c) the electric scalar potential of the plate.

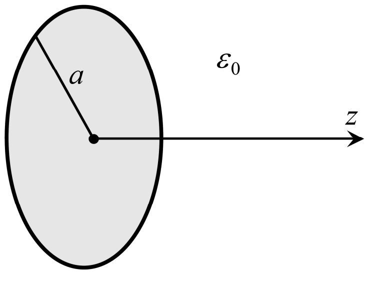


Figure 14.102. Isolated conducting circular plate

Solution:

a) Calculation of the constant k

The electric charge on the plate can be calculated by integrating the total surface charge density σ over one side of the plate.

$$\int_S \sigma \cdot dS = k \cdot Q \cdot \int_0^a \frac{2 \cdot \pi \cdot r \cdot dr}{\sqrt{a^2 - r^2}} = Q \quad ; \quad \sigma = k \cdot \frac{Q}{\sqrt{a^2 - r^2}} \tag{14.535}$$

It holds that:

$$\int_0^a \frac{r \cdot dr}{\sqrt{a^2 - r^2}} = -\sqrt{a^2 - r^2}\Big|_0^a = a \tag{14.536}$$

thus, it follows that:

$$k = \frac{1}{2 \cdot \pi \cdot a} \quad ; \quad \sigma = \frac{Q}{2 \cdot \pi \cdot a \cdot \sqrt{a^2 - r^2}} \tag{14.537}$$

b) Distribution of the electric scalar potential along the z-axis

It holds that:

$$\varphi(z) = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \int_S \frac{\sigma \cdot dS}{R} = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \int_0^a \frac{\sigma \cdot 2 \cdot \pi \cdot r \cdot dr}{\sqrt{r^2 + z^2}} \tag{14.538}$$

thus, it follows that:

$$\varphi(z) = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot a} \cdot \int_0^a \frac{r \cdot dr}{\sqrt{z^2 + r^2} \cdot \sqrt{a^2 - r^2}} \tag{14.539}$$

Furthermore, it holds that:

$$\int_0^a \frac{r \cdot dr}{\sqrt{z^2 + r^2} \cdot \sqrt{a^2 - r^2}} = \begin{pmatrix} a^2 - r^2 = t^2 \\ \\ -r \cdot dr = t \cdot dt \end{pmatrix}$$

$$= \int_0^a \frac{dt}{\sqrt{z^2 + a^2 - t^2}} = \left(\arcsin \frac{t}{\sqrt{a^2 + z^2}} \right)\Bigg|_0^a \tag{14.540}$$

$$= \arcsin \frac{a}{\sqrt{a^2 + z^2}} = \arctan \frac{a}{|z|}$$

thus, it follows that:

$$\varphi(z) = \frac{Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot a} \cdot \arctan \frac{a}{|z|} \tag{14.541}$$

c) The electric scalar potential of the plate

The electric scalar potential of the equipotential plate can be obtained from expression (14.541) for $z = 0$:

$$\Phi_{\mathrm{pl}} = \varphi(0) = \frac{Q}{8 \cdot \varepsilon_0 \cdot a} = 14.1176133 \text{ V} \tag{14.542}$$

Example 14.21. Two mutually parallel and identical thin-wire conductors of length ℓ are placed in the air. Let both conductors be uniformly charged with a positive electric charge of linear charge density λ. Let the distance between the conductors be d. Determine the expression for the repulsive force between the conductors.

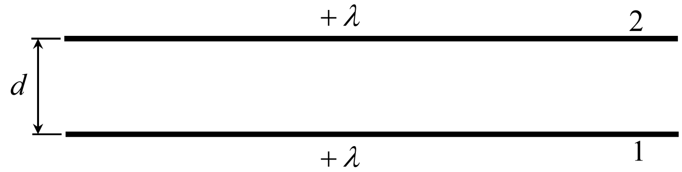


Figure 14.103. Two isolated charged thin-wire conductors

Solution:

Let us calculate the force with which conductor 1 repels conductor 2. Therefore, the local coordinate system (u, v) of conductor 1 is used (Figure 14.104).

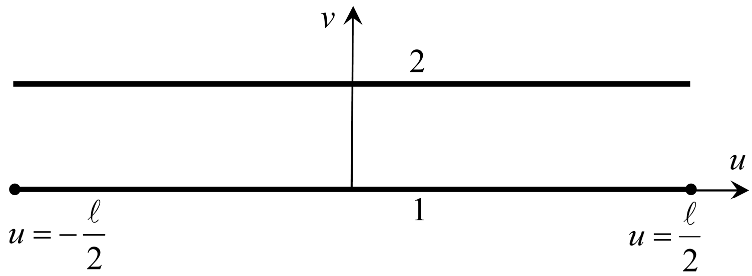


Figure 14.104. Local coordinate system (u, v)

According to expression (14.294), the distribution of the electric scalar potential due to its electric charge in the local coordinate system can be described by the following expression, suitable for differentiation:

$$\varphi = \frac{\lambda}{4 \cdot \pi \cdot \varepsilon_0} \cdot \left(\operatorname{arsinh} \frac{u + \frac{\ell}{2}}{v} - \operatorname{arsinh} \frac{u - \frac{\ell}{2}}{v} \right) \tag{14.543}$$

It then follows that the v-component of the electric field intensity, due to the electric charge of conductor 1, is given by the following expression:

$$E_{\mathrm{v}} = -\frac{\partial \varphi}{\partial v} = \frac{\lambda}{4 \cdot \pi \cdot \varepsilon_0 \cdot v} \cdot \left[\frac{u + \frac{\ell}{2}}{\sqrt{v^2 + \left(u + \frac{\ell}{2} \right)^2}} - \frac{u - \frac{\ell}{2}}{\sqrt{v^2 + \left(u - \frac{\ell}{2} \right)^2}} \right] \tag{14.544}$$

thus, the force on conductor 2 is:

$$\vec{F} = \vec{\mathrm{e}}_{\mathrm{v}} \cdot \int_{-\ell/2}^{\ell/2} E_{\mathrm{v}} \Big|_{v=d} \cdot \lambda \cdot du = \ldots = \frac{\lambda^2}{2 \cdot \pi \cdot \varepsilon_0 \cdot d} \cdot \left(\sqrt{d^2 + \ell^2} - d \right) \cdot \vec{\mathrm{e}}_{\mathrm{v}} \tag{14.545}$$

Example 14.22. A solid conducting sphere of radius $R = 0.1$ m is partially immersed, with one-sixth of its surface, in a medium with a different permittivity. Determine the electric capacitance of the solid sphere. Given: $\varepsilon_{\mathrm{r}1} = 2, \varepsilon_{\mathrm{r}2} = 4, \varepsilon_{\mathrm{r}3} = 6, \varepsilon_{\mathrm{r}4} = 3,\ \ \varepsilon_{\mathrm{r}4} = 3, \varepsilon_{\mathrm{r}5} = 5,\ \varepsilon_{\mathrm{r}6} = 10$.

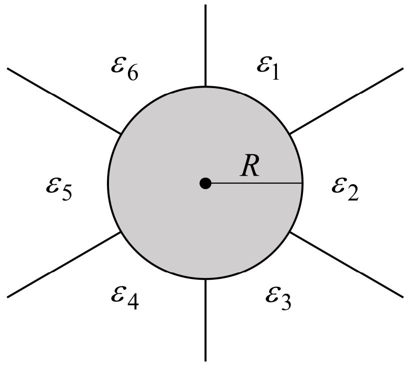


Figure 14.105. Solid conducting sphere surrounded by a heterogeneous dielectric

Solution:

At the boundary surfaces between two adjacent dielectrics, there is continuity of the electric field intensity, which has only a tangential component. It follows that the distribution of the electric scalar potential is as if the system consisted of a single homogeneous perfect LIH dielectric. Hence, the electric scalar potential distribution in the *i*-th dielectric is described by:

$$\varphi_{\mathrm{i}} = \frac{6 \cdot Q_{\mathrm{i}}}{4 \cdot \pi \cdot \varepsilon_{\mathrm{i}} \cdot r} \tag{14.546}$$

where Q_{i} is the electric charge associated with the i-th part of the sphere.

From expression (14.546), it follows that the electric scalar potential of the solid sphere is:

$$\varphi_{\mathrm{sphere}} = \frac{6 \cdot Q_{\mathrm{i}}}{4 \cdot \pi \cdot \varepsilon_{\mathrm{i}} \cdot R} = \frac{Q_{\mathrm{i}}}{C_{\mathrm{i}}} \tag{14.547}$$

where C_{i} is the electric capacitance of the i-th part of the sphere, which is given by the expression:

$$C_{\mathrm{i}} = \frac{4 \cdot \pi \cdot \varepsilon_{\mathrm{i}} \cdot R}{6} \tag{14.548}$$

It follows that the total capacitance of the solid sphere is:

$$C = \sum_{\mathrm{i=1}}^{6} \mathrm{C_i} = \frac{4 \cdot \pi \cdot R}{6} \cdot \sum_{\mathrm{i=1}}^{6} \varepsilon_{\mathrm{i}} \tag{14.549}$$

thus, it follows that:

$$C = 4 \cdot \pi \cdot \varepsilon_0 \cdot \frac{\varepsilon_{\mathrm{r1}} + \ldots + \varepsilon_{\mathrm{r6}}}{6} \cdot R = 55.6325028\,\mathrm{pF} \tag{14.550}$$

From expression (14.549), it follows that the permittivity of the equivalent perfect LIH dielectric is equal to the arithmetic mean of the permittivities of all six dielectrics, since the capacitance of the sphere in a perfect LIH dielectric of permittivity ε is given by the expression:

$$C = 4 \cdot \pi \cdot \varepsilon \cdot R \tag{14.551}$$

The following expressions also hold:

$$\frac{Q_{\mathrm{i}}}{\varepsilon_{\mathrm{i}}} = k \quad ; \quad \sum_{\mathrm{i=1}}^{6} \mathrm{Q_i} = k \cdot \sum_{\mathrm{i=1}}^{6} \varepsilon_{\mathrm{i}} = \mathrm{Q} \tag{14.552}$$

where Q is the total electric charge on the conducting sphere.

From expression (14.552), it follows that:

$$Q_{\mathrm{i}} = \frac{\varepsilon_{\mathrm{i}}}{\varepsilon_1 + \ldots + \varepsilon_6} \cdot Q = \frac{\varepsilon_{\mathrm{ri}}}{\varepsilon_{\mathrm{r1}} + \ldots + \varepsilon_{\mathrm{r6}}} \cdot Q \quad ; \quad \mathrm{i} = 1, 2, ..., 6 \tag{14.553}$$

Example 14.23. Let a point electric charge q be located between two conducting planes at $z = \pm\, h$. Assume that both planes are held at zero electric scalar potential. Determine the analytical expression (an infinite sum) that describes the distribution of the electric scalar potential in the region between the conducting planes. Solve the problem using a direct application of the method of images.

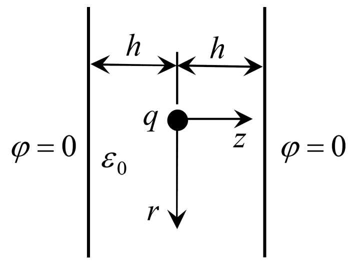


Figure 14.106. Point electric charge between two conducting planes

Solution:

The problem is axisymmetric, so a cylindrical coordinate system (r, z) is used. According to the method of images, there is an infinite number of image charges (Figure 14.106).

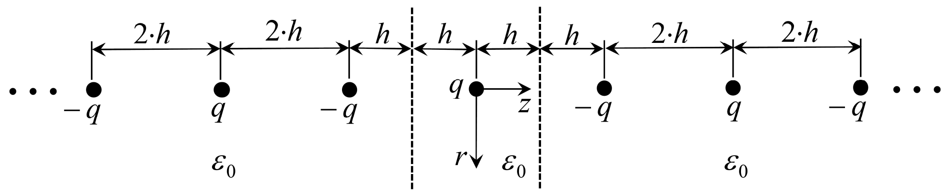


Figure 14.107. Real electric charge and image charges

The distribution of the electric scalar potential in the region between the conducting planes is described by the expression:

$$\varphi = \frac{q}{4\cdot\pi\cdot\varepsilon_0}\cdot\frac{1}{\sqrt{r^2+z^2}} + \frac{q}{4\cdot\pi\cdot\varepsilon_0}\cdot\sum_{\mathrm{k}=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+4\cdot k\cdot h)^2}} + \frac{1}{\sqrt{r^2+(z-4\cdot k\cdot h)^2}}\right)$$
$$-\frac{q}{4\cdot\pi\cdot\varepsilon_0}\cdot\sum_{\mathrm{k}=1}^{\infty}\left(\frac{1}{\sqrt{r^2+(z+4\cdot k\cdot h-2\cdot h)^2}} + \frac{1}{\sqrt{r^2+(z-4\cdot k\cdot h+2\cdot h)^2}}\right) \tag{14.554}$$

Example 14.24. Two conducting equipotential half-planes are located in air and are connected at an angle $\alpha = \pi/6$. Let the electric scalar potential of the first half-plane be $\varphi_1 = 1\ \mathrm{kV}$, and the scalar potential of the second half-plane be $\varphi_2 = 0.1\ \mathrm{kV}$. Determine the distribution of the electric scalar potential in the region between the half-planes, which corresponds to the angle α.

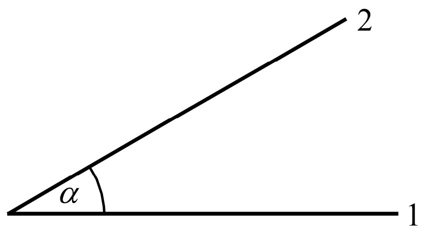


Figure 14.108. Two conducting equipotential half-planes

Solution:

The solution should be sought in a cylindrical coordinate system, and the distribution of the electric scalar potential depends only on the angle ϕ. In this case, the Laplace differential equation takes the form:

$$\Delta\varphi = \frac{1}{r^2}\cdot\frac{\partial^2\varphi}{\partial\phi^2} = 0 \tag{14.555}$$

and its solution is:

$$\varphi = C_1\cdot\phi + C_2 \tag{14.556}$$

with the given boundary conditions:

$$\varphi\big|_{\phi=0} = \varphi_1 = 1\ \text{kV} \quad ; \quad \varphi\big|_{\phi=\pi/6} = \varphi_2 = 0.1\ \text{kV} \tag{14.557}$$

After determining the constants C_1 and C_2 based on expressions (14.556) and (14.557), the final expression describing the distribution of the electric scalar potential is obtained:

$$\varphi = \frac{6}{\pi}\cdot(\varphi_2 - \varphi_1)\cdot\phi + \varphi_1 = 1 - 5.4\cdot\frac{\phi}{\pi}\ \text{kV} \tag{14.558}$$

Example 14.25. Determine the electric scalar potential function inside a system consisting of two conducting half-planes held at zero electric scalar potential, separated by a distance of a = 0.1 m, and closed on one side by a strip along which the potential distribution is given by the expression:
$\varphi = \varphi(0, y) = 100\cdot\sin(\pi\cdot y/a) + 50\cdot\sin(3\cdot\pi\cdot y/a)$ V.

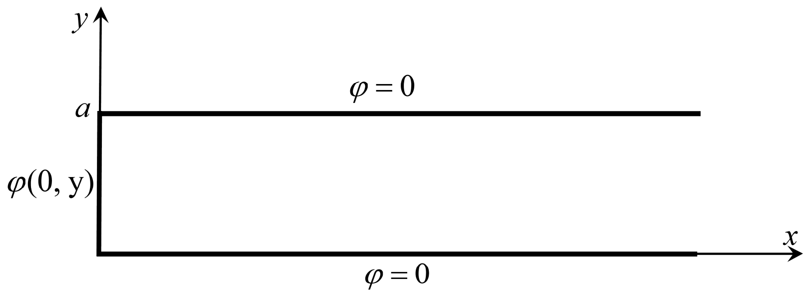


Figure 14.109. Two conducting half-planes and a strip

Solution:

This is a 2D problem in the (x, y) coordinate system, where the distribution of the electric scalar potential is described by the Laplace differential equation, and its general solution that can satisfy the boundary conditions is given by:

$$\varphi = \sum_{n=1}^{\infty}\left(A_n\cdot e^{-\frac{n\cdot\pi\cdot x}{a}} + B_n\cdot e^{\frac{n\cdot\pi\cdot x}{a}}\right)\cdot\left(C_n\cdot\sin\frac{n\cdot\pi\cdot y}{a} + D_n\cdot\cos\frac{n\cdot\pi\cdot y}{a}\right) \tag{14.559}$$

where the coefficients B_n; $n \geq 1$ are equal to zero because the electric scalar potential must remain finite as $x \to \infty$. Moreover, the coefficients D_n; $n \geq 1$ are zero since the given boundary condition $\varphi(0, y)$ contains only sine functions. It follows that the distribution of the electric scalar potential is described by the expression:

$$\varphi = \sum_{n=1}^{\infty} G_n\cdot e^{-\frac{n\cdot\pi\cdot x}{a}}\cdot\sin\frac{n\cdot\pi\cdot y}{a} \tag{14.560}$$

and after satisfying the given boundary condition, the desired electric scalar potential function is obtained:

$$\varphi = 100 \cdot e^{-\frac{\pi \cdot x}{a}} \cdot \sin\frac{\pi \cdot y}{a} + 50 \cdot e^{-\frac{3 \cdot \pi \cdot x}{a}} \cdot \sin\frac{3 \cdot \pi \cdot y}{a} \tag{14.561}$$

By substituting the given numerical value for the parameter a =0.1 m, it follows that:

$$\varphi = 100 \cdot e^{-10 \cdot \pi \cdot x} \cdot \sin(10 \cdot \pi \cdot y) \ + \ 50 \cdot e^{-30 \cdot \pi \cdot x} \cdot \sin(30 \cdot \pi \cdot y) \ \text{V} \tag{14.562}$$

Example 14.26. Based on the solution of the Laplace differential equation in the 2D rectangular coordinate system (x, y), determine the electric scalar potential function inside the rectangle shown in the figure. Let the following be given: $\varphi(x,b) = 10 \cdot \sin(\pi \cdot x/a) + 20 \cdot \sin(3 \cdot \pi \cdot x/a)$ V, a = 0.2 m, b = 0.1 m.

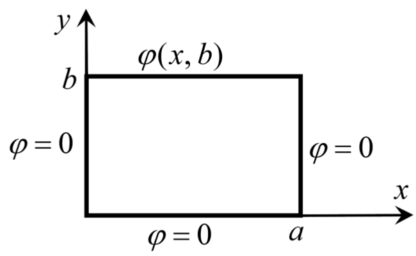


Figure 14.110. Geometry of the 2D problem and boundary conditions

Solution:

This is a 2D problem in the (x, y) coordinate system, where the distribution of the electric scalar potential is described by the Laplace differential equation, and its general solution that can satisfy the boundary conditions is given by:

$$\varphi = \sum_{n=1}^{\infty} \left(A_n \cdot \sin\frac{n \cdot \pi \cdot x}{a} + B_n \cdot \cos\frac{n \cdot \pi \cdot x}{a} \right) \cdot \left(C_n \cdot \sinh\frac{n \cdot \pi \cdot y}{a} + D_n \cdot \cosh\frac{n \cdot \pi \cdot y}{a} \right) \tag{14.563}$$

where the coefficients $B_n; n \geq 1$ are equal to zero because the given boundary condition contains only sine functions. Moreover, the coefficients $D_n; n \geq 1$ are zero as required by the boundary condition $\varphi(x, 0) = 0$. It follows that the distribution of the electric scalar potential is described by the expression:

$$\varphi = \sum_{n=1}^{\infty} G_n \cdot \sin\frac{n \cdot \pi \cdot x}{a} \cdot \sinh\frac{n \cdot \pi \cdot y}{a} \tag{14.564}$$

From the given boundary condition:

$$\varphi(x,b) = 10 \cdot \sin(\pi \cdot x/a) + 20 \cdot \sin(3 \cdot \pi \cdot x/a) \tag{14.565}$$

it follows that:

$$\varphi = G_1 \cdot \sin\frac{\pi \cdot x}{a} \cdot \sinh\frac{\pi \cdot y}{a} + G_3 \cdot \sin\frac{3 \cdot \pi \cdot x}{a} \cdot \sinh\frac{3 \cdot \pi \cdot y}{a} \tag{14.566}$$

From expressions (14.565) and (14.566), it follows that:

$$10 = G_1 \cdot \sinh\frac{\pi \cdot b}{a} \qquad \Rightarrow \qquad G_1 = \frac{10}{\sinh\frac{\pi \cdot b}{a}} \tag{14.567}$$

$$20 = G_3 \cdot \sinh\frac{3\cdot\pi\cdot b}{a} \qquad \Rightarrow \qquad G_3 = \frac{20}{\sinh\frac{3\cdot\pi\cdot b}{a}} \tag{14.568}$$

thus it is:

$$\varphi = 10\cdot\frac{\sin\frac{\pi\cdot x}{a}\cdot\sinh\frac{\pi\cdot y}{a}}{\sinh\frac{\pi\cdot b}{a}} + 20\cdot\frac{\sin\frac{3\cdot\pi\cdot x}{a}\cdot\sinh\frac{3\cdot\pi\cdot y}{a}}{\sinh\frac{3\cdot\pi\cdot b}{a}} \tag{14.569}$$

By substituting the given numerical values a = 0.2 m, b =0.1 m, it follows that:

$$\varphi = 10\cdot\frac{\sin(5\cdot\pi\cdot x)\cdot\sinh(5\cdot\pi\cdot y)}{\sinh\frac{\pi}{2}} + 20\cdot\frac{\sin(15\cdot\pi\cdot x)\cdot\sinh(15\cdot\pi\cdot y)}{\sinh\frac{3\cdot\pi}{2}} \tag{14.570}$$

Example 14.27. An electric dipole is centered at the origin of the coordinate system and is located in the air. A sphere is placed around the dipole, centered at the origin, with a radius of r_0 = 1 m. Determine the electric flux through the portion of the sphere located in the first octant of the Cartesian coordinate system (x, y, z). Let q = 1 μC, a = 0.01 m. Assume that $r_0 >> a$.

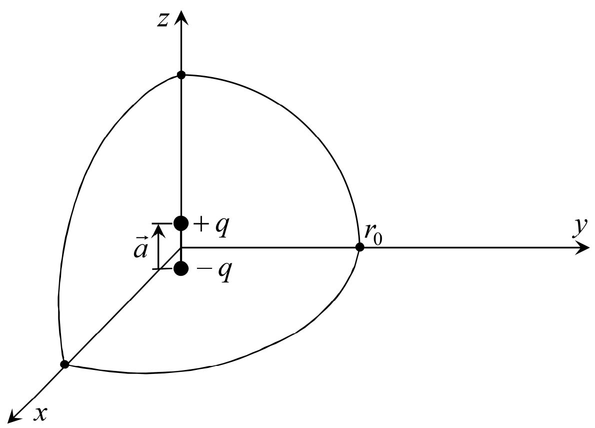


Figure 14.111. Electric dipole and one-eighth of a spherical surface

The electric scalar potential of an electric dipole located at the center of a spherical coordinate system (r, ϑ, ϕ) is given by expression (14.81), which in the air is:

$$\varphi = \frac{p\cdot\cos\vartheta}{4\cdot\pi\cdot\varepsilon_0\cdot r^2} = \frac{q\cdot a\cdot\cos\vartheta}{4\cdot\pi\cdot\varepsilon_0\cdot r^2} \tag{14.571}$$

thus, the r-component of the electric field intensity is given by:

$$E_r = -\frac{\partial\varphi}{\partial r} = \frac{p\cdot\cos\vartheta}{2\cdot\pi\cdot\varepsilon_0\cdot r^3} = \frac{q\cdot a\cdot\cos\vartheta}{2\cdot\pi\cdot\varepsilon_0\cdot r^3} \tag{14.572}$$

The electric flux passing through the given one-eighth of the spherical surface is generated by the normal component of the electric field intensity, which is the r-component of the electric field intensity. It holds that:

$$\Phi = \int_S \vec{D}\cdot d\vec{S} = \int_S \varepsilon_0\cdot E_r\cdot dS = \frac{q\cdot a}{2\cdot\pi\cdot r_0^3}\cdot\int_0^{\pi/2} d\phi\cdot\int_0^{\pi/2}\cos\vartheta\cdot r_0^2\cdot\sin\vartheta\cdot d\vartheta \tag{14.573}$$

and therefore:

$$\Phi = \frac{q \cdot a}{4 \cdot r_0} \cdot \int_0^{\pi/2} \cos\vartheta \cdot \sin\vartheta \cdot d\vartheta = \frac{q \cdot a}{8 \cdot r_0} = 1.25\ \text{nC} \tag{14.574}$$

Example 14.28. Two isolated solid conducting spheres with radii $r_1 = 0.1$ m and $r_2 = 0.15$ m are located in the air at a distance of $d = 5$ m from each other. Determine: a) the self and mutual capacitance coefficients, b) the partial capacitances, c) the mutual electric capacitance between the solid spheres, d) the electric capacitance of solid sphere 2 if solid sphere 1 is grounded, e) the electric capacitance of solid sphere 1 if solid sphere 2 is grounded. Assume that $d >> r_1$, $d >> r_2$.

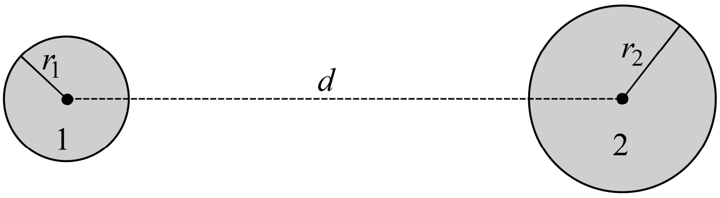


Figure 14.112. Two isolated solid conducting spheres

Solution:

The electric scalar potentials of the solid spheres are given by the following expressions:

$$\varphi_1 = \alpha_{1,1} \cdot Q_1 + \alpha_{1,2} \cdot Q_2 = \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_1} + \frac{Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot d} \tag{14.575}$$

$$\varphi_2 = \alpha_{2,1} \cdot Q_1 + \alpha_{2,2} \cdot Q_2 = \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot d} + \frac{Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_2} \tag{14.576}$$

where $\alpha_{\mathrm{i,k}}$ are the potential coefficients.
The system of linear equations (14.575) and (14.576) in matrix notation is given by:

$$\{\varphi\} = [\alpha] \cdot \{Q\} \tag{14.577}$$

where the matrix of potential coefficients is given by:

$$[\alpha] = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \begin{bmatrix} \dfrac{1}{r_1} & \dfrac{1}{d} \\ \dfrac{1}{d} & \dfrac{1}{r_2} \end{bmatrix} \tag{14.578}$$

a) Calculation of the capacitance coefficients

The matrix of capacitance coefficients is equal to the inverse of the matrix of potential coefficients:

$$[\beta] = [\alpha]^{-1} = \frac{4 \cdot \pi \cdot \varepsilon_0}{\dfrac{1}{r_1 \cdot r_2} - \dfrac{1}{d^2}} \cdot \begin{bmatrix} \dfrac{1}{r_2} & -\dfrac{1}{d} \\ -\dfrac{1}{d} & \dfrac{1}{r_1} \end{bmatrix} \tag{14.579}$$

and therefore:

$$[\beta]=\frac{4\cdot\pi\cdot\varepsilon_0\cdot r_1\cdot r_2\cdot d^2}{d^2-r_1\cdot r_2}\cdot\begin{bmatrix}\frac{1}{r_2} & -\frac{1}{d}\\ -\frac{1}{d} & \frac{1}{r_1}\end{bmatrix} \tag{14.580}$$

The self and mutual capacitance coefficients are:

$$\beta_{1,1}=\frac{4\cdot\pi\cdot\varepsilon_0\cdot r_1\cdot d^2}{d^2-r_1\cdot r_2}=1.11331805\times10^{-11}\ \text{F} \tag{14.581}$$

$$\beta_{2,2}=\frac{4\cdot\pi\cdot\varepsilon_0\cdot r_2\cdot d^2}{d^2-r_1\cdot r_2}=1.66997707\times10^{-11}\ \text{F} \tag{14.582}$$

$$\beta_{1,2}=\beta_{2,1}=-\frac{4\cdot\pi\cdot\varepsilon_0\cdot r_1\cdot r_2\cdot d}{d^2-r_1\cdot r_2}=-3.33995414\times10^{-13}\ \text{F} \tag{14.583}$$

b) Calculation of the partial capacitances

The partial capacitances are:

$$C_{1,1}=\beta_{1,1}+\beta_{1,2}=1.07991851\times10^{-11}\ \text{F} \tag{14.584}$$

$$C_{2,2}=\beta_{2,1}+\beta_{2,2}=1.63657753\times10^{-11}\ \text{F} \tag{14.585}$$

$$C_{1,2}=C_{2,1}=-\beta_{1,2}=3.33995414\times10^{-13}\ \text{F} \tag{14.586}$$

c) Calculation of the mutual electric capacitance between the solid spheres

The mutual electric capacitance between the solid spheres, according to Figure 14.113, is given by the expression:

$$C=C_{1,2}+\frac{C_{1,1}\cdot C_{2,2}}{C_{1,1}+C_{2,2}}=\ldots=6.84006184\times10^{-12}\ \text{F} \tag{14.587}$$

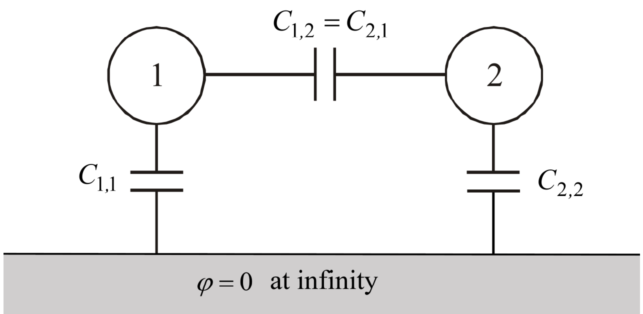


Figure 14.113. Partial capacitances of two isolated solid spheres

d) Calculation of the electric capacitance of solid sphere 2 if solid sphere 1 is grounded

If solid sphere 1 is grounded (Figure 14.114), its electric scalar potential is equal to zero, so the electric capacitance of solid sphere 2 is:

$$C_2=C_{1,2}+C_{2,2}=C_{2,1}+C_{2,2}=\beta_{2,2}=1.66997707\times10^{-11}\ \text{F} \tag{14.588}$$

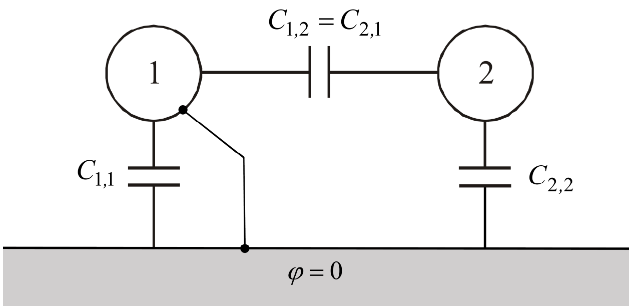


Figure 14.114. The electric scalar potential of solid sphere 1 is equal to zero

e) Calculation of the electric capacitance of solid sphere 2 if solid sphere 1 grounded

If solid sphere 2 is grounded (Figure 14.115), its electric scalar potential is equal to zero, so the electric capacitance of solid sphere 1 is:

$$C_1 = C_{1,2} + C_{1,1} = \beta_{1,1} = 1.11331805 \times 10^{-11} \text{ F} \tag{14.589}$$

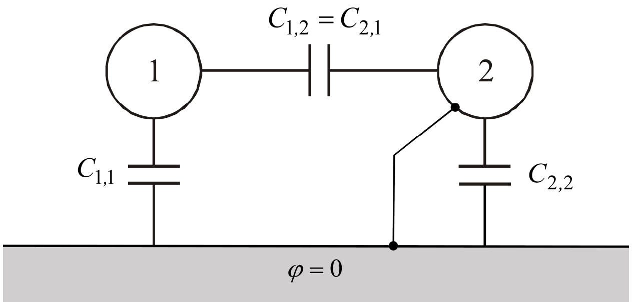


Figure 14.115. The electric scalar potential of solid sphere 2 is equal to zero

Example 14.29. Two conducting spheres with radii r_1 = 0.05 m and r_2 = 0.15 m are located in the unbounded air. Assume that the space between them is filled with a perfect LIH dielectric with a given relative permittivity $\varepsilon_r = 10$. Calculate all partial capacitances.

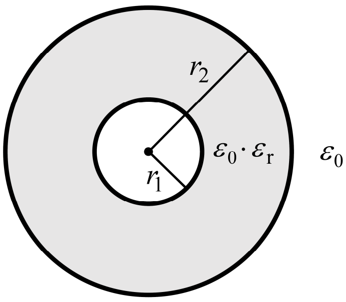


Figure 14.116. Two conducting spheres in the unbounded air

Solution:

In order to calculate all partial capacitances, it must be assumed that the inner sphere has an electric charge Q_1, and the outer sphere has an electric charge Q_2.
The electric field intensity is described by the expressions:

$$E_1 = \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r \cdot r^2} \quad \text{for} \quad r_1 \le r \le r_2 \tag{14.590}$$

$$E_2 = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r^2} \qquad \text{for} \quad r > r_2 \tag{14.591}$$

The electric scalar potential of sphere 2 is given by the expression:

$$\varphi_2 = \int_{r_2}^{\infty} E_2 \cdot dr = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_2} \tag{14.592}$$

whereas the electric scalar potential of sphere 1 is given by the expression:

$$\varphi_1 = \varphi_2 + \int_{r_1}^{r_2} E_1 \cdot dr = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_2} + \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r}} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tag{14.593}$$

Expressions (14.592) and (14.593) in matrix notation are written as:

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} \cdot \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \quad ; \quad \{\varphi\} = [\alpha] \cdot \{Q\} \tag{14.594}$$

where the potential coefficient matrix is given by the expression:

$$[\alpha] = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \begin{bmatrix} \dfrac{r_2 + (\varepsilon_\mathrm{r} - 1) \cdot r_1}{\varepsilon_\mathrm{r} \cdot r_1 \cdot r_2} & \dfrac{1}{r_2} \\ \dfrac{1}{r_2} & \dfrac{1}{r_2} \end{bmatrix} \tag{14.595}$$

Furthermore, it holds that:

$$\{Q\} = [\beta] \cdot \{\varphi\} = [\alpha]^{-1} \cdot \{\varphi\} \tag{14.596}$$

where the capacitance coefficient matrix is:

$$[\beta] = [\alpha]^{-1} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{bmatrix} = \frac{1}{\alpha_{1,1} \cdot \alpha_{2,2} - \alpha_{1,2} \cdot \alpha_{2,1}} \cdot \begin{bmatrix} \alpha_{2,2} & -\alpha_{1,2} \\ -\alpha_{2,1} & \alpha_{1,1} \end{bmatrix} \tag{14.597}$$

It follows that:

$$[\beta] = 4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r} \cdot \frac{r_1 \cdot r_2^2}{r_2 - r_1} \cdot \begin{bmatrix} \dfrac{1}{r_2} & -\dfrac{1}{r_2} \\ -\dfrac{1}{r_2} & \dfrac{r_2 + (\varepsilon_\mathrm{r} - 1) \cdot r_1}{\varepsilon_\mathrm{r} \cdot r_1 \cdot r_2} \end{bmatrix} \tag{14.598}$$

The partial capacitance matrix is given by the expression:

$$[C] = \begin{bmatrix} C_{1,1} & C_{1,2} \\ C_{2,1} & C_{2,2} \end{bmatrix} = \begin{bmatrix} \beta_{1,1} + \beta_{1,2} & -\beta_{1,2} \\ -\beta_{2,1} & \beta_{2,1} + \beta_{2,2} \end{bmatrix} \tag{14.599}$$

It follows that:

$$[C]=4\cdot\pi\cdot\varepsilon_0\cdot\varepsilon_r\cdot\frac{r_1\cdot r_2^2}{r_2-r_1}\cdot\begin{bmatrix}0 & \dfrac{1}{r_2}\\ \dfrac{1}{r_2} & \dfrac{r_2-r_1}{\varepsilon_r\cdot r_1\cdot r_2}\end{bmatrix} \tag{14.600}$$

Therefore, the partial capacitances are (Figure 14.117):

$$C_{1,1}=0 \tag{14.601}$$

$$C_{1,2}=C_{2,1}=4\cdot\pi\cdot\varepsilon_0\cdot\varepsilon_r\cdot\frac{r_1\cdot r_2}{r_2-r_1}=8.3448754\text{ pF} \tag{14.602}$$

$$C_{2,2}=4\cdot\pi\cdot\varepsilon_0\cdot r_2=16.6897508\text{ pF} \tag{14.603}$$

The result would be the same if the inner sphere were a solid conducting sphere.

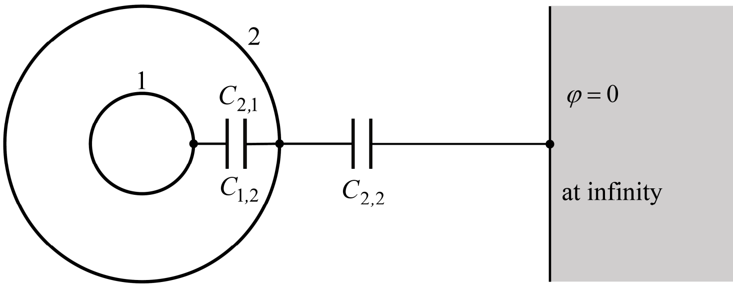


Figure 14.117. Partial capacitances

Example 14.30. A solid conducting sphere of radius r_1 = 0.02 m and a concentric hollow conducting sphere with inner radius r_2 = 0.05 m and outer radius r_3 = 0.06 are isolated in the air. A dielectric with relative permittivity $\varepsilon_r=10$ fills the space between the spheres. Calculate: a) all partial capacitances, b) the total capacitance between the spheres if sphere 1 is grounded, c) the total capacitance between the spheres if sphere 2 is grounded.

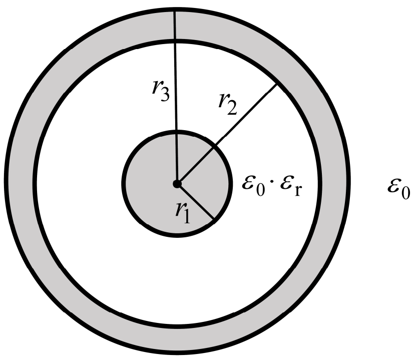


Figure 14.118. Two concentric conducting spheres

Solution:

a) Calculation of the partial capacitances

In order to calculate all partial capacitances, it must be assumed that the inner solid sphere (sphere 1) has an electric charge $Q_1,$ and the outer hollow sphere (sphere 2) has an electric charge $Q_2.$ In order for the electric field intensity inside sphere 2 to be zero, its inner spherical surface ($r=r_2$) must have a charge $-Q_1,$ and its outer spherical surface ($r=r_3$) must have a charge $Q_1+Q_2.$

In dielectrics, the electric field intensity is described by the expressions:

$$E_1 = \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r} \cdot r^2} \qquad \text{for} \quad r_1 \le r \le r_2 \tag{14.604}$$

$$E_2 = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r^2} \qquad \text{for} \quad r > r_3 \tag{14.605}$$

The electric scalar potential of hollow sphere 2 is given by the expression:

$$\varphi_2 = \int_{r_3}^{\infty} E_2 \cdot dr = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_3} \tag{14.606}$$

whereas the electric scalar potential of solid sphere 1 is given by the expression:

$$\varphi_1 = \varphi_2 + \int_{r_1}^{r_2} E_1 \cdot dr = \frac{Q_1 + Q_2}{4 \cdot \pi \cdot \varepsilon_0 \cdot r_3} + \frac{Q_1}{4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r}} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \tag{14.607}$$

Expressions (14.606) and (14.607) in matrix notation are written as:

$$\begin{Bmatrix} \varphi_1 \\ \varphi_2 \end{Bmatrix} = \begin{bmatrix} \alpha_{1,1} & \alpha_{1,2} \\ \alpha_{2,1} & \alpha_{2,2} \end{bmatrix} \cdot \begin{Bmatrix} Q_1 \\ Q_2 \end{Bmatrix} \quad ; \quad \{\varphi\} = [\alpha] \cdot \{Q\} \tag{14.608}$$

where the potential coefficient matrix is given by the expression:

$$[\alpha] = \frac{1}{4 \cdot \pi \cdot \varepsilon_0} \cdot \begin{bmatrix} \dfrac{r_3 \cdot (r_2 - r_1) + \varepsilon_\mathrm{r} \cdot r_1 \cdot r_2}{\varepsilon_\mathrm{r} \cdot r_1 \cdot r_2 \cdot r_3} & \dfrac{1}{r_3} \\ \dfrac{1}{r_3} & \dfrac{1}{r_3} \end{bmatrix} \tag{14.609}$$

Furthermore, it holds that:

$$\{Q\} = [\beta] \cdot \{\varphi\} = [\alpha]^{-1} \cdot \{\varphi\} \tag{14.610}$$

where the capacitance coefficient matrix is:

$$[\beta] = [\alpha]^{-1} = \begin{bmatrix} \beta_{1,1} & \beta_{1,2} \\ \beta_{2,1} & \beta_{2,2} \end{bmatrix} = \frac{1}{\alpha_{1,1} \cdot \alpha_{2,2} - \alpha_{1,2} \cdot \alpha_{2,1}} \cdot \begin{bmatrix} \alpha_{2,2} & -\alpha_{1,2} \\ -\alpha_{2,1} & \alpha_{1,1} \end{bmatrix} \tag{14.611}$$

It follows that:

$$[\beta] = 4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r} \cdot \frac{r_1 \cdot r_2 \cdot r_3}{r_2 - r_1} \cdot \begin{bmatrix} \dfrac{1}{r_3} & -\dfrac{1}{r_3} \\ -\dfrac{1}{r_3} & \dfrac{r_3 \cdot (r_2 - r_1) + \varepsilon_\mathrm{r} \cdot r_1 \cdot r_2}{\varepsilon_\mathrm{r} \cdot r_1 \cdot r_2 \cdot r_3} \end{bmatrix} \tag{14.612}$$

The partial capacitance matrix is given by the expression (14.599), so it is:

$$[C] = 4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_\mathrm{r} \cdot \frac{r_1 \cdot r_2 \cdot r_3}{r_2 - r_1} \cdot \begin{bmatrix} 0 & \dfrac{1}{r_3} \\ \dfrac{1}{r_3} & \dfrac{r_2 - r_1}{\varepsilon_\mathrm{r} \cdot r_1 \cdot r_2} \end{bmatrix} \tag{14.613}$$

Therefore, the partial capacitances are (Figure 14.117):

$$C_{1,1} = 0 \tag{14.614}$$

$$C_{1,2} = C_{2,1} = 4 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r \cdot \frac{r_1 \cdot r_2}{r_2 - r_1} = 37.0883352 \text{ pF} \tag{14.615}$$

$$C_{2,2} = 4 \cdot \pi \cdot \varepsilon_0 \cdot r_3 = 6.67590034 \text{ pF} \tag{14.616}$$

Note: The result would be the same if the inner solid sphere were a hollow conducting sphere.

b) Calculation of the electric capacitance between the spheres if sphere 1 is grounded

If sphere 1 is grounded, then its electric scalar potential is equal to zero, and the electric capacitance between the spheres is:

$$C = C_{2,1} + C_{2,2} = \beta_{2,2} = 43{,}76423554 \text{ pF} \tag{14.617}$$

c) Calculation of the electric capacitance between the spheres if sphere 2 is grounded

If sphere 2 is grounded, then its electric scalar potential is equal to zero, and the electric capacitance between the spheres is:

$$C = C_{1,2} = -\beta_{1,2} = 37{,}0883352 \text{ pF} \tag{14.618}$$

Example 14.31. Two mutually parallel thin-wire conductors, of length $\ell = 10$ m and radius $r_0 = 5$ mm, are located in the air above the ground at heights $h_1 = 5$ m and $h_2 = 7$ m. Let the distance between the conductors be $d = 3$ m. Using the average potential method, calculate all capacitance coefficients and the total capacitance between the conductors. Let it be $r_0 << d$.

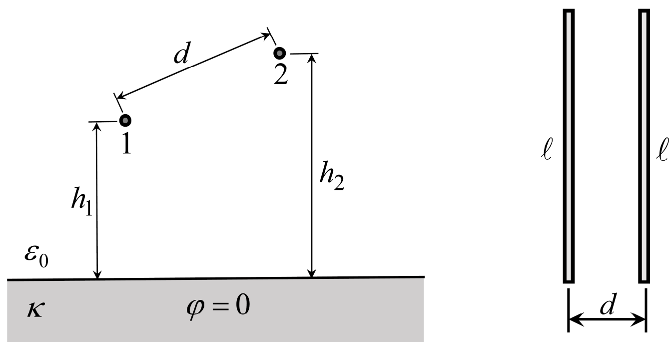


Figure 14.119. Two mutually parallel thin-wire conductors above the ground

Solution:

In rorder to calculate all partial capacitances using the average potential method, it must be assumed that both conductors are charged with electric charges Q_1 and Q_2, which are uniformly distributed along the axes of the conductors in the numerical model.

The influence of the ground on the distribution of the electric scalar potential can be taken into account by satisfying the boundary condition at the ground surface using the method of images. According to the method of images, two charged conductors in the air (a dielectric half-space) are replaced by four conductors in an unbounded perfect LIH dielectric whose permittivity is equal to the permittivity of air (Figure 14.120). In this case, the image charge has the opposite polarity to the corresponding real charge.

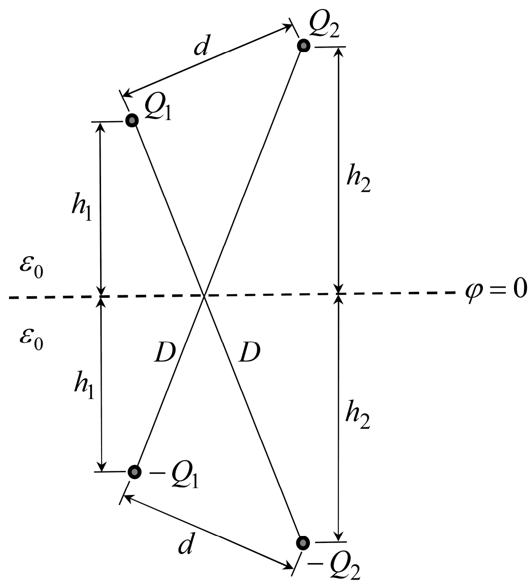


Figure 14.120. Real conductors and their mirror images

The electric scalar potentials of the conductors are described by the expressions::

$$\Phi_1 = \alpha_{1,1} \cdot Q_1 + \alpha_{1,2} \cdot Q_2 \quad ; \quad \Phi_2 = \alpha_{2,1} \cdot Q_1 + \alpha_{2,2} \cdot Q_2 \tag{14.619}$$

which can be written in matrix notation as follows:

$$\{\Phi\} = [\alpha] \cdot \{Q\} \tag{14.620}$$

The potential coefficient matrix is described by the expression:

$$[\alpha] = \begin{bmatrix} G(\ell, r_0) - G(\ell, 2 \cdot h_1) & G(\ell, d) - G(\ell, D) \\ G(\ell, d) - G(\ell, D) & G(\ell, r_0) - G(\ell, 2 \cdot h_2) \end{bmatrix} \tag{14.621}$$

where:

$$G(\ell, v) = \frac{1}{2 \cdot \pi \cdot \varepsilon_0 \cdot \ell^2} \cdot \left[\ell \cdot \ln \frac{\sqrt{\ell^2 + v^2} + \ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{14.622}$$

The capacitance coefficient matrix is described by the expression:

$$[\beta] = [\alpha]^{-1} = \frac{1}{\det[\alpha]} \begin{bmatrix} G(\ell, r_0) - G(\ell, 2 \cdot h_2) & G(\ell, D) - G(\ell, d) \\ G(\ell, D) - G(\ell, d) & G(\ell, r_0) - G(\ell, 2 \cdot h_1) \end{bmatrix} \tag{14.623}$$

where:

$$\begin{aligned} \det[\alpha] = & \left[G(\ell, r_0) - G(\ell, 2 \cdot h_1) \right] \cdot \left[G(\ell, r_0) - G(\ell, 2 \cdot h_2) \right] \\ & - \left(G(\ell, d) - G(\ell, D) \right)^2 \end{aligned} \tag{14.624}$$

If the given data are substituted, it follows that:

$$G(\ell, r_0) = 1.311202842 \times 10^{10} \quad ; \quad G(\ell, 2 \cdot h_1) = 8.397249834 \times 10^8 \tag{14.625}$$

$$G(\ell, 2 \cdot h_2) = 6.181018902 \times 10^8 \quad ; \quad G(\ell, d) = 2.111833483 \times 10^9 \tag{14.626}$$

$$D=\sqrt{(h_1+h_2)^2+d^2-(h_1-h_2)^2}=\sqrt{d^2+4\cdot h_1\cdot h_2}=12.20655562\text{ m} \tag{14.627}$$

$$G(\ell, D)=7.015641548\times 10^8 \tag{14.628}$$

$$[\alpha]=\begin{bmatrix} 1.227230344 & 0.1410269328 \\ 0.1410269328 & 1.249392653 \end{bmatrix}\cdot 10^{10} \tag{14.629}$$

$$[\beta]=[\alpha]^{-1}=\begin{bmatrix} 1.249392653 & -0.1410269328 \\ -0.1410269328 & 1.227230344 \end{bmatrix}\cdot\frac{10^{-10}}{1.51340398} \tag{14.630}$$

$$\beta_{1,1}=82.555132\times 10^{-12}\text{ F} \tag{14.631}$$

$$\beta_{2,1}=\beta_{1,2}=-9.318525305\times 10^{-12}\text{ F} \tag{14.632}$$

$$\beta_{2,2}=81.0907306\times 10^{-12}\text{ F} \tag{14.633}$$

and the partial capacitances are:

$$C_{1,1}=\beta_{1,1}+\beta_{1,2}=73.2366067\text{ pF} \tag{14.634}$$

$$C_{1,2}=C_{2,1}=-\beta_{1,2}=9.318525305\text{ pF} \tag{14.635}$$

$$C_{2,2}=\beta_{2,1}+\beta_{2,2}=71.7722053\text{ pF} \tag{14.636}$$

The total capacitance between the conductors is:

$$C=C_{1,2}+\frac{C_{1,1}\cdot C_{2,2}}{C_{1,1}+C_{2,2}}=45.56703117\text{ pF} \tag{14.637}$$

Example 14.32. Two isolated, infinitely long, mutually parallel cylindrical conductors with radii $R_1 = 0.1$ m and $R_2 = 0.05$ m are placed in the air, with their centers separated by a distance of $d = 0.5$ m. Assume the solid cylinders are charged with constant linear charge densities $\lambda_1 = -\lambda_2 = \lambda = 1$ μC/m. Calculate the electric scalar potential of each cylinder and the per-unit-length capacitance between them.

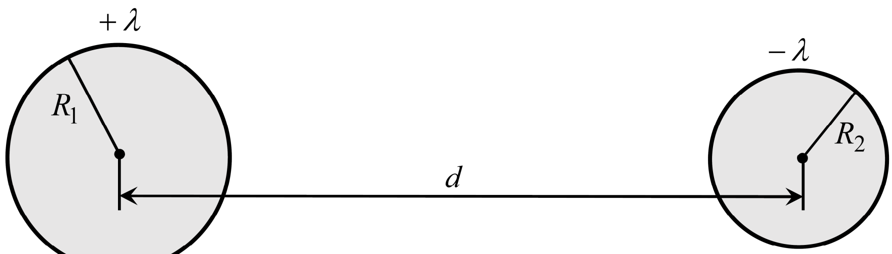


Figure 14.121. Two infinitely long mutually parallel cylindrical conductors

Solution:

The problem is solved based on Figure 14.122. The equivalent charges are placed on imaginary infinitely long, straight, thin-wire conductors.

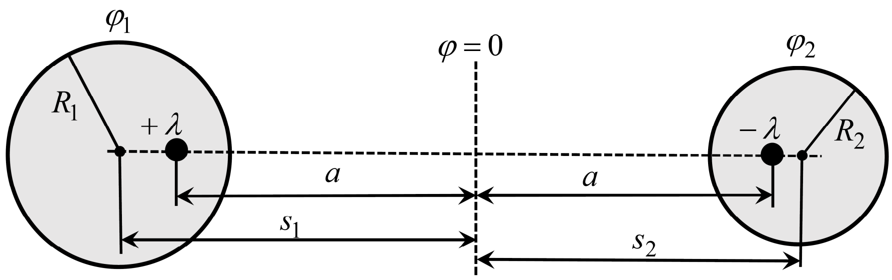


Figure 14.122. Graphical representation of the parameters required for solving the problem

The following expressions apply:

$$s_1^2 - a^2 = R_1^2 \tag{14.638}$$

$$s_2^2 - a^2 = R_2^2 \tag{14.639}$$

$$s_1 + s_2 = d \tag{14.640}$$

From expressions (14.638) and (14.639), it follows that:

$$s_1^2 - s_2^2 = (s_1 - s_2) \cdot (s_1 + s_2) = R_1^2 - R_2^2 \tag{14.641}$$

from which it follows that:

$$s_1 - s_2 = \frac{R_1^2 - R_2^2}{d} \tag{14.642}$$

From the system of two linear equations (14.640) and (14.642), it follows that:

$$s_1 = \frac{d}{2} + \frac{R_1^2 - R_2^2}{2 \cdot d} = 0.2575 \text{ m} \tag{14.643}$$

$$s_2 = d - s_1 = 0.2425 \text{ m} \tag{14.644}$$

$$a = \sqrt{s_1^2 - R_1^2} = 0.23728938 \text{ m} \tag{14.645}$$

The electric scalar potentials of the cylindrical conductors are described by the expressions:

$$\varphi_1 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{s_1 + a}{R_1} = 2.87452266 \times 10^4 \text{ V} \tag{14.646}$$

$$\varphi_2 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{s_2 - a}{R_2} = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{R_2}{s_2 + a} = -4.06475369 \times 10^4 \text{ V} \tag{14.647}$$

The per-unit-length capacitance between two cylindrical conductors is:

$$C = \frac{\lambda}{U} = \frac{\lambda}{\varphi_1 - \varphi_2} = \frac{2 \cdot \pi \cdot \varepsilon_0}{\ln \frac{(s_1 + a) \cdot (s_2 + a)}{R_1 \cdot R_2}} = 14.41072454 \text{ pF/m} \tag{14.648}$$

Example 14.33. Between two infinitely long, mutually parallel, eccentric conducting cylinders with radii R_1 = 0.2 m and R_2 = 0.1 m, air is present, and the distance between their centers is d = 0.05 m. Calculate the per-unit-length electric capacitance between the conductors.

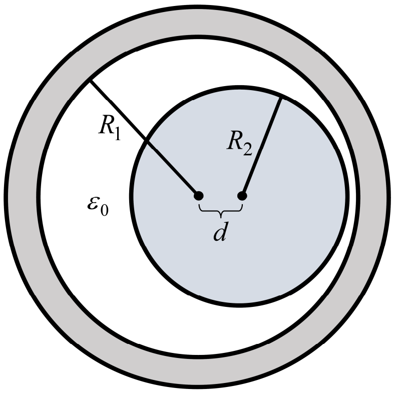


Figure 14.123. Two infinitely long eccentric cylindrical conductors

Solution:

The problem can be solved with reference to Figure 14.124. The equivalent charges are placed on imaginary, infinitely long, straight, thin-wire conductors.

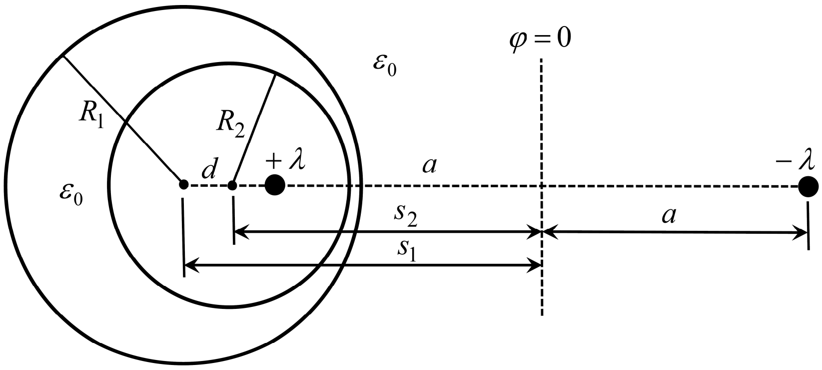


Figure 14.124. Graphical representation of the parameters required for solving the problem

The following expressions apply:

$$s_1^2 - a^2 = R_1^2 \quad ; \quad s_2^2 - a^2 = R_2^2 \quad ; \quad s_1 - s_2 = d \tag{14.649}$$

from which it follows that:

$$s_1 = \frac{d}{2} + \frac{R_1^2 - R_2^2}{2 \cdot d} = 0.325 \text{ m} \tag{14.650}$$

$$s_2 = s_1 - d = 0.275 \text{ m} \tag{14.651}$$

$$a = \sqrt{s_1^2 - R_1^2} = 0.2561737691 \text{ m} \tag{14.652}$$

The electric scalar potentials of the cylindrical conductors are described by the expressions:

$$\varphi_1 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{s_1 + a}{R_1} \tag{14.653}$$

$$\varphi_2 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{s_2 + a}{R_2} \tag{14.654}$$

and the electric voltage between the cylindrical conductors is given by the expression:

$$U = \varphi_2 - \varphi_1 = \frac{\lambda}{2 \cdot \pi \cdot \varepsilon_0} \cdot \ln \frac{(s_2 + a) \cdot R_1}{(s_1 + a) \cdot R_2} \tag{14.655}$$

The per-unit-length capacitance between two cylindrical conductors is:

$$C = \frac{\lambda}{U} = \frac{\lambda}{\varphi_2 - \varphi_1} = \frac{2 \cdot \pi \cdot \varepsilon_0}{\ln \frac{(s_2 + a) \cdot R_1}{(s_1 + a) \cdot R_2}} = 92.23099936 \text{ pF/m} \tag{14.656}$$

15. STATIONARY CURRENT FIELD

An electrostatic field is created by stationary electric charges in a perfect dielectric. A stationary current field is generated by electric charges moving uniformly within a conducting medium. In other words, it is an electromagnetic field produced in a conductor by time-invariant electric currents, also referred to in the literature as time-independent or steady electric currents.

The vector of the surface density of the stationary electric current in a conducting medium is given by:

$$\vec{J} \equiv \vec{J}_{\mathrm{c}} = N \cdot q \cdot \vec{v} \tag{15.1}$$

where:

N - the number of free charge carriers per cubic meter,

q - the electric charge of a single free charge carrier,

$\vec{v}$ - the velocity vector of free charge carriers.

From expression (15.1), it follows that:

$$\vec{v} = \frac{\vec{J}}{N \cdot q} \tag{15.2}$$

Therefore, according to expression (8.1), the Lorentz force on a free electric charge carrier is given by:

$$\vec{F} = q \cdot \vec{E} + q \cdot \left(\vec{v} \times \vec{B}\right) = q \cdot \vec{E} + \frac{1}{N} \cdot \left(\vec{J} \times \vec{B}\right) = q \cdot \vec{E}_{\mathrm{equ}} \tag{15.3}$$

where the equivalent electric field intensity vector is:

$$\vec{E}_{\mathrm{equ}} = \vec{E} + \frac{1}{N \cdot q} \cdot \left(\vec{J} \times \vec{B}\right) \tag{15.4}$$

therefore, in the case of *solid* and *liquid* conducting media:

$$\vec{J} = \kappa \cdot \vec{E}_{\mathrm{equ}} = \kappa \cdot \left(\vec{E} + \frac{1}{N \cdot q} \cdot \left(\vec{J} \times \vec{B}\right) \right) \tag{15.5}$$

from which it follows that:

$$\vec{E} = \frac{\vec{J}}{\kappa} - \frac{\vec{J} \times \vec{B}}{N \cdot q} \tag{15.6}$$

where κ is the electric conductivity of the conducting medium.

In the case of good conductors, it can be assumed that:

$$\vec{E} = \frac{\vec{J}}{\kappa} \quad ; \quad \vec{J} = \kappa \cdot \vec{E} \tag{15.7}$$

because in good conductors:

$$\frac{J}{\kappa} >> \frac{J \cdot B}{N \cdot q} \tag{15.8}$$

This means that it is assumed the distribution of a stationary electric current in good conductors does not depend on the static magnetic field generated by that current. Such an electric field is called a *stationary current field* or *static current field*.

Therefore, stationary electric currents flowing through good conductors create a stationary current field and a magnetostatic field, which, with a very good approximation, can be considered independent of each other. The stationary current field generates the magnetostatic field, but the influence of the magnetostatic field on the stationary current field is neglected.

15.1. Stationary Electric Charge at the Boundary Between Two Media

A stationary electric charge can be located at the boundary between two LIH media, i.e., at the boundary between two LIH conductors carrying stationary electric currents, as well as at the boundary between a conductor and a dielectric (Figure 15.1).

Conclusion: At the boundary between a conductor and a dielectric, in the case of a stationary current field as well as in the case of an electrostatic field, there is a stationary electric charge.

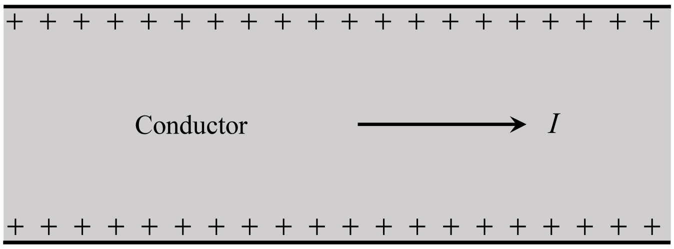


Figure 15.1. Stationary electric charge on the surface of a conductor surrounded by a dielectric

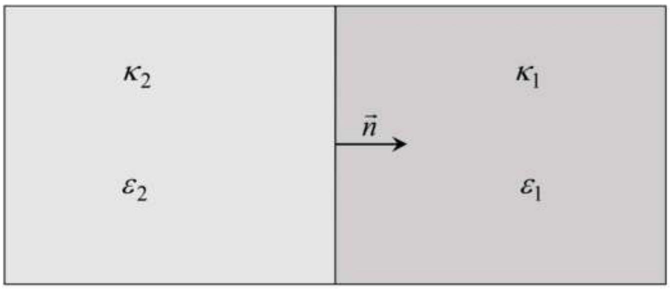


Figure 15.2. Two conducting LIH media

At the boundary between conducting LIH medium 1, which has the characteristics κ_1 and ε_1, and conducting LIH medium 2, which has the characteristics κ_2 and ε_2 (Figure 15.2), the following boundary conditions must be satisfied, according to the expressions given in subchapter 11.1:

$$\vec{n}\times\left(\vec{E}_1-\vec{E}_2\right)=0 \quad ; \quad \vec{n}\cdot\left(\vec{J}_1-\vec{J}_2\right)=0 \quad ; \quad \vec{n}\cdot\left(\vec{D}_1-\vec{D}_2\right)=\sigma \tag{15.9}$$

or, alternatively written:

$$E_{\mathrm{t1}}=E_{\mathrm{t2}}=E_{\mathrm{t}} \quad ; \quad J_{\mathrm{n1}}=J_{\mathrm{n2}}=J_{\mathrm{n}} \quad ; \quad D_{\mathrm{n1}}-D_{\mathrm{n2}}=\sigma \tag{15.10}$$

From expression (15.10), it follows that:

$$D_{\mathrm{n1}}-D_{\mathrm{n2}}=\left(\frac{\varepsilon_1}{\kappa_1}-\frac{\varepsilon_2}{\kappa_2}\right)\cdot J_{\mathrm{n}}=\sigma \tag{15.11}$$

Thus, for:

$$\frac{\varepsilon_1}{\kappa_1}>\frac{\varepsilon_2}{\kappa_2} \tag{15.12}$$

the surface density of the the stationary electric charge $\sigma>0$ (Figure 15.3), whereas for:

$$\frac{\varepsilon_1}{\kappa_1}<\frac{\varepsilon_2}{\kappa_2} \tag{15.13}$$

the surface density of the stationary electric charge $\sigma<0$.

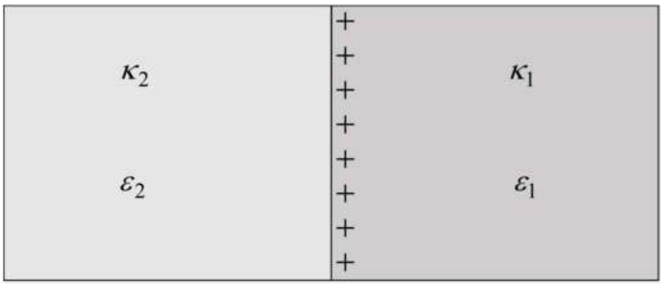


Figure 15.3. Positive stationary electric charge at the boundary of two conducting LIH media

15.2. Maxwell's Equations of a Stationary Current Field

According to expressions (6.47) and (6.48), a stationary current field in a stationary conducting medium is described by two of Maxwell's differential equations:

$$\nabla \times \vec{E} = 0 \tag{15.14}$$

$$\nabla \cdot \vec{J} = -\nabla \cdot \vec{J}_{\mathrm{s}} = g_{\mathrm{st}} \tag{15.15}$$

where the newly introduced quantity g_{st} represents the volume density of the stationary electric current leaked by an independent source into the surrounding conducting medium.

From Maxwell's differential equations of the stationary current field (15.14) and (15.15), or alternatively from expressions (6.95) and (6.96), it follows that Maxwell's integral equations of the stationary current field are given by the following expressions:

$$\oint_C \vec{E} \cdot d\vec{\ell} = 0 \tag{15.16}$$

$$\oint_{\partial V} \vec{J} \cdot d\vec{S} = \int_V g_{\mathrm{st}} \cdot dV = I_{\mathrm{st}} \tag{15.17}$$

where I_{st} is the stationary electric current leaked by an independent source into the conducting medium within volume V. The curve C is illustrated in Figure 1.4, whereas the volume V and the closed surface ∂V are shown in Figure 1.3.

Maxwell's differential equation (15.15) is also the differential continuity equation in a conducting medium. Maxwell's integral equation (15.17) is likewise the integral continuity equation in a conducting medium, as well as the general form of Kirchhoff's first law, which includes an independent electric current source.

15.3. Analogy Between Static Fields

There is an analogy between the stationary current field in a conducting LIH medium and the electrostatic field in a perfect LIH dielectric, as presented in Table 15.1.
The following quantities of the stationary current field and the electrostatic field are analogous:

$$\vec{J} \Leftrightarrow \vec{D} \quad ; \quad I \Leftrightarrow Q \quad ; \quad \kappa \Leftrightarrow \varepsilon \quad ; \quad g_{\mathrm{st}} \Leftrightarrow \rho_{\mathrm{s}} \tag{15.18}$$

In the case of linear isotropic regions without field sources, there is an analogy between the stationary current field, the electrostatic field, and the magnetostatic field, where the magnetostatic field is expressed using the magnetic scalar potential (Table 15.2). In the case of the magnetostatic field, a source-free region is referred to as a current-free region.

Table 15.1. Analogy between the electrostatic field and the stationary current field

Electrostatic field	Stationary current field
$\nabla\times\vec{E}=0$	$\nabla\times\vec{E}=0$
$\nabla\cdot\vec{D}=\rho_{\mathrm{s}}$	$\nabla\cdot\vec{J}=g_{\mathrm{st}}$
$\vec{D}=\varepsilon\cdot\vec{E}$	$\vec{J}=\kappa\cdot\vec{E}$
$\Delta\varphi=-\dfrac{\rho_{\mathrm{s}}}{\varepsilon}$	$\Delta\varphi=-\dfrac{g_{\mathrm{st}}}{\kappa}$
$\vec{E}=-\nabla\varphi$	$\vec{E}=-\nabla\varphi$

Table 15.2. Analogy between static fields in a source-free region

Electrostatic field	Stationary current field	Magnetostatic field
$\nabla\times\vec{E}=0$	$\nabla\times\vec{E}=0$	$\nabla\times\vec{H}=0$
$\nabla\cdot\vec{D}=0$	$\nabla\cdot\vec{J}=0$	$\nabla\cdot\vec{B}=0$
$\vec{D}=\varepsilon\cdot\vec{E}$	$\vec{J}=\kappa\cdot\vec{E}$	$\vec{B}=\mu\cdot\vec{H}$
$\Delta\varphi=0$	$\Delta\varphi=0$	$\Delta\varphi_{\mathrm{m}}=0$
$\vec{E}=-\nabla\varphi$	$\vec{E}=-\nabla\varphi$	$\vec{H}=-\nabla\varphi_{\mathrm{m}}$

It is important to emphasize that the stationary current field is a conservative field, just like the electrostatic field, and that the electric voltage between two points A and B is equal to the electric scalar potential difference at those two points:

$$U_{\mathrm{AB}}=\int_{A}^{B}\vec{E}\cdot d\vec{\ell}=\varphi_{\mathrm{A}}-\varphi_{\mathrm{B}} \tag{15.19}$$

In the general case, the distribution of stationary electric current sources may be such that current leaks from the volume V (with volume density g_{st}), from the surface S (with the normal component of surface density J_{n}), from the curve C (with linear density τ), and there may also be a set of point electric current sources. Analogous to expression (6.14), the general integral equation for the electric scalar potential is given by:

$$\varphi=\frac{1}{4\cdot\pi\cdot\kappa}\cdot\left(\int_{V}\frac{g_{\mathrm{st}}\cdot dV}{r}+\int_{S}\frac{J_{\mathrm{n}}\cdot dS}{r}+\int_{C}\frac{\tau\cdot d\ell}{r}+\sum_{\mathrm{i=1}}^{\mathrm{n}}\frac{I_{\mathrm{i}}}{r_{\mathrm{i}}}\right) \tag{15.20}$$

where r is the distance between the source point and the field point.

15.4. Point Source of Stationary Electric Current

Analogous to expressions (14.9) - (14.11), for a point source of stationary electric current with intensity I in an unbounded conducting LIH medium, the following holds:

$$g_{\mathrm{st}} = I \cdot \delta(\vec{r}) \quad ; \quad \Delta\varphi = -\frac{I}{\kappa} \cdot \delta(\vec{r}) \quad ; \quad \varphi = \frac{I}{4 \cdot \pi \cdot \kappa \cdot r} \tag{15.21}$$

where $\delta(\vec{r})$ is the Dirac delta function associated with the point at the origin of the spherical coordinate system, where the independent field source is located.

A point source of stationary electric current with intensity I in an unbounded LIH conducting medium of electrical conductivity κ generates a spherically symmetric field (Figure 15.4). According to expression (15.17), the following holds:

$$\oint_{\partial V} \vec{J} \cdot d\vec{S} = \kappa \cdot E \cdot S = I \cdot \int_{V} \delta(\vec{r}) \cdot dV = I \tag{15.22}$$

where, according to Figure 15.4, ∂V is a sphere centered at the origin of the spherical coordinate system, and it follows that the electric field intensity is given by the expression:

$$E = \frac{I}{4 \cdot \pi \cdot \kappa \cdot r^2} \tag{15.23}$$

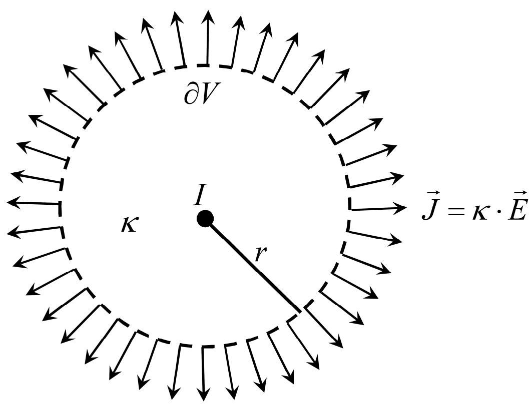


Figure 15.4. Electric field intensity of a point source of stationary electric current

The electric field intensity around a point source of stationary electric current has only a radial component, i.e., it holds that:

$$\vec{E} = -\nabla\varphi = -\frac{d\varphi}{dr} \cdot \vec{\mathrm{e}}_{\mathrm{r}} \tag{15.24}$$

from which it follows that:

$$\varphi = -\int \vec{E} \cdot d\vec{\ell} = -\int E \cdot dr = -\frac{I}{4 \cdot \pi \cdot \kappa} \cdot \int \frac{dr}{r^2} = \frac{I}{4 \cdot \pi \cdot \kappa \cdot r} \tag{15.25}$$

if the common assumption is made that the electric scalar potential is zero at infinity.

15.5. Grounding Resistance of a Solid Perfectly Conducting Sphere in an Unbounded LIH Soil

Let a solid perfectly conducting sphere of radius r_0 be placed in an unbounded LIH soil with electrical conductivity κ (Figure 15.5). Let the sphere leak a stationary electric current I into the surrounding soil from its surface. The objective is to derive the expression for the grounding resistance of such a sphere, valid for a stationary current field.

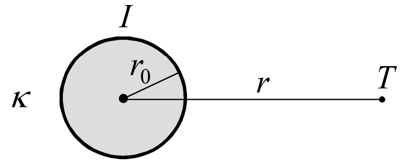


Figure 15.5. Solid perfectly conducting sphere in an unbounded LIH soil

In the region outside the sphere, the electric field intensity and the distribution of the electric scalar potential are described by expressions (15.23) and (15.25), which also apply in the case of a point source of stationary electric current.

Inside the solid perfectly conducting sphere, the electric field intensity is zero, and the electric scalar potential within the sphere (including its surface) is constant and is given by the following expression:

$$\varphi_{\text{sphere}} = \frac{I}{4 \cdot \pi \cdot \kappa \cdot r_0} \tag{15.26}$$

The grounding resistance of a solid perfectly conducting sphere in an unbounded LIH medium is:

$$R_{\text{sphere}} = \frac{\varphi_{\text{sphere}}}{I} = \frac{1}{4 \cdot \pi \cdot \kappa \cdot r_0} \tag{15.27}$$

If the same sphere is placed in an unbounded ideal LIH dielectric with permittivity ε, and is charged with a charge Q, then its electric scalar potential is:

$$\varphi_{\text{sphere}} = \frac{Q}{4 \cdot \pi \cdot \varepsilon \cdot r_0} \tag{15.28}$$

whereas its capacitance is:

$$C_{\text{sphere}} = \frac{Q}{\varphi_{\text{sphere}}} = 4 \cdot \pi \cdot \varepsilon \cdot r_0 \tag{15.29}$$

It follows that:

$$R_{\text{sphere}} \cdot C_{\text{sphere}} = \frac{\varepsilon}{\kappa} \quad ; \quad R_{\text{sphere}} = \frac{\varepsilon}{\kappa} \cdot \frac{1}{C_{\text{sphere}}} \tag{15.30}$$

Expression (15.30) is independent of the shape of the grounding electrode, so the analogy between the electrostatic and stationary current field can be used in numerical modelling of equipotential grounding systems.

15.6. Method of Images for Stationary Electric Current

Let a point source of stationary electric current with intensity I be located in front of a conducting half-space at a distance a. Let the point source of electric current be placed in the half-space with electrical conductivity κ_1, whereas the remaining part of the space has electrical conductivity κ_2 (Figure 15.6). For simplicity, let the point source of electric current be located on the x-axis of a rectangular coordinate system (x, y, z).

The expressions for the distribution of the electric scalar potential in both half-spaces can be determined using the method of images. The positions of the image electric currents are chosen consistently, and their magnitudes are determined based on satisfying the boundary conditions (Figures 15.7 and 15.8).

According to Figure 15.7, the electric scalar potential in conducting medium 1, which is considered unbounded, is generated by the real electric current I and the image electric current I_2, whose position is the mirror image of the real current source. According to Figure 15.8, the electric scalar potential in conducting medium 2, which is also considered unbounded, is generated by the image current I_1, which is located at the position of the real electric current source.

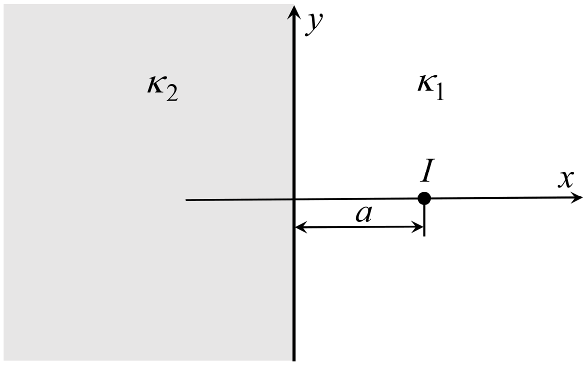


Figure 15.6. Point current source in front of the boundary between two conducting half-spaces

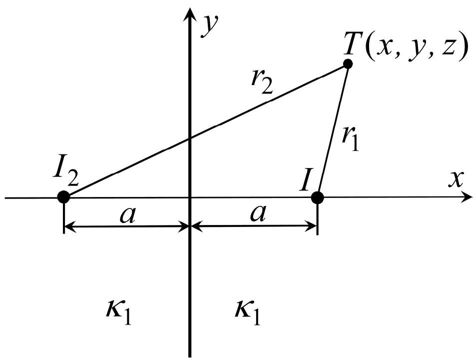


Figure 15.7. Point current source and its image when the electric scalar potential is calculated in conducting medium 1, which is considered unbounded

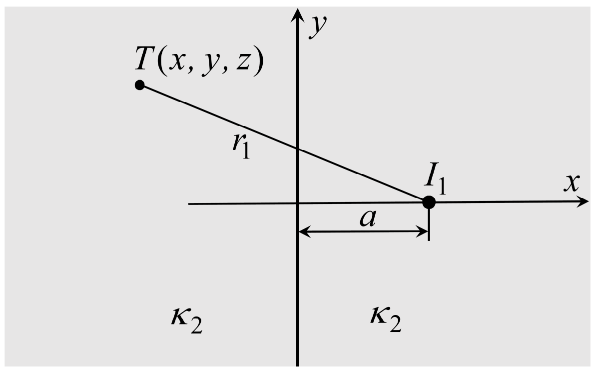


Figure 15.8. Image point current source when the electric scalar potential is calculated in conducting medium 2, which is considered unbounded

The distribution of the electric scalar potential in conducting medium 1 $(x \geq 0)$, according to Figure 15.7, is described by the following expression:

$$\varphi_1 = \frac{1}{4 \cdot \pi \cdot \kappa_1} \cdot \left(\frac{I}{r_1} + \frac{I_2}{r_2} \right) \tag{15.31}$$

whereas the distribution of the electric scalar potential in conducting medium 2 $(x \leq 0)$, according to Figure 15.8, is described by the following expression:

$$\varphi_2 = \frac{I_1}{4 \cdot \pi \cdot \kappa_2 \cdot r_1} \tag{15.32}$$

where:

$$r_1 = \sqrt{(x-a)^2 + y^2 + z^2} \quad ; \quad r_2 = \sqrt{(x+a)^2 + y^2 + z^2} \tag{15.33}$$

Expressions (15.31) and (15.32) must satisfy the following boundary conditions:

$$\varphi_1\big|_{x=0} = \varphi_2\big|_{x=0} \tag{15.34}$$

$$\kappa_1 \cdot \frac{\partial \varphi_1}{\partial x}\bigg|_{x=0} = \kappa_2 \cdot \frac{\partial \varphi_2}{\partial x}\bigg|_{x=0} \tag{15.35}$$

If expressions (15.31) and (15.32) are substituted into the boundary condition (15.34), taking into account expression (15.33), the following expression is obtained:

$$\frac{1}{\kappa_1} \cdot (I + I_2) = \frac{1}{\kappa_2} \cdot I_1 \tag{15.36}$$

Similarly, if expressions (15.31) and (15.32) are substituted into boundary condition (15.35), taking into account expression (15.33), the following expression is obtained:

$$I - I_2 = I_1 \tag{15.37}$$

From the system of two linear equations (15.36) and (15.37), it follows that:

$$I_2 = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \cdot I = k_\mathrm{R} \cdot I \quad ; \quad k_\mathrm{R} = \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \tag{15.38}$$

$$I_1 = \frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot I = k_\mathrm{T} \cdot I \quad ; \quad k_\mathrm{T} = \frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \tag{15.39}$$

where k_R is the reflection factor, and k_T is the transmission factor, for which the following holds:

$$k_\mathrm{R} + k_\mathrm{T} = 1 \tag{15.40}$$

In the special case when $\kappa_2 \to 0$, the following holds:

$$\lim_{\kappa_2 \to 0} k_\mathrm{R} = 1 \quad \Rightarrow \quad \lim_{\kappa_2 \to 0} I_2 = I \tag{15.41}$$

According to expressions (15.31) and (15.38), the distribution of the electric scalar potential in conducting medium 1 is described by the following expression:

$$\varphi_1 = \frac{I}{4 \cdot \pi \cdot \kappa_1} \cdot \left(\frac{1}{r_1} + \frac{k_\mathrm{R}}{r_2} \right) \tag{15.42}$$

whereas, according to expressions (15.32) and (15.39), the distribution of the electric scalar potential in conducting medium 2 is described by the following expression:

$$\varphi_2 = \frac{k_\mathrm{T} \cdot I}{4 \cdot \pi \cdot \kappa_2 \cdot r_1} \tag{15.43}$$

15.7. Grounding Resistance of a Solid Perfectly Conducting Sphere on the Soil Surface and Potential Distribution in the Air

Let a solid perfectly conducting sphere of radius r_0 be located on the surface of an LIH soil with electrical conductivity κ, such that one hemisphere is buried in the soil (Figure 15.9). The sphere leaks a stationary electric current I into the surrounding soil through the half of its surface that is in direct contact with the soil. The objective is to derive an expression for the grounding resistance of such a sphere, valid for a stationary current field.

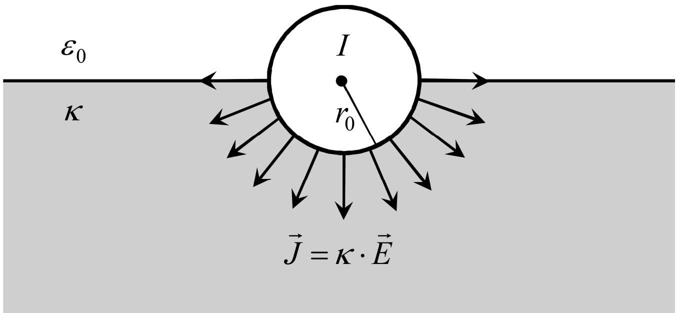


Figure 15.9. Solid perfectly conducting sphere on the surface of LIH soil

The surface density of the electric current over the hemisphere that is in direct contact with the soil is equal in magnitude to that in the case of a sphere buried in an unbounded LIH soil, which leaks a stationary electric current 2·*I*. The boundary condition at the soil surface (outside the sphere) requires that no electric current flows from the soil into the air, i.e., at the soil surface:

$$\frac{\partial \varphi}{\partial n} = 0 \tag{15.44}$$

and this boundary condition is satisfied by mirroring the electric current leaked by the hemisphere into the soil with respect to the soil surface (Figure 15.10), and the solution is valid only in the lower half-space occupied by the soil. In this case, the reflection factor is given by the expression:

$$k_R = \frac{\kappa - \kappa_{air}}{\kappa_1 + \kappa_{air}} = 1 \quad ; \quad \kappa_{air} = 0 \tag{15.45}$$

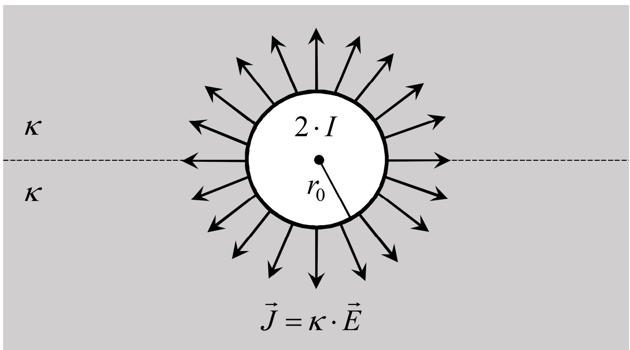


Figure 15.10. Hemisphere and its image with respect to the surface of LIH soil

According to expression (15.26), the electric scalar potential of the sphere is:

$$\varphi_{sphere} = \frac{2 \cdot I}{4 \cdot \pi \cdot \kappa \cdot r_0} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot r_0} \tag{15.46}$$

thus, the grounding resistance of the solid perfectly conducting sphere on the surface of LIH soil is:

$$R_{sphere} = \frac{\varphi_{sphere}}{I} = \frac{1}{2 \cdot \pi \cdot \kappa \cdot r_0} \tag{15.47}$$

From expressions (15.27) and (15.47), it follows that the grounding resistance of a sphere on the surface of LIH soil is twice that of a sphere buried in unbounded LIH soil. This is physically reasonable, since when the sphere is on the soil surface, only half of the surrounding unbounded soil conducts the electric current.

It follows that the distribution of the electric scalar potential in the soil is described by the expression:

$$\varphi = \frac{2 \cdot I}{4 \cdot \pi \cdot \kappa \cdot r} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot r} \tag{15.48}$$

where r is the distance from the center of the sphere to the field point in the soil. The distribution of the electric scalar potential in the air is described by the following expression:

$$\varphi = \lim_{\kappa_{\text{air}} \to 0} \frac{k_{\text{T}} \cdot I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot r} = \lim_{\kappa_{\text{air}} \to 0} \frac{I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot r} \cdot \frac{2 \cdot \kappa_{\text{air}}}{\kappa_{\text{air}} + \kappa} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot r} \tag{15.49}$$

where r is the distance from the center of the sphere to the field point in the air. From expressions (15.48) and (15.49), it follows that, in this particular case, the distribution of the electric scalar potential in both the air and the soil is described by the same expression.

The electric scalar potential must be continuous throughout the entire space, and the electric scalar potential in the air is produced by a stationary electric charge located at the boundary between the conductor (the sphere or the soil) and the dielectric (air). In the general case, the electrostatic field in the air is defined by the electric scalar potential at the air-soil boundary, which generates a stationary current field in the soil. In this particular case, due to the central symmetry of the sphere, the continuity of the potential can be ensured by an electric charge Q located at the boundary between the sphere and the air (Figure 15.11). There is no stationary electric charge at the boundary between the soil and the air because, according to expressions (15.48) and (15.50), the normal component of the electric field intensity at the soil surface is zero in both media.

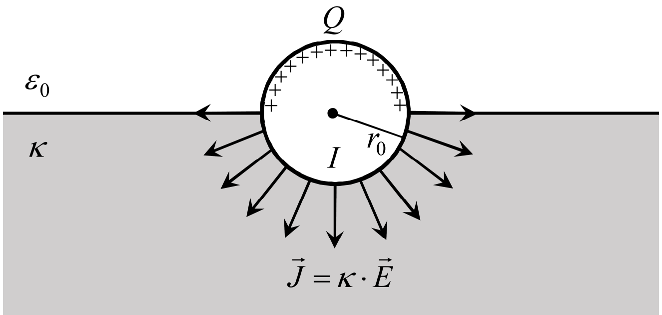


Figure 15.11. Stationary electric charge at the boundary between the sphere and air

In order to ensure the continuity of the electric scalar potential, the distribution of the electric scalar potential in the air must be described by the following expression (Figure 15.12):

$$\varphi = \frac{2 \cdot Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot r} = \frac{Q}{2 \cdot \pi \cdot \varepsilon_0 \cdot r} \equiv \frac{I}{2 \cdot \pi \cdot \kappa \cdot r} \quad ; \quad \frac{Q}{\varepsilon_0} = \frac{I}{\kappa} \tag{15.50}$$

where r is the distance from the center of the sphere to the field point in the air.

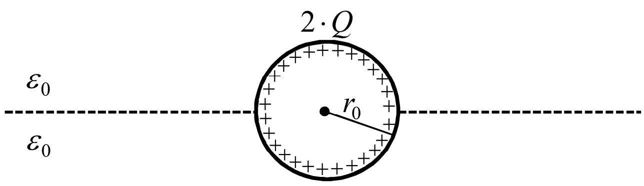


Figure 15.12. Real and image electric charge when the distribution of the electric scalar potential is calculated in the air

Expression (15.50) can also be derived from Gauss's law, taking into account the central symmetry of the electric field in the entire space (Figure 15.13). In this case, Gauss's law is given by:

$$\oint_S \vec{D} \cdot d\vec{S} = \varepsilon_0 \cdot E \cdot 2 \cdot \pi \cdot r^2 = Q \tag{15.51}$$

from which it follows that the electric field intensity vector in the air, in the spherical coordinate system, is described by the following expression:

$$\vec{E} = \frac{Q}{2 \cdot \pi \cdot \varepsilon_0 \cdot r^2} \cdot \vec{e}_\mathrm{r} \tag{15.52}$$

thus, the distribution of the electric scalar potential in the air is described by the following expression:

$$\varphi = \int_r^\infty E \cdot dr = \frac{Q}{2 \cdot \pi \cdot \varepsilon_0 \cdot r} = \frac{2 \cdot Q}{4 \cdot \pi \cdot \varepsilon_0 \cdot r} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot r} \tag{15.53}$$

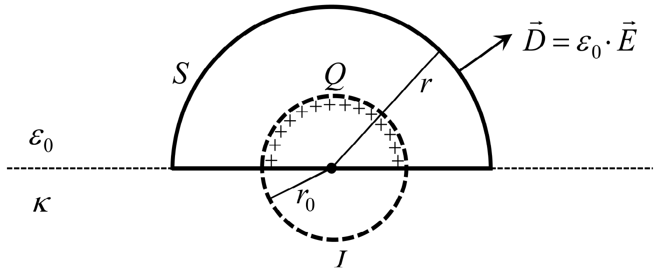


Figure 15.13. Stationary electric charge and integration surface S enclosing the upper hemisphere located in the air

15.8. Grounding Resistance of a Solid Perfectly Conducting Sphere Buried in the Soil and Potential Distribution in the Air

Let a solid perfectly conducting sphere of radius *a* be buried in LIH soil of electrical conductivity κ, at a depth *h* below the soil surface (Figure 15.14). The sphere leaks a stationary electric current *I* into the surrounding soil from its surface, and let $r_0 << h$. The conductor supplying current to the buried sphere is insulated from the soil, meaning that the current supply path is not taken into account. The task is to calculate the grounding resistance of such a sphere, valid for a stationary current field.

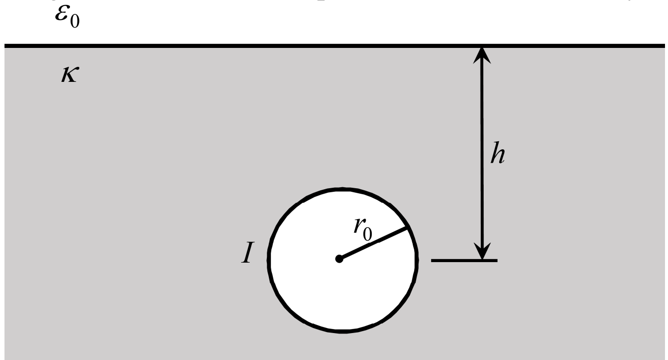


Figure 15.14. Solid perfectly conducting sphere buried in LIH soil

The boundary condition at the soil surface is the requirement that no electric current flows from the soil into the air, as described by expression (15.44). According to the method of images, this boundary condition is satisfied by reflecting the electric current leaked by the sphere into the soil with respect to the soil surface (Figure 15.15), and the resulting solution is valid only in the soil. In this case, the reflection factor is equal to one.

According to the method of images, the sphere in LIH soil is replaced by two spheres in an unbounded conducting LIH medium of the same conductivity κ. If the assumption that $r_0 << h$ is taken into account, then the distribution of the electric scalar potential in the soil is given by the expression:

$$\varphi = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{R} + \frac{1}{R_\mathrm{s}} \right) \tag{15.54}$$

where R is the distance from the field point in the soil to the center of the real sphere, whereas R_s is the distance from the field point to the image sphere.

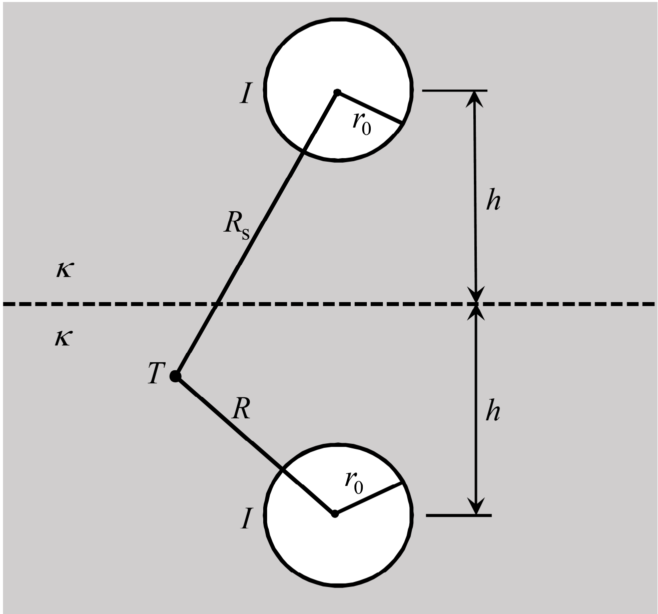


Figure 15.15. Real sphere and its image with respect to the soil surface

The electric scalar potential of the sphere is described by the following expression:

$$\varphi_{\text{sphere}} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{r_0} + \frac{1}{2 \cdot h} \right) \tag{15.55}$$

According to expression (15.55), the contribution of the electric current from the image sphere to the electric scalar potential of the real sphere is approximated by the contribution of the electric current from the image sphere to the electric scalar potential at the center of the real sphere.

The distribution of the electric scalar potential in the air is described by the expression (Figure 15.16):

$$\varphi = \lim_{\kappa_{\text{air}} \to 0} \frac{k_T \cdot I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot R} = \lim_{\kappa_{\text{air}} \to 0} \frac{I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot R} \cdot \frac{2 \cdot \kappa_{\text{air}}}{\kappa_{\text{air}} + \kappa} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot R} \tag{15.56}$$

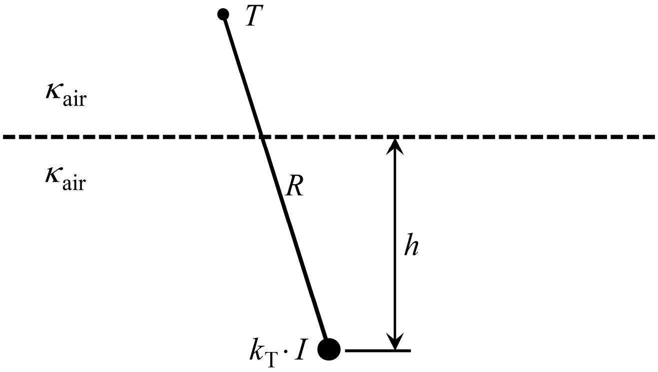


Figure 15.16. Image point source of electric current generating the electric scalar potential in the air

Thus, the distribution of the electric scalar potential in the air can be calculated by assuming that the entire space is filled with soil of electrical conductivity κ, and that a sphere located at the position of the real one leaks an electric current $2 \cdot I$ into the unbounded soil (Figure 15.17).

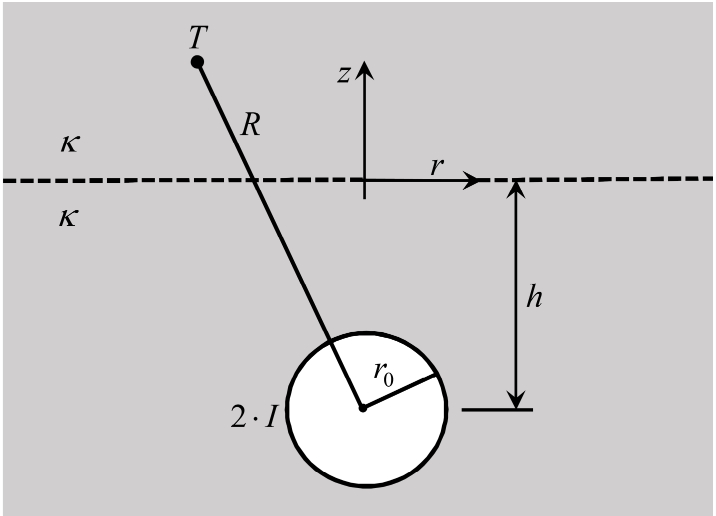


Figure 15.17. Image source of electric current for calculating the electric scalar potential in the air

The surface density of the stationary electric charge on the soil surface follows from the boundary condition:

$$\sigma = D_{\mathrm{z,air}}\Big|_{z=0} - D_{\mathrm{z,soil}}\Big|_{z=0} = \varepsilon_0 \cdot E_{\mathrm{z,air}}\Big|_{z=0} \quad ; \quad D_{\mathrm{z,soil}}\Big|_{z=0} = 0 \tag{15.57}$$

From expression (15.56), it is straightforward to obtain that the axisymmetric electric field intensity in the air, in the cylindrical coordinate system (r, ϕ, z) shown in Figure 15.17, is described by the expression:

$$\vec{E}_{\mathrm{air}} = = \frac{I \cdot \left[r \cdot \vec{\mathrm{e}}_{\mathrm{r}} + (z+h) \cdot \vec{\mathrm{e}}_{\mathrm{z}}\right]}{2 \cdot \pi \cdot \kappa \cdot R^3} \tag{15.58}$$

where, according to Figure 15.17:

$$R = \sqrt{r^2 + (z+h)^2} \tag{15.59}$$

From expressions (15.58) and (15.59), it follows that the z-component of the electric field intensity in the air is described by the expression:

$$E_{\mathrm{z,\,air}} = \frac{I \cdot (z+h)}{2 \cdot \pi \cdot \kappa \cdot R^3} = \frac{I \cdot (z+h)}{2 \cdot \pi \cdot \kappa \cdot \left[r^2 + (z+h)^2\right]^{3/2}} \tag{15.60}$$

If expression (15.60) is substituted into expression (15.57), the final expression for the surface density of the stationary electric charge on the soil surface is obtained as follows:

$$\sigma = \varepsilon_0 \cdot E_{\mathrm{z,air}}\Big|_{z=0} = \frac{\varepsilon_0 \cdot I \cdot h}{2 \cdot \pi \cdot \kappa \cdot \sqrt{\left(r^2 + h^2\right)^3}} \tag{15.61}$$

If the surface charge density is integrated over the soil surface, the total amount of stationary electric charge at the soil-air boundary is obtained as follows:

$$Q = \int_0^{\infty} \sigma \cdot 2 \cdot \pi \cdot r \cdot dr = \frac{\varepsilon_0 \cdot I \cdot h}{\kappa} \cdot \int_0^{\infty} \frac{r \cdot dr}{\sqrt{\left(r^2 + h^2\right)^3}} = \frac{\varepsilon_0 \cdot I}{\kappa} \tag{15.62}$$

from which it follows that:

$$Q = \frac{\varepsilon_0 \cdot I}{\kappa} \tag{15.63}$$

The distribution of the electric scalar potential in the air, along the z-axis, is described by the integral:

$$\varphi(z) = \frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \int_S \frac{\sigma \cdot dS}{\sqrt{r^2 + z^2}} = \frac{1}{2 \cdot \pi \cdot \varepsilon_0} \cdot \int_0^{\infty} \frac{\sigma \cdot 2 \cdot \pi \cdot r \cdot dr}{\sqrt{r^2 + z^2}} \tag{15.64}$$

If expression (15.61) is substituted into expression (15.64), it follows that:

$$\varphi(z) = \frac{I \cdot h}{2 \cdot \pi \cdot \kappa} \cdot \int_0^{\infty} \frac{r \cdot dr}{\sqrt{\left(r^2 + h^2\right)^3} \cdot \sqrt{r^2 + z^2}} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot (z + h)} \tag{15.65}$$

which can also be obtained from expression (15.56), using the substitution:

$$R = z + h \tag{15.66}$$

15.9. Cylindrical Conductor in a Homogeneous Stationary Current Field

Let an infinitely long solid conducting cylinder (LIH medium 1) of radius a be placed in a stationary current field that is homogeneous far from the cylinder, whereas the surrounding medium is an unbounded conducting LIH medium (medium 2). To solve this problem, a cylindrical coordinate system with its origin on the axis of the conducting cylinder is appropriate (Figure 15.18).

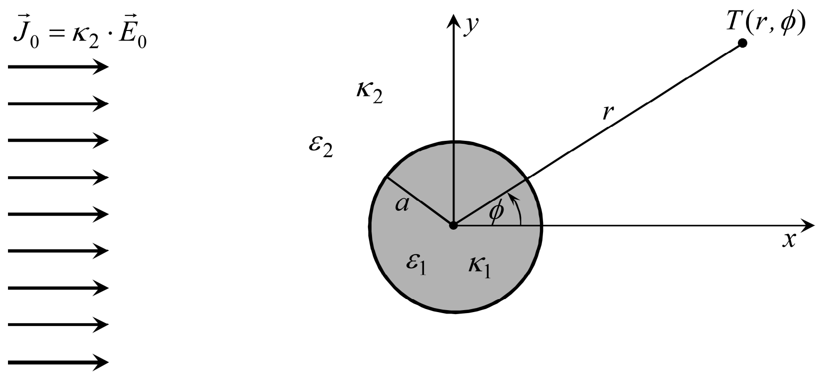


Figure 15.18. Solid conducting cylinder in a homogeneous stationary current field

Let the permittivities and electrical conductivities of both conducting media be given. Let a stationary electric current, perpendicular to the conducting cylinder, flow through both media such that the stationary current field is homogeneous far from the cylinder. The boundary condition at large distances from the cylinder is that there is no distortion of the field, i.e., it remains homogeneous. According to expression (14.183), the general solution of the Laplace differential equation in medium 2, which must satisfy this condition, is:

$$\varphi(r, \phi) = \sum_{\mathrm{n}=1}^{\infty} \left(A_{\mathrm{n}} \cdot r^n + B_{\mathrm{n}} \cdot \frac{1}{r^n} \right) \cdot \left[C_{\mathrm{n}} \cdot \sin(n \cdot \phi) + D_{\mathrm{n}} \cdot \cos(n \cdot \phi) \right] \tag{15.67}$$

The distribution of the electric scalar potential is axisymmetric with respect to the x-axis, i.e., it is an even function with respect to the angle ϕ. Therefore:

$$C_{\mathrm{n}} = 0 \quad ; \quad \forall \mathrm{n} \tag{15.68}$$

and the general solution of the Laplace differential equation (15.67) takes on a new form:

$$\varphi(r,\phi) = \sum_{\mathrm{n}=1}^{\infty} \left(A_{\mathrm{n}} \cdot r^{n} + B_{\mathrm{n}} \cdot \frac{1}{r^{n}} \right) \cdot \cos(n \cdot \phi) \tag{15.69}$$

The subsequent procedure for deriving the expression for the distribution of the electric scalar potential is similar to the one used in deriving the potential distribution for a solid dielectric sphere in a homogeneous electric field.

From the general solution of the Laplace differential equation (15.69), it follows that the initial expression for the distribution of the electric scalar potential in medium 2 is:

$$\varphi_2 = \sum_{\mathrm{n}=1}^{\infty} \left(A_{\mathrm{n}} \cdot r^{n} + B_{\mathrm{n}} \cdot \frac{1}{r^{n}} \right) \cdot \cos(n \cdot \phi) \tag{15.70}$$

It is sufficient to match the radial component of the electric field at infinity, which, according to Figure 15.19, is described by the following expression:

$$E_0 \cdot \cos\phi = - \left. \frac{\partial \varphi_2}{\partial r} \right|_{r \to \infty} \tag{15.71}$$

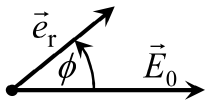


Figure 15.19. Radial component of the electric field intensity far from the cylinder

If expression (15.70) is substituted into expression (15.71), it follows that:

$$E_0 \cdot \cos\phi = - \left\{ \sum_{\mathrm{n}=1}^{\infty} \left(A_{\mathrm{n}} \cdot n \cdot r^{n-1} - B_{\mathrm{n}} \cdot \frac{n}{r^{n+1}} \right) \cdot \cos(n \cdot \phi) \right\}_{r \to \infty} \tag{15.72}$$

From expression (15.72) it follows that $n = 1$, which means that all coefficients A_{n} and B_{n} are equal to zero, except for the coefficients A_1 and B_1, so from expression (15.72) we obtain:

$$E_0 \cdot \cos\phi = - \left(A_1 - B_1 \cdot \frac{1}{r^2} \right)_{r \to \infty} \cdot \cos\phi = - A_1 \cdot \cos\phi \tag{15.73}$$

which means that:

$$A_1 = - E_0 \tag{15.74}$$

whereas the coefficient B_1 is currently unknown, and the term containing B_1 tends to zero as r approaches infinity.

Thus, the distribution of the electric scalar potential in medium 2 is described by the expression:

$$\varphi_2 = \left(- E_0 \cdot r + B_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{15.75}$$

and, to satisfy the boundary conditions between the two media, the electric scalar potential inside the solid conducting cylinder must be described by the expression:

$$\varphi_1 = \left(C_1 \cdot r + D_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{15.76}$$

From the condition that the electric scalar potential φ_1 on the axis of the solid conducting cylinder must be finite, it follows that the coefficient:

$$D_1 = 0 \tag{15.77}$$

and:

$$\varphi_1 = C_1 \cdot r \cdot \cos\phi \tag{15.78}$$

Thus, the distribution of the electric scalar potential in both media is described by expressions (15.75) and (15.78), where two coefficients are unknown, and they can be determined based on the satisfaction of the boundary conditions

$$\varphi_1\big|_{r=a} = \varphi_2\big|_{r=a} \tag{15.79}$$

$$-\kappa_1 \cdot \frac{\partial \varphi_1}{\partial r}\bigg|_{r=a} = -\kappa_2 \cdot \frac{\partial \varphi_2}{\partial r}\bigg|_{r=a} \tag{15.80}$$

which yields the following system of linear equations:

$$B_1 - a^2 \cdot C_1 = E_0 \cdot a^2 \qquad ; \qquad B_1 + \frac{\kappa_1}{\kappa_2} \cdot a^2 \cdot C_1 = -E_0 \cdot a^2 \tag{15.81}$$

from which it easily follows that:

$$B_1 = E_0 \cdot a^2 \cdot \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \qquad ; \qquad C_1 = -E_0 \cdot \frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \tag{15.82}$$

After satisfying all boundary conditions, according to expressions (15.75), (15.78), and (15.82), the potential distribution in both media is described by the following expressions:

$$\varphi_1 = -\frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot r \cdot \cos\phi \; = \; -\frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot x \quad ; \quad r \le a \tag{15.83}$$

$$\varphi_2 = -E_0 \cdot r \cdot \cos\phi \; + \; \frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \cdot \left(\frac{a}{r} \right)^2 \cdot E_0 \cdot r \cdot \cos\phi \quad ; \quad r \ge a \tag{15.84}$$

from which it follows that the electric scalar potential is equal to zero on the plane passing through the center of the cylinder for which $\phi = \pm\pi/2$, i.e., and that is the plane $x = 0$.

From expression (15.83), which describes the distribution of the electric scalar potential in medium 1, the expressions for the non-zero components of the electric field in medium 1 can be easily obtained, which, in the cylindrical coordinate system, are given by:

$$\vec{E}_1 = E_{1\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{1\phi} \cdot \vec{\mathrm{e}}_\phi \tag{15.85}$$

$$E_{1\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = \frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot \cos\phi \tag{15.86}$$

$$E_{1\phi} = -\frac{1}{r}\cdot\frac{\partial\varphi_1}{\partial\phi} = -\frac{2\cdot\kappa_2}{\kappa_1+\kappa_2}\cdot E_0\cdot\sin\phi \tag{15.87}$$

whereas in the Cartesian coordinate system:

$$\vec{E}_1 = E_{1x}\cdot\vec{\mathrm{i}} = \frac{2\cdot\kappa_2}{\kappa_1+\kappa_2}\cdot E_0\cdot\vec{\mathrm{i}} \tag{15.88}$$

from which it follows that, as expected, the magnitude of the electric field inside the solid cylinder has changed, but the electric field has remained homogeneous.

From expression (15.84), which describes the distribution of the electric scalar potential in medium 2, the expressions for the non-zero components of the electric field intensity in medium 2 can easily be obtained, which in the cylindrical coordinate system are given by:

$$\vec{E}_2 = E_{2\mathrm{r}}\cdot\vec{\mathrm{e}}_{\mathrm{r}} + E_{2\phi}\cdot\vec{\mathrm{e}}_{\phi} \tag{15.89}$$

$$E_{2\mathrm{r}} = -\frac{\partial\varphi_2}{\partial r} = E_0\cdot\cos\phi\cdot\left(1+\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\cdot\left(\frac{a}{r}\right)^2\right) \tag{15.90}$$

$$E_{2\phi} = -\frac{1}{r}\cdot\frac{\partial\varphi_2}{\partial\phi} = -E_0\cdot\sin\phi\cdot\left(1-\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}\cdot\left(\frac{a}{r}\right)^2\right) \tag{15.91}$$

The surface density of the stationary electric charge at the boundary between the two media follows from the expression:

$$D_{2\mathrm{r}}\big|_{r=a} - D_{1\mathrm{r}}\big|_{r=a} = \varepsilon_2\cdot E_{2\mathrm{r}}\big|_{r=a} - \varepsilon_1\cdot E_{1\mathrm{r}}\big|_{r=a} = \sigma \tag{15.92}$$

where ε_1 and ε_2 are the permittivities of the conducting media (Figure 15.18).

From expressions (15.86), (15.90), and (15.92), it follows that the surface density of the stationary electric charge is given by the expression:

$$\sigma = \left(\varepsilon_2+\varepsilon_2\cdot\frac{\kappa_1-\kappa_2}{\kappa_1+\kappa_2}-\varepsilon_1\cdot\frac{2\cdot\kappa_2}{\kappa_1+\kappa_2}\right)\cdot E_0\cdot\cos\phi \tag{15.93}$$

from which it follows that:

$$\sigma = \frac{\kappa_1\cdot\varepsilon_2-\kappa_2\cdot\varepsilon_1}{\kappa_1+\kappa_2}\cdot 2\cdot E_0\cdot\cos\phi \tag{15.94}$$

Definition: Let electric current lines be the lines along which electric current flows in a conducting medium. They are analogous to electric field lines in a dielectric. The electric field intensity is tangential to both the electric current lines and the electric field lines.

If $\kappa_1 > \kappa_2$, then the electric current lines have the same graphical appearance as the electric field lines shown in Figure 14.36. If $\kappa_1 < \kappa_2$, then the electric current lines have the same graphical appearance as the electric field lines shown in Figure 14.37.

Note: From the analogy between the electrostatic and stationary current field, it follows that, for the same geometry of the problem, the distribution of the electric scalar potential in the case of a solid dielectric cylinder surrounded by a perfect LIH dielectric and placed in a homogeneous electric field is described by expressions that can be obtained from expressions (15.83) and (15.84) by making the substitutions:

$$\kappa_1\to\varepsilon_1 \quad ; \quad \kappa_2\to\varepsilon_2 \tag{15.95}$$

15.10. Cylindrical Dielectric in a Homogeneous Stationary Current Field

Let an infinitely long solid dielectric cylinder (medium 1) of radius a be placed in a homogeneous stationary current field, whereas the surrounding space is filled with an unbounded conducting LIH medium (medium 2). A suitable coordinate system for solving this problem is the cylindrical coordinate system with its origin on the axis of the dielectric cylinder (Figure 15.20).

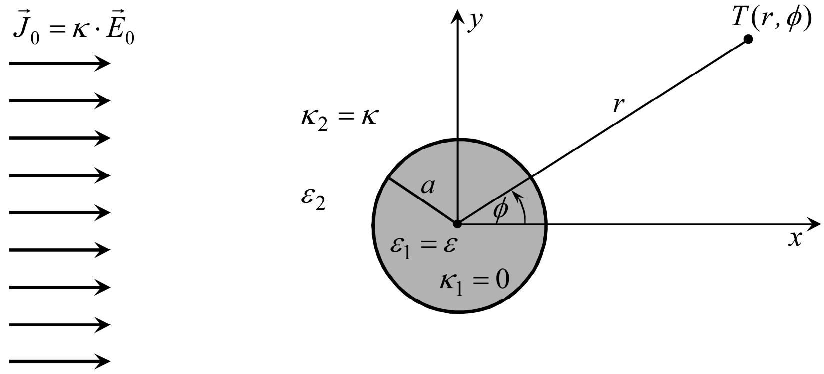


Figure 15.20. Solid dielectric cylinder in a homogeneous stationary current field

The expressions for the distribution of the electric scalar potential in both media can be obtained in two ways:

- In the same manner as for the solid conducting cylinder in a homogeneous stationary current field (subchapter 15.8), leading to expressions (15.75) and (15.78), which are given by:

$$\varphi_1 = C_1 \cdot r \cdot \cos\phi \tag{15.96}$$

$$\varphi_2 = \left(- E_0 \cdot r + B_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{15.97}$$

  and then the unknown constants B_1 and C_1 are determined from the boundary conditions:

$$\varphi_1 \big|_{r=a} = \varphi_2 \big|_{r=a} \tag{15.98}$$

$$- \kappa_2 \cdot \frac{\partial \varphi_2}{\partial r} \bigg|_{r=a} = - \kappa \cdot \frac{\partial \varphi_2}{\partial r} \bigg|_{r=a} = 0 \tag{15.99}$$

  where the boundary condition (15.99) means that the current field does not enter the solid dielectric cylinder, i.e., the electric current lines bypass the dielectric.

- Using the solutions for the distribution of the electric scalar potential (15.83) and (15.84), which apply to a solid conducting cylinder (medium 1) in a homogeneous stationary current field, with the substitutions:

$$\kappa_2 \to \kappa \quad ; \quad \kappa_1 \to 0 \tag{15.100}$$

Using either of the two described methods, the expressions for the distribution of the electric scalar potential in both media can be easily obtained and are given by:

$$\varphi_1 = - 2 \cdot E_0 \cdot r \cdot \cos\phi = - 2 \cdot E_0 \cdot x \quad ; \quad r \le a \tag{15.101}$$

$$\varphi_2 = - E_0 \cdot r \cdot \left(1 + \frac{a^2}{r^2} \right) \cdot \cos\phi = - E_0 \cdot x \cdot \left(1 + \frac{a^2}{r^2} \right) \quad ; \quad r \ge a \tag{15.102}$$

from which it follows that the electric scalar potential is equal to zero on the plane that passes through the center of the cylinder, where $\phi = \pm \pi / 2$, and that is the plane $x = 0$.

According to expressions (15.101) and (15.102), the distribution of the electric scalar potential in the solid dielectric cylinder, as well as in the surrounding conducting medium, does not depend on the properties of these two media. Physically, it is important only that medium 1 is a perfect LIH dielectric, and medium 2 is a conducting LIH medium.

From expression (15.101), which describes the distribution of the electric scalar potential in medium 1, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 1. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_1 = E_{1\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{1\phi} \cdot \vec{\mathrm{e}}_\phi \tag{15.103}$$

$$E_{1\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = 2 \cdot E_0 \cdot \cos\phi \tag{15.104}$$

$$E_{1\phi} = -\frac{1}{r} \cdot \frac{\partial \varphi_1}{\partial \phi} = -2 \cdot E_0 \cdot \sin\phi \tag{15.105}$$

whereas in the Cartesian coordinate system:

$$\vec{E}_1 = E_{1\mathrm{x}} \cdot \vec{\mathrm{i}} = 2 \cdot E_0 \cdot \vec{\mathrm{i}} \qquad (15106)$$

from which it follows that the electric field intensity in the solid cylinder has doubled in magnitude, but the electric field has remained homogeneous.

From expression (15.102), which describes the distribution of the electric scalar potential in medium 2, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 2. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\phi} \cdot \vec{\mathrm{e}}_\phi \tag{15.107}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\phi \cdot \left(1 - \frac{a^2}{r^2}\right) \tag{15.108}$$

$$E_{2\phi} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \phi} = -E_0 \cdot \sin\phi \cdot \left(1 + \frac{a^2}{r^2}\right) \tag{15.109}$$

The surface density of the stationary electric charge at the boundary of two media follows from the expression:

$$D_{2\mathrm{r}}\big|_{r=a} - D_{1\mathrm{r}}\big|_{r=a} = -\varepsilon_1 \cdot E_{1\mathrm{r}}\big|_{r=a} = -\varepsilon \cdot E_{1\mathrm{r}}\big|_{r=a} = \sigma \tag{15.110}$$

From expressions (15.104) and (15.110), it follows that the surface density of the stationary electric charge is described by the expression:

$$\sigma = -2 \cdot \varepsilon \cdot E_0 \cdot \cos\phi \tag{15.111}$$

Thus, on the left side of the cylinder, the surface charge is positive, whereas on the right side of the cylinder it is negative. This surface charge creates an electric field inside the solid dielectric cylinder.

15.11. Solid Perfectly Conducting Cylinder in a Homogeneous Stationary Current Field

Let an infinitely long solid perfectly conducting cylinder (medium 1) with radius a be placed in a homogeneous stationary current field, whereas the surrounding medium around the cylinder is an unbounded conducting LIH medium (medium 2). For solving this problem, a cylindrical coordinate system with the origin on the axis of the conducting cylinder is appropriate (Figure 15.21).

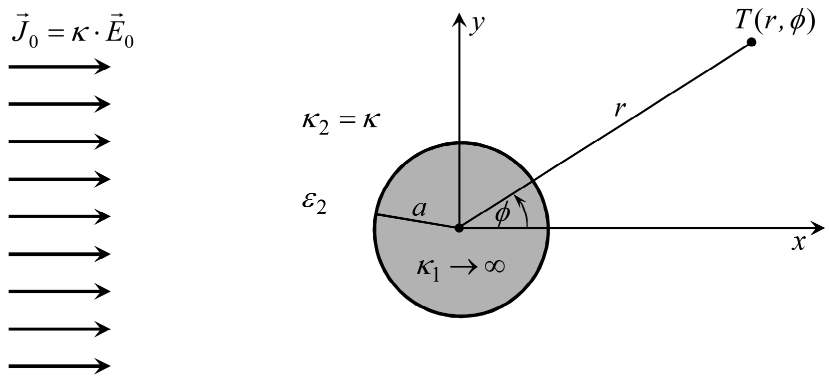


Figure 15.21. Solid perfectly conducting cylinder in a homogeneous stationary current field

The expressions for the distribution of the electric scalar potential in both media can be obtained in two ways:

- In the same manner as for the solid conducting cylinder in a homogeneous stationary current field (subchapter 15.8), leading to expression (15.75), which is given by:

$$\varphi_2 = \left(-E_0 \cdot r + B_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{15.112}$$

  and then the unknown constant B_1 is determined from the boundary condition:

$$\varphi_1 \big|_{r=a} = \varphi_2 \big|_{r=a} = 0 \tag{15.113}$$

  because in the solid perfectly conducting cylinder is:

$$\varphi_1 = \text{const.} = 0 \quad ; \quad \vec{E}_1 = 0 \tag{15.114}$$

  where, in this special case, the constant electric scalar potential of the solid perfectly conducting cylinder is equal to zero.

- Using the solutions for the distribution of the electric scalar potential (15.83) and (15.84), which apply to a solid conducting cylinder (medium 1) in a homogeneous stationary current field, with the substitutions:

$$\kappa_2 \to \kappa \quad ; \quad \kappa_1 \to \infty \tag{15.115}$$

Using either of the two described methods, the expressions for the distribution of the electric scalar potential in both media can be easily obtained and are given by:

$$\varphi_1 = 0 \quad ; \quad r \le a \tag{15.116}$$

$$\varphi_2 = -E_0 \cdot r \cdot \left(1 - \frac{a^2}{r^2} \right) \cdot \cos\phi = -E_0 \cdot x \cdot \left(1 - \frac{a^2}{r^2} \right) \quad ; \quad r \ge a \tag{15.117}$$

from which it follows that the electric scalar potential is equal to zero on the plane that passes through the center of the cylinder, where $\phi = \pm \pi / 2$, and that is the plane $x = 0$.

According to expressions (15.115) and (15.116), the distribution of the electric scalar potential in both media does not depend on the properties of the two media. Physically, it is only important that medium 1 is a perfectly conducting medium, and medium 2 is a conducting LIH medium.

From expression (15.116), which describes the distribution of the electric scalar potential in medium 2, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 2. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_2 = E_{2r} \cdot \vec{e}_r + E_{2\phi} \cdot \vec{e}_\phi \tag{15.118}$$

$$E_{2r} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\phi \cdot \left(1 + \frac{a^2}{r^2}\right) \tag{15.119}$$

$$E_{2\phi} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \phi} = -E_0 \cdot \sin\phi \cdot \left(1 - \frac{a^2}{r^2}\right) \tag{15.120}$$

The surface density of the stationary electric charge at the boundary of two media follows from the expression:

$$D_{2r}\big|_{r=a} - D_{1r}\big|_{r=a} = D_{2r}\big|_{r=a} = \varepsilon_2 \cdot E_{2r}\big|_{r=a} = \sigma \tag{15.121}$$

From expressions (15.119) and (15.121), it follows that the surface density of the stationary electric charge is described by the expression:

$$\sigma = 2 \cdot \varepsilon_2 \cdot E_0 \cdot \cos\phi \tag{15.122}$$

from which it follows that the surface charge on the left side of the cylinder is negative, whereas on the right side it is positive.

There is an analogy between a solid perfectly conducting cylinder in a conducting medium and a solid conducting cylinder in a dielectric medium. In these cases, the electric current lines and electric field lines, as well as the equipotential lines, have the same graphical appearance as the electric field lines and equipotential lines in the case of a solid conducting sphere in a dielectric medium, as shown in Figure 14.40.

15.12. Solid Conducting Sphere in a Homogeneous Stationary Current Field

Let a solid conducting sphere (medium 1), with radius a, be placed in a homogeneous stationary current field, whereas the surrounding space around the sphere is filled with an unbounded conducting LIH medium (medium 2). To solve this problem, a spherical coordinate system is used (Figure 15.22).

There is an analogy between a solid conducting sphere in a stationary current field and a solid dielectric sphere in a homogeneous electrostatic field. It is only necessary to replace the permittivities of the media with the electrical conductivities of the media.

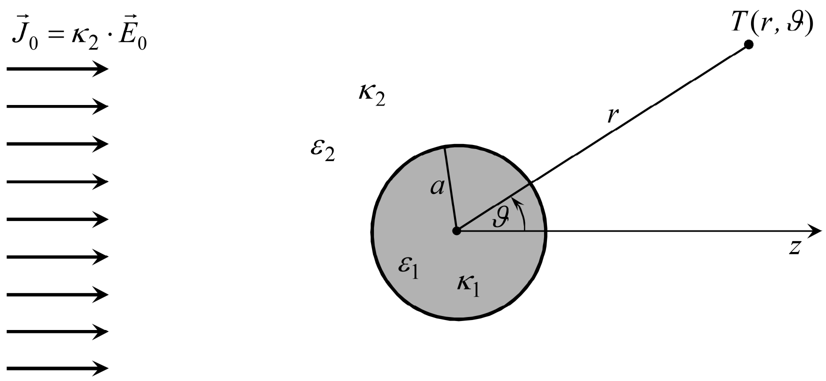


Figure 15.22. Solid conducting sphere in a homogeneous stationary current field

According to the analogous expressions (14.217) and (14.218), the distribution of the electric scalar potential in both media is given by the following expressions:

$$\varphi_1 = -\frac{3 \cdot \kappa_2}{\kappa_1 + 2 \cdot \kappa_2} \cdot E_0 \cdot r \cdot \cos\vartheta = -\frac{3 \cdot \kappa_2}{\kappa_1 + 2 \cdot \kappa_2} \cdot E_0 \cdot z \quad ; \quad r \le a \tag{15.123}$$

$$\varphi_2 = -E_0 \cdot r \cdot \cos\vartheta + \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot \left(\frac{a}{r}\right)^3 \cdot E_0 \cdot r \cdot \cos\vartheta \quad ; \quad r \geq a \tag{15.124}$$

from which it follows that the electric scalar potential is equal to zero at the center of the sphere, or on the plane passing through the center of the sphere where $\vartheta = \pi/2$, and that is the plane $z = 0$.

From expression (15.123), which describes the distribution of the electric scalar potential in medium 1, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 1. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_1 = E_{1\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{1\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{15.125}$$

$$E_{1\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = \frac{3\cdot\kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot E_0 \cdot \cos\vartheta \tag{15.126}$$

$$E_{1\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_1}{\partial \vartheta} = -\frac{3\cdot\kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot E_0 \cdot \sin\vartheta \tag{15.127}$$

whereas in the Cartesian coordinate system:

$$\vec{E}_1 = E_{1z} \cdot \vec{\mathrm{k}} = \frac{3\cdot\kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot E_0 \cdot \vec{\mathrm{k}} \tag{15.128}$$

From expression (15.124), which describes the distribution of the electric scalar potential in medium 2, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 2. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{15.129}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\vartheta \cdot \left(1 + 2 \cdot \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot \left(\frac{a}{r}\right)^3\right) \tag{15.130}$$

$$E_{2\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \vartheta} = -E_0 \cdot \sin\vartheta \cdot \left(1 - \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\cdot\kappa_2} \cdot \left(\frac{a}{r}\right)^3\right) \tag{15.131}$$

The surface density of the stationary electric charge at the boundary of two media follows from the expression:

$$D_{2\mathrm{r}}\big|_{r=a} - D_{1\mathrm{r}}\big|_{r=a} = \varepsilon_2 \cdot E_{2\mathrm{r}}\big|_{r=a} - \varepsilon_1 \cdot E_{1\mathrm{r}}\big|_{r=a} = \sigma \tag{15.132}$$

where ε_1 and ε_2 are the permittivities of the conducting media (Figure 15.12).

From expressions (15.126), (15.130), and (15.132), it follows that the surface density of the stationary electric charge is described by the expression:

$$\sigma = \left(\varepsilon_2 + 2\cdot\varepsilon_2 \cdot \frac{\kappa_1 - \kappa_2}{\kappa_1 + 2\cdot\kappa_2} - \varepsilon_1 \cdot \frac{3\cdot\kappa_2}{\kappa_1 + 2\cdot\kappa_2}\right) \cdot E_0 \cdot \cos\vartheta \tag{15.133}$$

from which it follows that:

$$\sigma = \frac{\kappa_1 \cdot \varepsilon_2 - \kappa_2 \cdot \varepsilon_1}{\kappa_1 + 2\cdot\kappa_2} \cdot 3 \cdot E_0 \cdot \cos\vartheta \tag{15.134}$$

15.13. Solid Dielectric Sphere in a Homogeneous Stationary Current Field

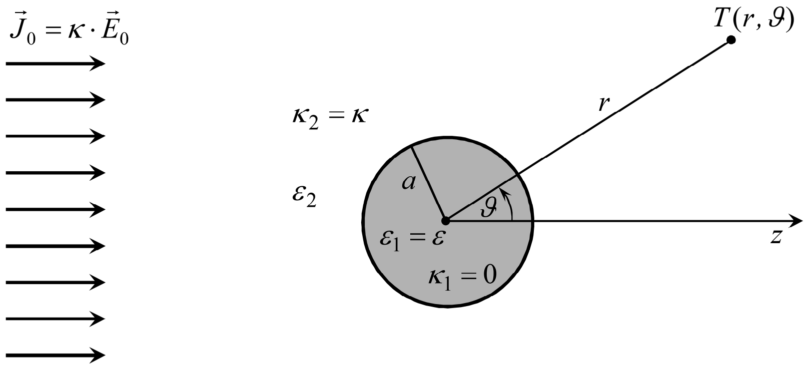


Figure 15.23. Solid dielectric sphere in a homogeneous stationary current field

Let a solid dielectric sphere (medium 1) of radius a be placed in a homogeneous stationary current field, whereas the space around the sphere is filled with an unbounded conducting LIH medium (medium 2). A spherical coordinate system with its origin at the center of the sphere is used (Figure 15.23).

The expressions for the electric scalar potentials in both media can be obtained in two ways:

- In the same manner as for a solid dielectric sphere in a homogeneous electrostatic field (subchapter 14.18), up to expressions (14.212) and (14.209), which are given by:

$$\varphi_1 = C_1 \cdot r \cdot \cos\vartheta \qquad (15.135)$$

$$\varphi_2 = \left(- E_0 \cdot r + B_1 \cdot \frac{1}{r^2} \right) \cdot \cos\vartheta \qquad (15.136)$$

  and after that, the unknown constants B_1 and C_1 are determined from the boundary conditions:

$$\varphi_1\big|_{r=a} = \varphi_2\big|_{r=a} \qquad (15.137)$$

$$- \kappa_2 \cdot \frac{\partial \varphi_2}{\partial r}\bigg|_{r=a} = - \kappa \cdot \frac{\partial \varphi_2}{\partial r}\bigg|_{r=a} = 0 \qquad (15.138)$$

  where the boundary condition (15.138) means that the current field does not penetrate the solid dielectric sphere, i.e., the electric current lines bypass the dielectric.

- Using the solutions for the distribution of the electric scalar potential (15.123) and (15.124), which are valid for a solid conducting sphere (medium 1) in a homogeneous stationary current field, with the substitutions:

$$\kappa_2 \to \kappa \quad ; \quad \kappa_1 \to 0 \qquad (15.139)$$

Using either of the two described methods, the expressions for the distribution of the electric scalar potential in both media can be easily obtained and are given by:

$$\varphi_1 = -\frac{3}{2} \cdot E_0 \cdot r \cdot \cos\vartheta = -\frac{3}{2} \cdot E_0 \cdot z \quad ; \quad r \le a \qquad (15.140)$$

$$\varphi_2 = - E_0 \cdot \left(1 + \frac{a^3}{2 \cdot r^3} \right) \cdot r \cdot \cos\vartheta \quad ; \quad r \ge a \qquad (15.141)$$

from which it follows that the electric scalar potential is equal to zero on the plane that passes through the center of the sphere, where $\vartheta = \pi / 2$, and that is the plane $z = 0$.

According to expressions (15.140) and (15.141), the distribution of the electric scalar potential in both media does not depend on the properties of the two media. Physically, it is important only that medium 1 is a perfect LIH dielectric, and medium 2 is a conducting LIH medium.

From expression (15.140), which describes the distribution of the electric scalar potential in medium 1, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 1. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_1 = E_{1\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{1\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{15.142}$$

$$E_{1\mathrm{r}} = -\frac{\partial \varphi_1}{\partial r} = \frac{3}{2} \cdot E_0 \cdot \cos\vartheta \tag{15.143}$$

$$E_{1\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_1}{\partial \vartheta} = -\frac{3}{2} \cdot E_0 \cdot \sin\vartheta \tag{15.144}$$

whereas in the Cartesian coordinate system:

$$\vec{E}_1 = E_{1z} \cdot \vec{\mathrm{k}} = \frac{3}{2} \cdot E_0 \cdot \vec{\mathrm{k}} \tag{15145}$$

from which it follows that the electric field intensity in the sphere has increased in magnitude, but the electric field remains homogeneous.

From expression (15.141), which describes the distribution of the electric scalar potential in medium 2, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 2. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{15.146}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\vartheta \cdot \left(1 - \frac{a^3}{r^3}\right) \tag{15.147}$$

$$E_{2\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \vartheta} = -E_0 \cdot \sin\vartheta \cdot \left(1 + \frac{a^3}{2 \cdot r^3}\right) \tag{15.148}$$

The surface density of the stationary electric charge at the boundary of two media follows from the expression:

$$D_{2\mathrm{r}}\big|_{r=a} - D_{1\mathrm{r}}\big|_{r=a} = -\varepsilon_1 \cdot E_{1\mathrm{r}}\big|_{r=a} = -\varepsilon \cdot E_{1\mathrm{r}}\big|_{r=a} = \sigma \tag{15.149}$$

From expressions (15.143) and (15.149), it follows that the surface density of the stationary electric charge is described by the expression:

$$\sigma = -\frac{3}{2} \cdot \varepsilon \cdot E_0 \cdot \cos\vartheta \tag{15.150}$$

which means that the surface charge on the left side of the sphere is positive, whereas on the right side it is negative. This surface charge creates an electric field inside the dielectric sphere.

15.14. Solid Perfectly Conducting Sphere in a Homogeneous Stationary Current Field

Let a solid perfectly conducting sphere (medium 1) of radius a be placed in a homogeneous stationary current field, whereas the surrounding space is filled with an unbounded conducting LIH medium (medium 2). To solve this problem, a spherical coordinate system with its origin at the center of the sphere is suitable (Figure 15.24).

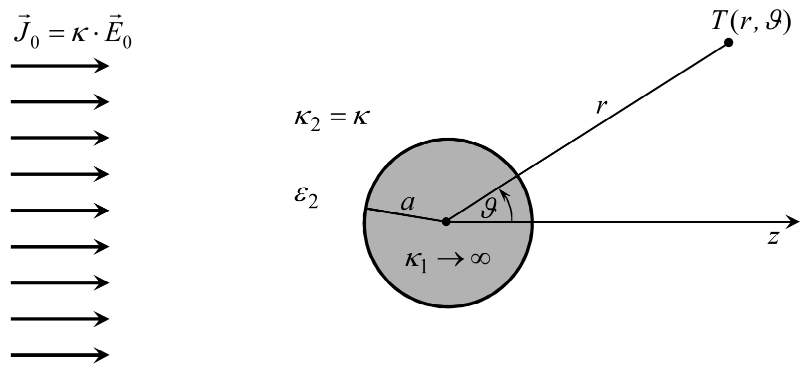


Figure 15.24. Solid perfectly conducting sphere in a homogeneous stationary current field

The expressions for the distribution of the electric scalar potential in both media can be obtained in two ways:

- In the same manner as for the solid dielectric sphere in a homogeneous electrostatic field (subchapter 14.18), up to expression (14.209), which is described by:

$$\varphi_2 = \left(-E_0 \cdot r + B_1 \cdot \frac{1}{r^2} \right) \cdot \cos\vartheta \tag{15.151}$$

  and after that, the unknown constant B_1 can be determined from the boundary condition:

$$\varphi_1\big|_{r=a} = \varphi_2\big|_{r=a} = 0 \tag{15.152}$$

  because in a solid perfectly conducting sphere:

$$\varphi_1 = \text{const.} = 0 \quad ; \quad \vec{E}_1 = 0 \tag{15.153}$$

- Using the solutions for the distribution of the electric scalar potential (15.123) and (15.124), which hold for a solid conducting sphere (medium 1) in a homogeneous stationary current field, with substitutions:

$$\kappa_2 \to \kappa \quad ; \quad \kappa_1 \to \infty \tag{15.154}$$

Using either of the two described methods, the expressions for the distribution of the electric scalar potential in both media can be easily obtained and are given by:

$$\varphi_1 = 0 \quad ; \quad r \le a \tag{15.155}$$

$$\varphi_2 = -E_0 \cdot r \cdot \cos\vartheta + \left(\frac{a}{r}\right)^3 \cdot E_0 \cdot r \cdot \cos\vartheta \quad ; \quad r \ge a \tag{15.156}$$

from which it follows that the electric scalar potential is equal to zero on the plane that passes through the center of the sphere, where $\vartheta = \pi/2$, and that is the plane $z = 0$.

According to expressions (15.155) and (15.156), the distribution of the electric scalar potential in both media does not depend on the properties of the two media. Physically, it is important only that medium 1 is a solid, perfectly conducting medium, and medium 2 is a conducting LIH medium.

From expression (15.156), which describes the distribution of the electric scalar potential in medium 2, it is easy to obtain the expressions for the non-zero components of the electric field intensity in medium 2. These expressions in the cylindrical coordinate system are as follows:

$$\vec{E}_2 = E_{2\mathrm{r}} \cdot \vec{\mathrm{e}}_\mathrm{r} + E_{2\vartheta} \cdot \vec{\mathrm{e}}_\vartheta \tag{15.157}$$

$$E_{2\mathrm{r}} = -\frac{\partial \varphi_2}{\partial r} = E_0 \cdot \cos\vartheta \cdot \left(1 + 2 \cdot \frac{a^3}{r^3}\right) \tag{15.158}$$

$$E_{2\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_2}{\partial \vartheta} = -E_0 \cdot \sin\vartheta \cdot \left(1 - \frac{a^3}{r^3}\right) \tag{15.159}$$

The surface density of the stationary electric charge at the boundary of two media follows from the expression:

$$D_{2r}\big|_{r=a} - D_{1r}\big|_{r=a} = D_{2r}\big|_{r=a} = \varepsilon_2 \cdot E_{2r}\big|_{r=a} = \sigma \tag{15.160}$$

From expressions (15.158) and (15.160), it follows that the surface density of the stationary electric charge is described by the expression:

$$\sigma = 3 \cdot \varepsilon_2 \cdot E_0 \cdot \cos\vartheta \tag{15.161}$$

which means that the surface charge on the left side of the sphere is negative, whereas on the right side it is positive.

There is an analogy between a solid perfectly conducting sphere in a conducting medium and a solid conducting sphere in a dielectric medium. In these cases, the electric current lines, electric field lines, and equipotential lines have the same graphical appearance as shown in Figure 14.40.

15.15. Thin-Wire Conductor in a Conducting Medium and the Average Potential Method

A straight segment of a solid cylindrical conductor with radius r_0 and length ℓ, in a surrounding conducting medium, leaks a stationary electric current *I* (Figure 15.25). Let the surface density of the electric current be constant along the surface of the conductor's segment. The conductor segment is located in an unbounded conducting LIH medium with electrical conductivity *κ*.

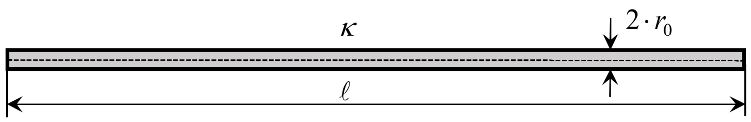


Figure 15.25. A straight segment of a cylindrical conductor in an unbounded conducting LIH medium

If $r_0 << \ell$, the thin-wire approximation can be used, which assumes that the electric current originates from the axis of the conductor segment, and the distribution of the electric scalar potential is sought at points outside the conductor segment, including the surface of the conductor segment. The potential inside the conductor segment is equal to the potential on its surface. Since in the numerical model the stationary electric current *I* leaks from the axis of the conductor segment, it is described by the linear density of the electric current:

$$\tau = \frac{I}{\ell} = \text{const.} \tag{15.162}$$

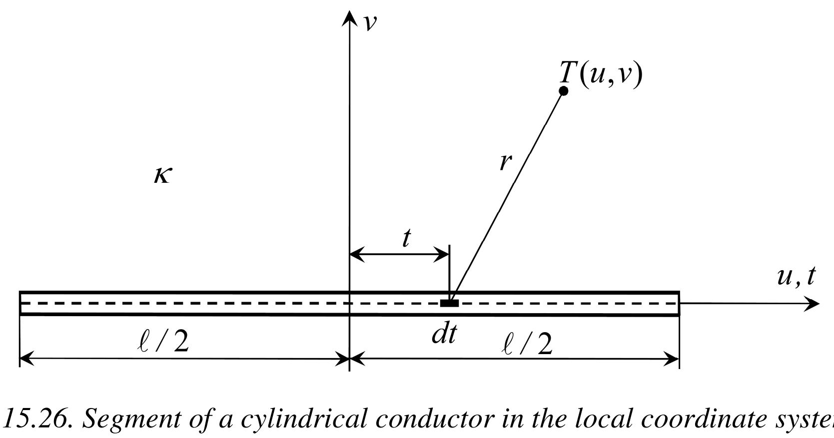


Figure 15.26. Segment of a cylindrical conductor in the local coordinate system (u, v)

For the sake of simplicity, the segment is given in the local 2D coordinate system (u, v), in which the axis of the conductor segment and the field point $T(u, v)$ lie. The position of the segment in the (x, y, z) coordinate system is arbitrary. According to Figure 15.26, the point $T(u, v)$ at which the electric scalar potential is computed can be in the first or second quadrant of the local coordinate system (u, v).

There is an analogy between the segment of a solid cylindrical conductor in an unbounded conducting LIH medium, which leaks a stationary electric current I into the surrounding conducting medium, and the segment of a solid cylindrical conductor in an unbounded perfect LIH dielectric, which is charged with charge Q. Since the analogous electrostatic problem is described in detail in subchapter 14.24, the segment of the cylindrical conductor in the unbounded conducting LIH medium is briefly described in this subchapter.

The distribution of the electric scalar field around the conductor segment is described by the expression:

$$\varphi = \frac{\tau}{4 \cdot \pi \cdot \kappa} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{dt}{\sqrt{(u-t)^2 + v^2}} = \frac{I}{4 \cdot \pi \cdot \kappa \cdot \ell} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{dt}{\sqrt{(u-t)^2 + v^2}} \tag{15.163}$$

and after integration, it is obtained that:

$$\varphi = \frac{I}{4 \cdot \pi \cdot \kappa \cdot \ell} \cdot \left(\operatorname{arsinh} \frac{u + \frac{\ell}{2}}{v} - \operatorname{arsinh} \frac{u - \frac{\ell}{2}}{v} \right) \tag{15.164}$$

or alternatively written as:

$$\varphi = \frac{I}{4 \cdot \pi \cdot \kappa \cdot \ell} \cdot \ln \frac{\sqrt{v^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{v^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{15.165}$$

The distribution of the electric scalar potential is axisymmetric with respect to the local axis u, and the equipotential surfaces are rotational ellipsoids with foci at the ends of the conductor's axis. This means that in the local coordinate system (u, v), the equipotential lines are ellipses with foci at the ends of the conductor's axis (Figure 14.54).

The distribution of the electric scalar potential along the mantle of a conductor segment can be written as:

$$\varphi = \varphi(u, r_0) = \frac{I}{4 \cdot \pi \cdot \kappa \cdot \ell} \cdot \ln \frac{\sqrt{r_0^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{r_0^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{15.166}$$

According to the average potential method, the electric scalar potential of a conductor segment $\boldsymbol{\Phi}$ is approximated by the average potential along the mantle of the conductor segment:

$$\Phi = \Phi_{\text{av}} = \frac{1}{\ell} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \varphi(u, r_0) \cdot du = P(\ell, r_0) \cdot I \tag{15.167}$$

where $P(\ell, r_0)$ is the *self-resistance* of the conductor segment in the unbounded conducting LIH medium, obtained by the average potential method. It holds that:

$$P(\ell, v) = \frac{1}{2 \cdot \pi \cdot \kappa \cdot \ell^2} \cdot \left[\ell \cdot \operatorname{arsinh}\frac{\ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{15.168}$$

or alternatively written as:

$$P(\ell, v) = \frac{1}{2 \cdot \pi \cdot \kappa \cdot \ell^2} \cdot \left[\ell \cdot \ln\frac{\sqrt{\ell^2 + v^2} + \ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{15.169}$$

It also holds that:

$$P(\ell, v) = \frac{\varepsilon}{\kappa} \cdot G(\ell, v) \tag{15.170}$$

where the auxiliary function $G(\ell, v)$ is defined by expressions (14.300) and (14.301), and holds for a segment of a cylindrical conductor in an unbounded perfect LIH dielectric.

Thus, in an unbounded conducting LIH medium, the following holds for a segment of the conductor:

$$\Phi = \Phi_{\text{av}} = P(\ell, r_0) \cdot I = R \cdot I \tag{15.171}$$

where $R = P(\ell, r_0)$ is the self-resistance of the segment of the cylindrical conductor in an unbounded conducting LIH medium, obtained using the average potential method. This resistance is defined between the conductor segment and a reference point at infinity, or, alternatively, it represents the contribution of a unit segment current to the average potential of the same segment.

15.16. Grounding Resistance of a Horizontal Conductor Segment in LIH Soil

Let a segment of a cylindrical thin-wire conductor of radius r_0 and length ℓ be buried in LIH soil with electrical conductivity κ (Figure 15.27). If this conductor segment leaks a stationary electric current I into the soil, it acts as an elementary horizontal ground electrode. Therefore, in this particular case, we refer to the grounding resistance of the conductor segment instead of its self-resistance. The horizontal ground electrode can be a conducting strip, cylindrical conductor, or wire, and can be implemented as radial conductors, a ring conductor, a grounding grid, or a combination of these.

The goal is to derive an expression for the grounding resistance of the given equipotential segment of the cylindrical conductor using the average potential method.

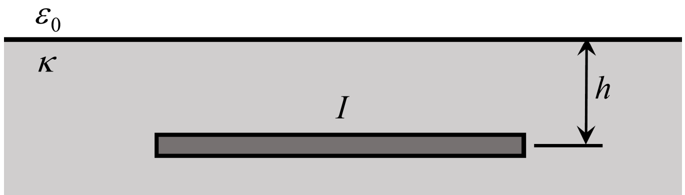


Figure 15.27. Horizontal segment of a cylindrical conductor in LIH soil

For the calculation of the *self-resistance* of the cylindrical conductor segment using the average potential method, it should be assumed in the numerical model that the conductor segment uniformly leaks a stationary electric current I into the soil from its axis.

By the method of images, one segment of the conductor in the LIH half-space with electrical conductivity κ is replaced by two conductor segments in an unbounded LIH medium with electrical conductivity κ (Figure 15.28).

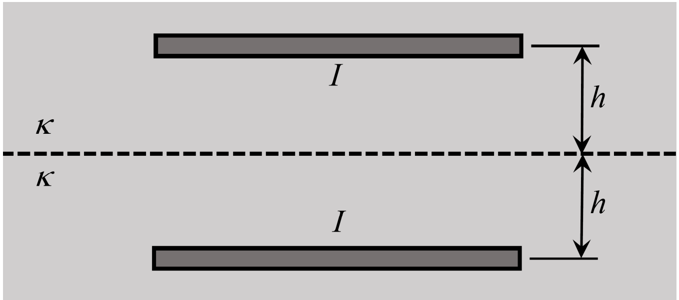


Figure 15.28. Real and image horizontal segment of a cylindrical conductor

The electric scalar potential of a segment of a cylindrical conductor, obtained by the average potential method, is described by the expression:

$$\Phi = \Phi_{\text{av}} = R \cdot I \quad ; \quad R = P(\ell, r_0) + P(\ell, 2 \cdot h) \tag{15.172}$$

where:

R - the *total self-resistance* of a cylindrical conductor segment in a heterogeneous medium consisting of soil and air,

$P(\ell, r_0)$ - the *self-resistance* of a cylindrical conductor segment in an unlimited conducting LIH medium, described by the expression (15.169),

$P(\ell, 2 \cdot h)$ - the *mutual resistance* of the real segment of a cylindrical conductor and the image segment of a cylindrical conductor in an unbounded conducting LIH medium, which is described by the expression (15.169).

Since the horizontal grounding conductor is approximated by only one straight thin-wire segment of a cylindrical conductor, then:

$$R_{\text{gr}} = R \tag{15.173}$$

where R_{gr} is grounding resistance of a cylindrical conductor segment.

15.17. Grounding Resistance of Two Parallel Conductor Segments in LIH Soil

Let there be two identical, mutually parallel segments of a cylindrical thin-wire conductor of radius r_0 and length ℓ, buried in LIH soil with electrical conductivity κ (Figure 15.29). Let these two conductor segments form an equipotential grounding system that leaks a stationary electric current I_{gr} into the surrounding soil. Assume that the current supply to the conductor segments is insulated from the soil. The goal is to derive an expression for the electric scalar potential and the grounding resistance of this equipotential grounding electrode using the average potential method.

The heterogeneity of the entire medium is accounted for by applying the method of images, whereby the conductor segments are mirrored. Two conductor segments in an LIH half-space with electrical conductivity κ are replaced by four conductor segments in an unbounded LIH medium with the same electrical conductivity κ (Figure 15.30).

The electric scalar potentials of the conductor segments are described by the following expressions:

$$\Phi_1 = R_{1,1} \cdot I_1 \;+\; R_{1,2} \cdot I_2 \tag{15.174}$$

$$\Phi_2 = R_{2,1} \cdot I_1 \;+\; R_{2,2} \cdot I_2 \tag{15.175}$$

where the self and mutual resistances of the conductor segments are described by:

$$R_{1,1} = P(\ell, r_0) + P(\ell, 2 \cdot h_1) \tag{15.176}$$

$$R_{1,2} = R_{2,1} = P(\ell, d) + P(\ell, D) \qquad (15.177)$$

$$R_{2,2} = P(\ell, r_0) + P(\ell, 2 \cdot h_2) \qquad (15.178)$$

where the function $P(\ell, v)$ is described by expression (15.169).

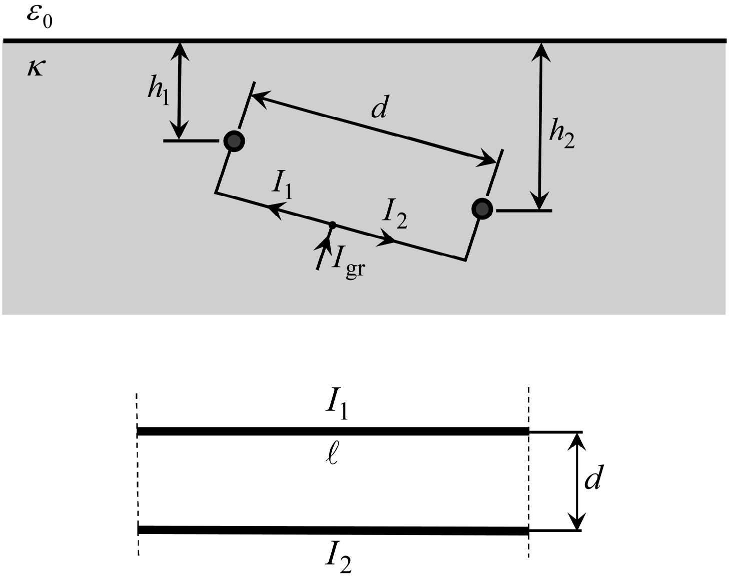


Figure 15.29. Two parallel segments of a thin-wire cylindrical conductor

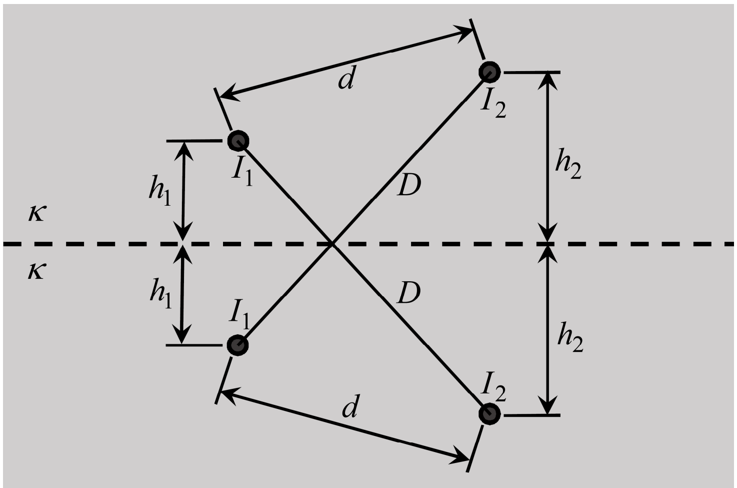


Figure 15.30. Real and image segments of the cylindrical conductor

Since the conductor segments form an equipotential grounding system, it follows that:

$$\Phi_1 = \Phi_2 = \Phi_{gr} \qquad (15.179)$$

$$I_1 + I_2 = I_{gr} \qquad (15.180)$$

The symmetric system of linear equations in matrix notation can be written as follows:

$$\begin{bmatrix} R_{1,1} & R_{1,2} & -1 \\ R_{2,1} & R_{2,2} & -1 \\ -1 & -1 & 0 \end{bmatrix} \cdot \begin{Bmatrix} I_1 \\ I_2 \\ \Phi_{\text{gr}} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -I_{\text{gr}} \end{Bmatrix} \tag{15.181}$$

In the special case when $h_1 = h_2$, it follows that:

$$I_1 = I_2 = \frac{I_{\text{gr}}}{2} \tag{15.182}$$

$$\Phi_1 = \Phi_2 = \Phi_{\text{gr}} = \frac{R_{1,1} + R_{1,2}}{2} \cdot I_{\text{gr}} \tag{15.183}$$

15.18. Resistance Between Two Parallel Conductor Segments in LIH Soil

Let there be two identical, mutually parallel segments of a cylindrical thin-wire conductor with radius r_0 and length ℓ buried in LIH soil with electrical conductivity κ (Figure 15.31). Let a stationary electric current I flow between these two equipotential conductor segments. Assume the current supply to the conductor segments is insulated from the soil. The goal is to derive an expression for the resistance between these two segments using the average potential method. In this case, each of these two conductor segments is a horizontal equipotential grounding electrode.

The heterogeneity of the entire medium is accounted for by applying the method of images. Two conductor segments in an LIH half-space with electrical conductivity κ are replaced by four conductor segments in an unbounded LIH medium with the same electrical conductivity κ (Figure 15.32).

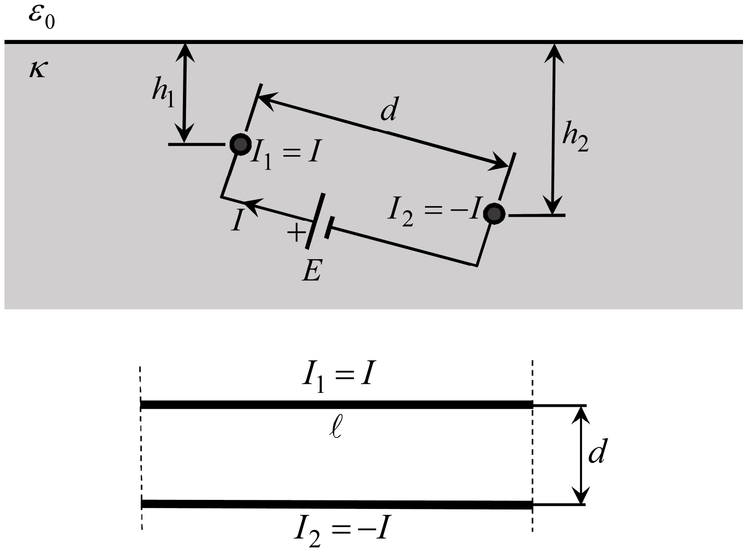


Figure 15.31. Two parallel segments of a cylindrical conductor

According to Figure 15.31, the horizontal grounding electrode 1 leaks a stationary electric current I into the soil, whereas the horizontal grounding electrode 2 receives a stationary electric current I from the soil, meaning that grounding electrode 2 leaks a stationary electric current $-I$ into the soil. Each of these two grounding electrodes is approximated in the numerical model by just one segment of a cylindrical conductor.

The electric scalar potentials of the conductor segments are described by the following expressions:

$$\Phi_1 = R_{1,1} \cdot I - R_{1,2} \cdot I = (R_{1,1} - R_{1,2}) \cdot I \tag{15.184}$$

$$\Phi_2 = R_{2,1} \cdot I - R_{2,2} \cdot I = (R_{2,1} - R_{2,2}) \cdot I \tag{15.185}$$

where the self and mutual resistances of the conductor segments, obtained using the average potential method, are described by the expressions (15.176) - (15.178).

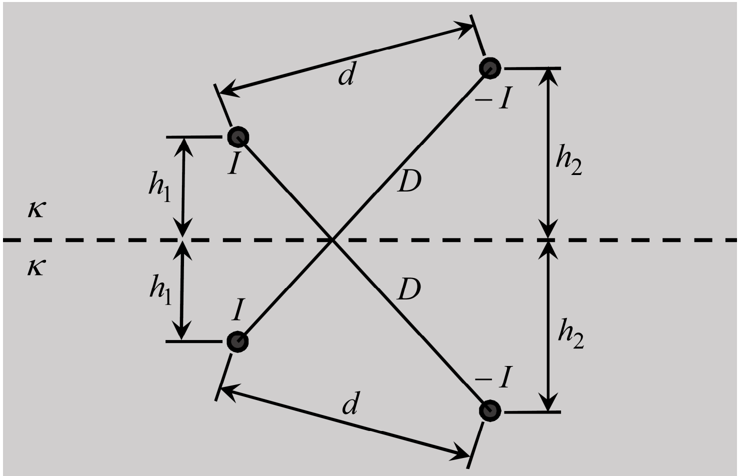


Figure 15.32. Real and image segments of the cylindrical conductor

The resistance between these two conductor segments is described by the following expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = R_{1,1} + R_{2,2} - 2 \cdot R_{1,2} \tag{15.186}$$

In the special case when $h_1 = h_2$, it follows that:

$$\Phi_2 = -\Phi_1 \quad ; \quad R = \frac{2 \cdot \Phi_1}{I} = 2 \cdot (R_{1,1} - R_{1,2}) \tag{15.187}$$

15.19. Grounding Resistance of a Vertical Conductor Segment in LIH Soil

Let there be a segment of a cylindrical thin-wire conductor with radius r_0 and length ℓ, buried in LIH soil with electrical conductivity κ (Figure 15.33). The conductor segment leaks a stationary electric current I into the soil. The goal is to derive an expression for the grounding resistance of this equipotential cylindrical conductor segment using the average potential method. For the solution of the problem, only the part of the conductor buried in the soil is relevant.

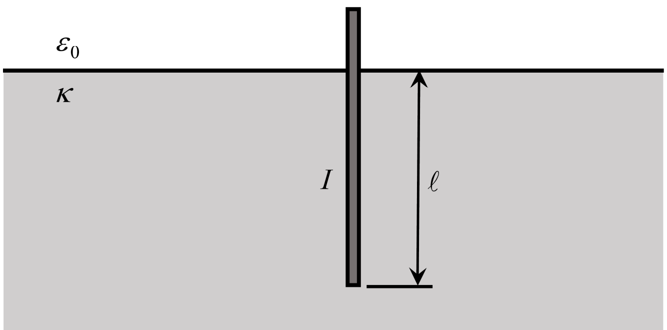


Figure 15.33. Vertical segment of a cylindrical conductor in LIH soil

By the method of images, one conductor segment in an LIH half-space with electrical conductivity κ is replaced by two conductor segments in an unbounded LIH medium with the same electrical conductivity κ (Figure 15.34).

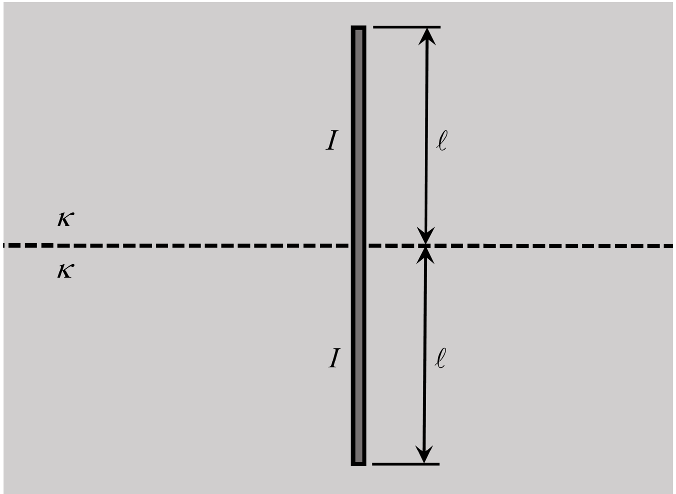


Figure 15.34. Real and image vertical segment of a cylindrical conductor

According to Figure 15.34, in this special case, the real and image segments of the conductor can be replaced in the numerical model by a single unified segment of a cylindrical conductor with radius r_0 and length $2 \cdot \ell$ in an LIH medium with electrical conductivity *κ*, which leaks a stationary electric current $2 \cdot I$.

The electric scalar potential of this unified segment, obtained by the average potential method, is described by the expression:

$$\Phi = I \cdot R = 2 \cdot I \cdot P(2 \cdot \ell, r_0) \tag{15.188}$$

where the function $P(\ell, v)$ is described by the expression (15.169).
It follows that the grounding resistance of the vertical conductor segment is described by:

$$R = \frac{1}{4 \cdot \pi \cdot \kappa \cdot \ell^2} \cdot \left[2 \cdot \ell \cdot \ln \frac{\sqrt{4 \cdot \ell^2 + r_0^2} + 2 \cdot \ell}{r_0} - \frac{4 \cdot \ell^2}{\sqrt{4 \cdot \ell^2 + r_0^2} \; + \; r_0} \right] \tag{15.189}$$

15.20. Numerical Modelling of Equipotential Grounding Grids

Let the equipotential grounding grid be located in LIH soil with electrical conductivity *κ*. In the numerical model, the conductors of the equipotential grounding grid (Figure 15.35) are divided into straight cylindrical segments.

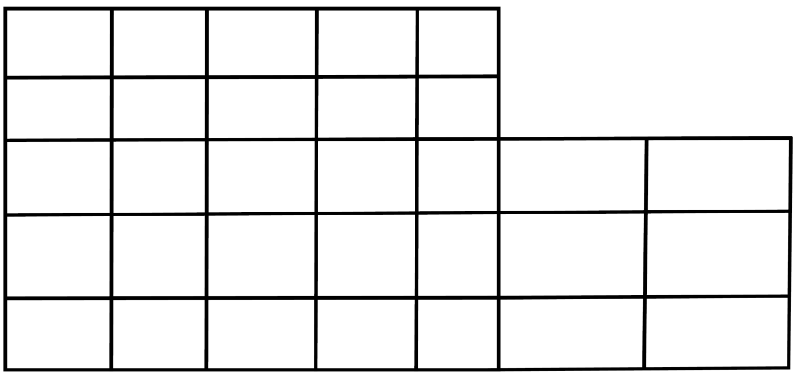


Figure 15.35. Illustrative example of an equipotential grounding grid

Let there be a total of *n* segments. Let the grounding current I_{gr} be known - this is the electric current that the grounding system leaks into the soil. Let the unknowns be: the electric currents of the segments

I_k; $k = 1, 2, ..., n$, and the electric scalar potential of the grounding grid Φ_{gr}. All electric scalar potentials of the conductor segments Φ_k; $k = 1, 2, ..., n$ are equal to the potential of the grounding grid.

Different weighted residual methods can be used for the numerical approximation of the electric scalar potential of segments, with specific cases including the point collocation method and the average potential method. For the average potential method, the system of linear equations is given by:

$$\sum_{k=1}^{n} R_{i,k} \cdot I_k = \Phi_i = \Phi_{gr} \quad ; \quad k = 1, 2, ..., n \tag{15.190}$$

$$\sum_{k=1}^{n} I_k = I_{gr} \tag{15.191}$$

which, in matrix notation, is given by:

$$\begin{bmatrix} R_{1,1} & \cdots & R_{1,n} & -1 \\ \vdots & \ddots & \vdots & \vdots \\ R_{n,1} & \cdots & R_{n,n} & -1 \\ -1 & \cdots & -1 & 0 \end{bmatrix} \begin{Bmatrix} I_1 \\ \vdots \\ I_n \\ \Phi_{gr} \end{Bmatrix} = \begin{Bmatrix} 0 \\ \vdots \\ 0 \\ -I_{gr} \end{Bmatrix} \tag{15.192}$$

where $R_{i,k} = R_{k,i}$; $i = 1, 2, ..., n$; $k = 1, 2, ..., n$ are the self and mutual resistances of the conductor segments. This, in fact, represents the contribution of the unit current of the k-th segment to the potential of the i-th segment, taking into account the heterogeneity of the entire space.

The self and mutual resistances $R_{i,k}$ in an unbounded LIH soil of electrical conductivity κ are analogous to the self and mutual potential coefficients $\alpha_{i,k}$ in an unbounded perfect LIH dielectric with permittivity ε. The relationship between them is described by the following expression:

$$R_{i,k} = \frac{\varepsilon}{\kappa} \cdot \alpha_{i,k} \tag{15.193}$$

The self-potential coefficients $\alpha_{i,i}$ are described by expression (14.303). In the special case of parallel conductor segments, as shown in Figure 14.57, the mutual potential coefficients $\alpha_{i,k}$ are described by expression (14.304). In the case of general parallelism of conductor segments, as shown in Figure 14.58, the mutual potential coefficients are described by expressions (14.305) - (14.307). If the conductor segments are non-parallel (Figure 14.59), then the mutual potential coefficients $\alpha_{i,k}$ are described by expressions (14.310) and (14.311).

If there are multiple equipotential grounding grids, then the system of linear equations (15.192) needs to be extended by adding one additional equation and one additional unknown for each new equipotential grounding grid.

The system of linear equations obtained using the average potential method is symmetric, whereas the point collocation method yields an asymmetric system of linear equations.

After solving the system of linear equations (15.192), the electric scalar potential can be computed at any point in the soil. The distribution of the electric scalar potential in the soil, or on the surface of the soil, is described by the following expression:

$$\varphi = \sum_{k=1}^{n} R_k \cdot I_k \tag{15.194}$$

where R_k is the resistance between the k-th segment and the field point, taking into account the heterogeneity of the medium. This, in fact, represents the contribution of the unit current of the k-th segment to the potential of the field point, considering the heterogeneity of the entire space.

Algorithms for numerical modelling of equipotential grounding electrodes in heterogeneous soil, such as horizontally stratified multilayer soil, as well as algorithms for numerical modelling of non-equipotential grounding electrodes in homogeneous and heterogeneous soil, are much more complex. Particularly complex is the time-harmonic and transient analysis of grounding systems.

15.21. Equivalent Radius of a Solid Rectangular Conductor

In the numerical model of the grounding grid, it is assumed that the conductor segments have a cylindrical cross-section. However, in practice, a galvanized steel strip with a rectangular cross-section is used, which, for the purposes of numerical modelling, can be approximated as a cylindrical conductor (Figure 15.36).

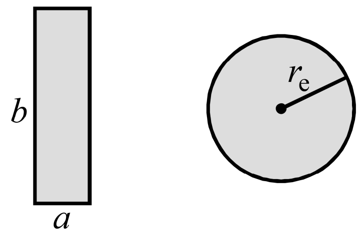


Figure 15.36. Rectangular conductor and equivalent cylindrical conductor

The equivalent radius of a rectangular conductor can be calculated using several criteria: the criterion of the same perimeter, the criterion of the same cross-sectional area, or, for example, the method of the average geometric distance. A more detailed numerical analysis based on the boundary element method shows that the equivalent radius can be well approximated using the criterion of the same perimeter, such that:

$$r_{\mathrm{e}} = \frac{a+b}{\pi} \tag{15.195}$$

15.22. Circular Perfectly Conducting Plate in an Unbounded Soil

Let a thin, circular, perfectly conducting plate with radius a be located in an unbounded LIH soil with electrical conductivity κ, and let it leak a stationary electric current with intensity I into the soil (Figure 15.37). It is easy to conclude that, under the given assumptions, such a plate is equipotential.

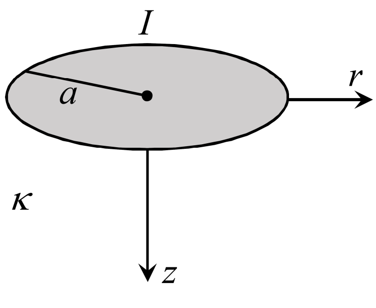


Figure 15.37. Circular perfectly conducting plate in an unbounded LIH soil

Since the distribution of the electric scalar potential is axially symmetric with respect to the z-axis, a 2D cylindrical coordinate system (r, z) is used. In the theoretical consideration, the starting point is an equipotential rotational ellipsoid created by a straight, thin-wire conductor segment of length ℓ, which uniformly leaks a stationary electric current with intensity I from its axis into the surrounding soil (Figure 15.38). The rotational ellipsoid is formed by rotating an ellipse around the z-axis, with the foci of the ellipse located at the ends of the conductor segment.

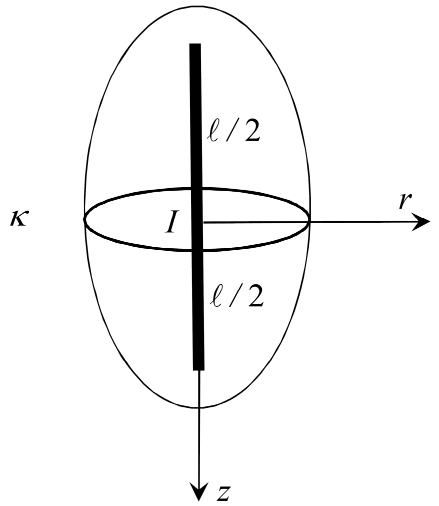


Figure 15.38. Equipotential rotational ellipsoid around the conductor segment

The distribution of the electric scalar potential around a conductor segment, which uniformly leaks a stationary electric current with intensity I from its axis in the numerical model, is described by expression (15.165), which can be written as follows:

$$\varphi = \frac{I}{4\cdot\pi\cdot\kappa\cdot\ell}\cdot\ln\frac{\sqrt{r^2+\left(z+\frac{\ell}{2}\right)^2}+z+\frac{\ell}{2}}{\sqrt{r^2+\left(z-\frac{\ell}{2}\right)^2}+z-\frac{\ell}{2}} \tag{15.196}$$

For the equipotential ellipse, which has a small semi-axis of length aa (the radius of the equipotential plate) and a large semi-axis of length b (Figure 15.39), the following holds:

$$\frac{\ell}{2} = \sqrt{b^2-a^2} \tag{15.197}$$

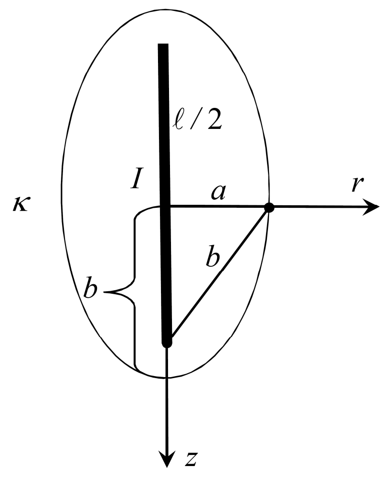


Figure 15.39. Major and minor semi-axes of the equipotential ellipse around the conductor segment

If b tends to zero, then the equipotential rotational ellipsoid, with a small semi-axis of length a, becomes an equipotential circular plate with radius a. Thus, according to expression (15.197), the following holds:

$$\lim_{b\to 0}\frac{\ell}{2} = \sqrt{-a^2} = j\cdot a \tag{15.198}$$

After substituting expression (15.198) into the expression for the distribution of the electric scalar potential (15.196), the resulting expression for the distribution of the electric scalar potential around a circular perfectly conducting plate in an unbounded LIH soil is:

$$\varphi = \frac{I}{j\cdot 8\cdot \pi\cdot \kappa\cdot a}\cdot \ln\frac{\sqrt{r^2+(z+j\cdot a)^2}\ +\ z+j\cdot a}{\sqrt{r^2+(z-j\cdot a)^2}\ +\ z-j\cdot a} \tag{15.199}$$

The distribution of the electric scalar potential around a circular plate is an even function with respect to the z-axis, so for further processing, expression (15.199) can be rewritten in the following form:

$$\varphi = \frac{I}{j\cdot 8\cdot \pi\cdot \kappa\cdot a}\cdot \ln\frac{\sqrt{r^2+(|z|+j\cdot a)^2}\ +|z|+j\cdot a}{\sqrt{r^2+(|z|-j\cdot a)^2}\ +|z|-j\cdot a} \tag{15.200}$$

from which it follows that:

$$\varphi = \frac{I}{j\cdot 8\cdot \pi\cdot \kappa\cdot a}\cdot \ln\frac{\sqrt{A+j\cdot 2\cdot a\cdot |z|}\ +|z|+j\cdot a}{\sqrt{A-j\cdot 2\cdot a\cdot |z|}\ +|z|-j\cdot a} \tag{15.201}$$

where:

$$A = r^2+z^2-a^2 \tag{15.202}$$

Let it be:

$$\sqrt{A+j\cdot 2\cdot a\cdot |z|}\ =\alpha+j\cdot\beta \quad ; \quad \sqrt{A-j\cdot 2\cdot a\cdot |z|}\ =\alpha-j\cdot\beta \tag{15.203}$$

from which, after squaring these two expressions, it follows that:

$$\alpha^2-\beta^2 = A \quad ; \quad \alpha\cdot\beta = |z|\cdot a \tag{15.204}$$

Furthermore, it is:

$$\alpha^4 - A\cdot\alpha^2 - z^2\cdot a^2 = 0 \quad ; \quad \alpha^2 = \frac{A+\sqrt{A^2+4\cdot a^2\cdot z^2}}{2} \tag{15.205}$$

from which it follows that:

$$\alpha = \sqrt{\frac{A+\sqrt{A^2+4\cdot a^2\cdot z^2}}{2}} \quad ; \quad \beta = \sqrt{\frac{-A+\sqrt{A^2+4\cdot a^2\cdot z^2}}{2}} \tag{15.206}$$

If the expressions labeled as (15.203) are substituted into expression (15.200), a new expression for the distribution of the electric scalar potential around the circular plate is obtained:

$$\varphi = \frac{I}{j\cdot 8\cdot \pi\cdot a\cdot \kappa}\cdot \ln\frac{(\alpha+|z|)+j\cdot(\beta+a)}{(\alpha+|z|)-j\cdot(\beta+a)} \tag{15.207}$$

from which it follows that:

$$\varphi = \frac{I}{j\cdot 8\cdot \pi\cdot a\cdot \kappa}\cdot \ln e^{j\cdot 2\cdot \arctan\frac{\beta+a}{\alpha+|z|}} = \frac{I}{4\cdot \pi\cdot a\cdot \kappa}\cdot \arctan\frac{\beta+a}{\alpha+|z|} \tag{15.208}$$

Since, according to (15.204):

$$\beta = \frac{|z| \cdot a}{\alpha} \tag{15.209}$$

it is easily obtained that:

$$\frac{\beta + a}{\alpha + |z|} = \frac{a}{\alpha} = \frac{\beta}{|z|} \tag{15.210}$$

From expressions (15.210) and (15.208), the following practical expressions for the distribution of the electric scalar potential around the circular perfectly conducting plate follow can be obtained:

$$\varphi = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arctan \frac{a}{\alpha} = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arcsin \frac{a}{\sqrt{a^2 + \alpha^2}} \tag{15.211}$$

$$\varphi = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arctan \frac{\beta}{|z|} = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arcsin \frac{\beta}{\sqrt{z^2 + \beta^2}} \tag{15.212}$$

where the auxiliary functions α and β are described by expression (15.206).

From expressions (15.206) and (15.211), it is easily obtained that for $z = 0$ & $r \geq a$:

$$\alpha = \sqrt{r^2 - a^2} \quad ; \quad \varphi = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arcsin \frac{a}{r} \tag{15.213}$$

whereas for $z = 0$ & $r \leq a$:

$$\alpha = 0 \quad ; \quad \varphi = \Phi_{\text{pl}} = \frac{I}{8 \cdot a \cdot \kappa} = \text{const.} \tag{15.214}$$

From expressions (15.206) and (15.211), it follows that for $r = 0$:

$$\alpha = |z| \quad ; \quad \beta = a \quad ; \quad \varphi = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arctan \frac{a}{|z|} \tag{15.215}$$

The expression for the distribution of the surface density of the electric current on one side of the equipotential circular plate can be obtained from the following expression:

$$J = -\kappa \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \leq a}} \frac{\partial \varphi}{\partial z} \quad ; \quad \varphi = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arctan \frac{\beta}{z} \tag{15.216}$$

It holds that:

$$\frac{\partial \varphi}{\partial z} = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \frac{1}{1 + \left(\dfrac{\beta}{z}\right)^2} \cdot \frac{z \cdot \dfrac{\partial \beta}{\partial z} - \beta}{z^2} = \frac{I}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \frac{z \cdot \dfrac{\partial \beta}{\partial z} - \beta}{\beta^2 + z^2} \tag{15.217}$$

$$J = -\kappa \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \leq a}} \frac{\partial \varphi}{\partial z} = -\frac{I}{4 \cdot \pi \cdot a} \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \leq a}} \frac{z \cdot \dfrac{\partial \beta}{\partial z} - \beta}{\beta^2 + z^2} = \frac{I}{4 \cdot \pi \cdot a} \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \leq a}} \frac{1}{\beta} \tag{15.218}$$

$$J = \frac{I}{4 \cdot \pi \cdot a} \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \le a}} \sqrt{\frac{2}{-A + \sqrt{A^2 + 4 \cdot a^2 \cdot z^2}}} = \frac{I}{4 \cdot \pi \cdot a} \cdot \sqrt{\frac{2}{2 \cdot |A|}} \tag{15.219}$$

and finally:

$$J = \frac{I}{4 \cdot \pi \cdot a \cdot \sqrt{a^2 - r^2}} \tag{15.220}$$

The expression (15.220) that describes the distribution of the surface density of the electric current on one side of the equipotential plate can also be obtained from expression (15.199), which has the following alternative form:

$$\varphi = \frac{I}{j \cdot 8 \cdot \pi \cdot \kappa \cdot a} \cdot \left(\operatorname{arsinh} \frac{z + j \cdot a}{r} - \operatorname{arsinh} \frac{z - j \cdot a}{r} \right) \tag{15.221}$$

It follows that:

$$\frac{\partial \varphi}{\partial z} = \frac{I}{j \cdot 8 \cdot \pi \cdot \kappa \cdot a} \cdot \left(\frac{1}{\sqrt{(z + j \cdot a)^2 + r^2}} - \frac{1}{\sqrt{(z - j \cdot a)^2 + r^2}} \right) \tag{15.222}$$

and it is:

$$J = -\kappa \cdot \lim_{\substack{z \to 0 \\ z > 0 \\ r \le a}} \frac{\partial \varphi}{\partial z} = \frac{j \cdot I}{8 \cdot \pi \cdot a} \cdot \left(\frac{1}{\sqrt{r^2 - a^2}} - \frac{1}{\sqrt{r^2 - a^2}} \right) \tag{15.223}$$

$$J = \frac{j \cdot I}{8 \cdot \pi \cdot a} \cdot \left(\frac{1}{\pm j \cdot \sqrt{a^2 - r^2}} - \frac{1}{\pm j \cdot \sqrt{a^2 - r^2}} \right) \tag{15.224}$$

By choosing the sign in expression (15.224) that gives the physical solution, which is the positive real solution, it follows that:

$$J = \frac{j \cdot I}{8 \cdot \pi \cdot a} \cdot \left(\frac{1}{j \cdot \sqrt{a^2 - r^2}} - \frac{1}{-j \cdot \sqrt{a^2 - r^2}} \right) = \frac{I}{4 \cdot \pi \cdot a \cdot \sqrt{a^2 - r^2}} \tag{15.225}$$

15.23. Circular Perfectly Conducting Plate on the Soil Surface

Let a thin, circular, perfectly conducting plate with radius a be located on the surface of the LIH soil with electrical conductivity κ, and let it leak a stationary electric current with intensity I into the soil (Figure 15.40). It is easy to conclude that, under the given assumptions, such a plate is equipotential.

According to the method of images, after the perfectly conducting plate is reflected with respect to the soil surface, a perfectly conducting plate in an unbounded LIH soil (Figure 15.37) is obtained, which leaks a stationary electric current with intensity $2 \cdot I$ into the soil. Since the plate is on the soil surface, the same expression describes the distribution of the electric scalar potential in both the air and the soil.

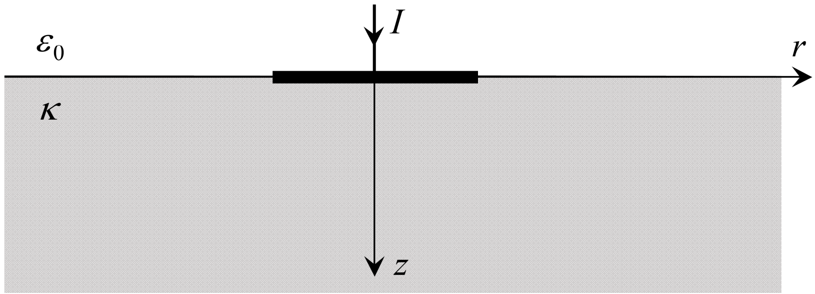


Figure 15.40. Circular perfectly conducting plate on the surface of the LIH soil

From expressions (15.211) and (15.212), which apply for half of the electric current, it follows that the distribution of the electric scalar potential around the circular perfectly conducting plate is described by the following expressions:

$$\varphi = \frac{2\cdot I}{4\cdot\pi\cdot a\cdot\kappa}\cdot\arctan\frac{a}{\alpha} = \frac{I}{2\cdot\pi\cdot a\cdot\kappa}\cdot\arcsin\frac{a}{\sqrt{a^2+\alpha^2}} \tag{15.226}$$

$$\varphi = \frac{2\cdot I}{4\cdot\pi\cdot a\cdot\kappa}\cdot\arctan\frac{\beta}{|z|} = \frac{I}{2\cdot\pi\cdot a\cdot\kappa}\cdot\arcsin\frac{\beta}{\sqrt{z^2+\beta^2}} \tag{15.227}$$

where the auxiliary functions α and β are described by expression (15.206).

According to expression (15.220), the surface density of the electric current that the circular equipotential plate leaks into the soil from one side is described by the following expression:

$$J = \frac{2\cdot I}{4\cdot\pi\cdot a\cdot\sqrt{a^2-r^2}} = \frac{I}{2\cdot\pi\cdot a\cdot\sqrt{a^2-r^2}} \tag{15.228}$$

and this means that the surface density of the stationary electric charge located on the other side of the circular equipotential plate is described by the following expression:

$$\sigma = \frac{Q}{2\cdot\pi\cdot a\cdot\sqrt{a^2-r^2}} \quad ; \quad \frac{Q}{\varepsilon_0} = \frac{I}{\kappa} \tag{15.229}$$

From expression (15.214), it follows that the electric scalar potential of the plate is:

$$\varphi = \Phi_{\text{pl}} = \frac{2\cdot I}{8\cdot a\cdot\kappa} = \frac{I}{4\cdot a\cdot\kappa} = \text{const.} \tag{15.230}$$

and thus the grounding resistance of the plate is:

$$R_{\text{pl}} = \frac{\Phi_{\text{pl}}}{I} = \frac{1}{4\cdot a\cdot\kappa} \tag{15.231}$$

From expression (15.213), it follows that the distribution of the electric scalar potential on the soil surface $(z=0\ \&\ r\geq a)$ is described by the following expression:

$$\varphi = \frac{I}{2\cdot\pi\cdot a\cdot\kappa}\cdot\arcsin\frac{a}{r} = \frac{2\cdot\Phi_{\text{pl}}}{\pi}\cdot\arcsin\frac{a}{r} \tag{15.232}$$

From expression (15.214), it follows that the distribution of the electric scalar potential along the z-axis $(r=0)$ is described by the following expression:

$$\varphi = \frac{I}{2\cdot\pi\cdot a\cdot\kappa}\cdot\arctan\frac{a}{|z|} = \frac{2\cdot\Phi_{\text{pl}}}{\pi}\cdot\arctan\frac{a}{|z|} \tag{15.233}$$

15.24. Solved Examples

Example 15.1. A solid perfectly conducting sphere of radius $r_0 = 1.25$ cm is buried at a depth $h = 3$ m below the surface of an LIH soil with electrical conductivity $\kappa = 0.1$ S/m. If the sphere leaks a stationary electric current of intensity I = 100 A into the soil, determine: a) the distribution of the electric field intensity in the soil, b) the distribution of the electric field intensity on the soil surface, c) the distribution of the electric scalar potential on the soil surface, d) the electric scalar potential of the sphere and the grounding resistance of the sphere. Assume that $h >> r_0$. Use a 2D cylindrical coordinate system (r, z).

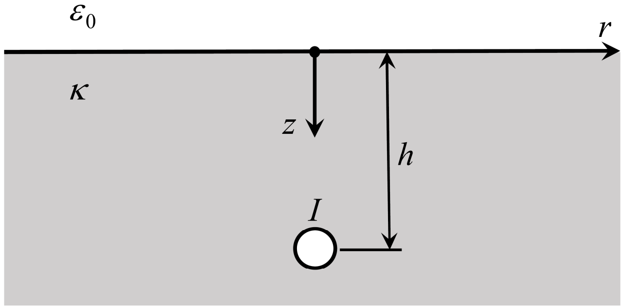


Figure 15.41. A solid perfectly conducting sphere buried in LIH soil

Solution:

a) The distribution of the electric field intensity in the soil

The method of images is used. With the introduced assumption that $h >> r_0$, both the real sphere and its image are approximated as point sources of stationary electric current.

If the solution is sought within the soil, the real sphere and its mirror image in an unbounded LIH soil with electrical conductivity κ both leak a stationary electric current I (Figure 15.42).

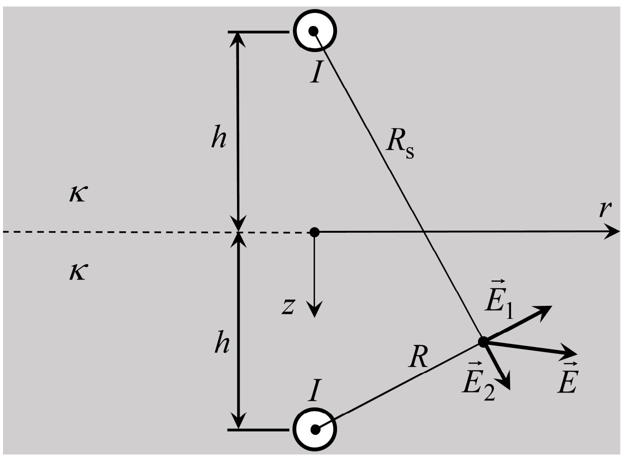


Figure 15.42. Real and image solid sphere in an unbounded LIH soil

The reflection factor is:

$$k_R = \frac{\kappa - \kappa_{air}}{\kappa + \kappa_{air}} = \frac{\kappa - 0}{\kappa + 0} = 1 \tag{15.234}$$

The electric field intensity vector in the soil ($z \geq 0$) due to the electric current from the real sphere is described by the following expression (Figure 15.42):

$$\vec{E}_1 = \frac{I}{4 \cdot \pi \cdot \kappa \cdot R^2} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z-h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{R} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z-h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + (z-h)^2\right)^3}} \tag{15.235}$$

whereas the electric field intensity vector in the soil due to the electric current from the image sphere is described by the following expression:

$$\vec{E}_2 = \frac{I}{4 \cdot \pi \cdot \kappa \cdot R_\mathrm{s}^2} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z+h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{R_\mathrm{s}} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z+h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + (z+h)^2\right)^3}} \tag{15.236}$$

and it follows that the total electric field intensity in the soil at the field point (r, z) is described by:

$$\vec{E} = \vec{E}_1 + \vec{E}_2 = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z-h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + (z-h)^2\right)^3}} + \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + (z+h) \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + (z+h)^2\right)^3}} \right) \tag{15.237}$$

b) The distribution of the electric field intensity on the soil surface

The electric current does not exit the soil, i.e., at the soil surface, the electric field intensity has no z-component (Figure 15.43).

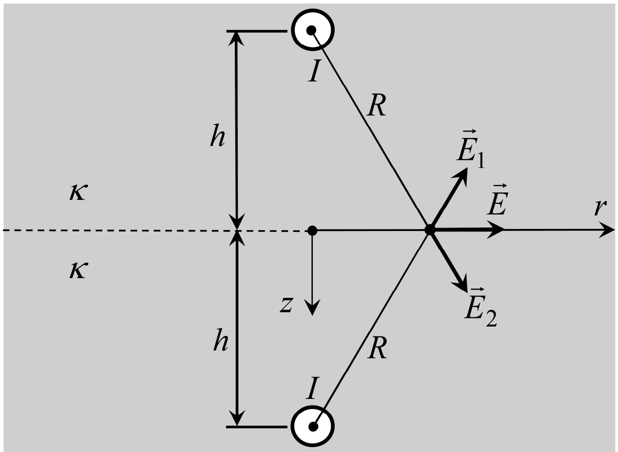


Figure 15.43. Electric field intensity vector on the soil surface

The electric field intensity vector on the soil surface due to the electric current from the real sphere is described by the following expression (Figure 15.43):

$$\vec{E}_1\Big|_{z=0} = \frac{I}{4 \cdot \pi \cdot \kappa \cdot R^2} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} - h \cdot \vec{\mathrm{e}}_\mathrm{z}}{R} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} - h \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + h^2\right)^3}} \tag{15.238}$$

whereas the electric field intensity vector on the soil surface due to the electric current from the image sphere is described by the following expression:

$$\vec{E}_2\Big|_{z=0} = \frac{I}{4 \cdot \pi \cdot \kappa \cdot R^2} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + h \cdot \vec{\mathrm{e}}_\mathrm{z}}{R} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\mathrm{e}}_\mathrm{r} + h \cdot \vec{\mathrm{e}}_\mathrm{z}}{\sqrt{\left(r^2 + h^2\right)^3}} \tag{15.239}$$

and it follows that the total electric field intensity on the soil surface, i.e., at the field point (r, 0), is described by:

$$\vec{E}\Big|_{z=0} = \vec{E}_1\Big|_{z=0} + \vec{E}_2\Big|_{z=0} = \frac{I}{2 \cdot \pi \cdot \kappa} \cdot \frac{r}{\sqrt{\left(r^2 + h^2\right)^3}} \cdot \vec{\mathrm{e}}_\mathrm{r} \tag{15.240}$$

c) The distribution of the electric scalar potential on the soil surface

According to Figure 15.42, the distribution of the electric scalar potential in the soil ($z \geq 0$) is described by the following expression:

$$\varphi = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{R} + \frac{1}{R_\mathrm{s}} \right) = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{\sqrt{r^2 + (z-h)^2}} + \frac{1}{\sqrt{r^2 + (z+h)^2}} \right) \tag{15.241}$$

and thus, the distribution of the electric scalar potential on the soil surface ($z = 0$) is described by:

$$\varphi\big|_{z=0} = 2 \cdot \frac{I}{4 \cdot \pi \cdot \kappa \cdot R} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot \sqrt{r^2 + h^2}} \tag{15.242}$$

d) The electric scalar potential of the sphere and the grounding resistance of the sphere

Assuming that $h >> r_0$, according to Figure 15.42, the electric scalar potential of the sphere is described by the following expression:

$$\varphi_\text{sphere} = \frac{I}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{r_0} + \frac{1}{2 \cdot h} \right) = 6.37946064 \text{ kV} \tag{15.243}$$

where the electric scalar potential due to the image solid sphere is calculated at the center of the sphere. The grounding resistance of the solid sphere is:

$$R_\text{sphere} = \frac{\varphi_\text{sphere}}{I} = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \left(\frac{1}{r_0} + \frac{1}{2 \cdot h} \right) = 63.7946064\ \Omega \tag{15.244}$$

Example 15.2. A solid perfectly conducting sphere of radius $r_0 = 1.25$ cm is buried at a depth below the surface of an LIH soil with electrical conductivity $\kappa = 0.1$ S/m. If the sphere leaks a stationary electric current of intensity I = 100 A into the soil, determine: a) the distribution of the electric scalar potential in the air, b) the distribution of the electric field intensity in the air, c) the distribution of the surface charge density on the soil surface. Assume that $h >> r_0$. Use a 2D cylindrical coordinate system (r, z). The solid perfectly conducting sphere buried in LIH soil is shown in Figure 15.41.

Solution:

a) The distribution of the electric scalar potential in the air

The method of images is used. With the introduced assumption that $h >> r_0$, both the real sphere and its image are approximated as point sources of stationary electric current.

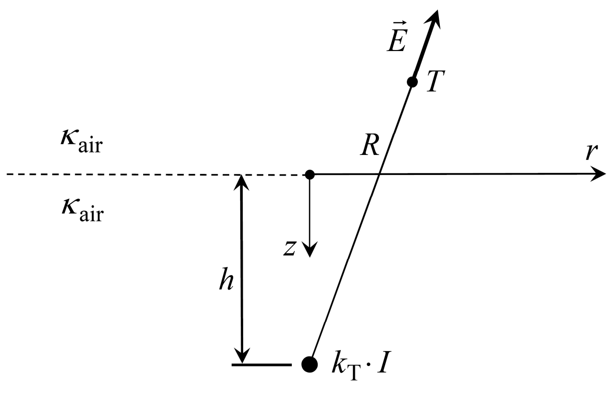


Figure 15.44. Image point source of electric current

The distribution of the electric scalar potential in the air is described by expression (15.56), which reads:

$$\varphi = \lim_{\kappa_{\text{air}} \to 0} \frac{k_{\text{T}} \cdot I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot R} = \lim_{\kappa_{\text{air}} \to 0} \frac{I}{4 \cdot \pi \cdot \kappa_{\text{air}} \cdot R} \cdot \frac{2 \cdot \kappa_{\text{air}}}{\kappa_{\text{air}} + \kappa} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot R} \tag{15.245}$$

where, according to Figure 15.44, the distance between the image point source of electric current and the field point in the air is described by the expression:

$$R = \sqrt{r^2 + (h - z)^2} \tag{15.246}$$

and the distribution of the electric scalar potential in the air ($z \le 0$) is described by the expression:

$$\varphi = \frac{2 \cdot I}{4 \cdot \pi \cdot \kappa \cdot R} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot R} = \frac{I}{2 \cdot \pi \cdot \kappa \cdot \sqrt{r^2 + (h - z)^2}} \tag{15.247}$$

If the given data are substituted, then it follows that:

$$\varphi = \frac{500}{\pi \cdot \sqrt{r^2 + (9 - z)^2}} \ \text{V} \tag{15.248}$$

b) The distribution of the electric field intensity in the air

Based on expressions (15.245) and (15.247), it is easy to conclude that the electric field intensity vector in the air ($z \le 0$) is described by the expression

$$\vec{E} = \frac{2 \cdot I}{4 \cdot \pi \cdot \kappa \cdot R^2} \cdot \frac{r \cdot \vec{\text{e}}_{\text{r}} - (h - z) \cdot \vec{\text{e}}_{\text{z}}}{R} = \frac{I}{2 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\text{e}}_{\text{r}} - (h - z) \cdot \vec{\text{e}}_{\text{z}}}{\sqrt{\left[r^2 + (h - z)^2\right]^3}} \tag{15.249}$$

If the given data are substituted, then it follows that:

$$\vec{E} = \frac{500}{\pi} \cdot \frac{r \cdot \vec{\text{e}}_{\text{r}} - (3 - z) \cdot \vec{\text{e}}_{\text{z}}}{\sqrt{\left[r^2 + (3 - z)^2\right]^3}} \ \frac{\text{V}}{\text{m}} \tag{15.250}$$

c) The distribution of the surface density of the stationary electric charge on the soil surface

From expression (15.249), it follows that the electric field intensity in the air at the soil surface ($z = 0$) is described by the expression:

$$\vec{E}_{\text{air}}\Big|_{z=0} = \frac{I}{2 \cdot \pi \cdot \kappa} \cdot \frac{r \cdot \vec{\text{e}}_{\text{r}} - h \cdot \vec{\text{e}}_{\text{z}}}{\sqrt{\left(r^2 + h^2\right)^3}} \tag{15.251}$$

At the soil-air boundary, the tangential component of the electric field intensity vector is continuous, whereas the normal component of the electric field intensity vector is zero in the soil and non-zero in the air due to the accumulation of stationary electric charge at the soil-air boundary. It holds that:

$$\vec{n} \cdot \left(\vec{D}_{\text{air}} - \vec{D}_{\text{soil}}\right) = -\vec{\text{e}}_{\text{z}} \cdot \left(\vec{D}_{\text{air}} - \vec{D}_{\text{soil}}\right) = -\vec{\text{e}}_{\text{z}} \cdot \vec{D}_{\text{air}} = \sigma \tag{15.252}$$

and therefore it holds that:

$$\sigma = -\varepsilon_0 \cdot E_{\text{z,air}}\Big|_{z=0} = \frac{\varepsilon_0 \cdot I \cdot h}{2 \cdot \pi \cdot \kappa \cdot \sqrt{\left(r^2 + h^2\right)^3}} \tag{15.253}$$

If the given data are substituted, then it follows that:

$$\sigma = \frac{1500 \cdot \varepsilon_0}{\pi \cdot \sqrt{\left(r^2 + 9\right)^3}} \ \frac{\text{C}}{\text{m}^2} \tag{15.254}$$

Example 15.3. Two ends of an electrical network are grounded via three solid perfectly conducting spheres of radius r_0, each half-buried in a homogeneous soil with electrical conductivity $\kappa = 0.01$ S/m. Determine the total resistance between the grounded ends of the network, assuming that a, $b \gg r_0$. Neglect the resistance of the connecting conductors. Let it be: $r_0 = 0.5$ m, $a = 5$ m, $b = 250$ m.

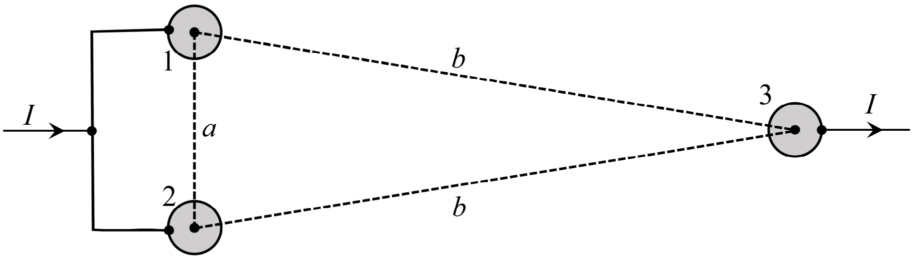


Figure 15.45. Solid perfectly conducting spheres half-buried in LIH soil

Solution:

Due to symmetry, it holds that:

$$\Phi_1 = \Phi_2 \quad ; \quad I_1 = I_2 = \frac{I}{2} \quad ; \quad I_3 = -I \tag{15.255}$$

According to subchapter 15.7, using the method of images, a solid perfectly conducting hemisphere on the surface of a homogeneous medium (soil) that leaks an electric current *I* into the soil is equivalent to a solid perfectly conducting sphere in an unbounded LIH medium with soil conductivity, which leaks a stationary electric current 2·*I* into that medium.

It follows that:

$$\Phi_1 = \Phi_2 = \frac{I}{4\cdot\pi\cdot\kappa}\cdot\left(\frac{1}{r_0}+\frac{1}{a}-\frac{2}{b}\right) \tag{15.256}$$

whereas:

$$\Phi_3 = \frac{I}{4\cdot\pi\cdot\kappa}\cdot\left(\frac{1}{b}+\frac{1}{b}-\frac{2}{r_0}\right) \tag{15.257}$$

and the resistance between the grounded ends of the network is:

$$R = \frac{\Phi_1 - \Phi_3}{I} = \frac{1}{4\cdot\pi\cdot\kappa}\cdot\left(\frac{1}{a}-\frac{4}{b}+\frac{3}{r_0}\right) = 49.2107084\ \Omega \tag{15.258}$$

Example 15.4. The grounding electrode is formed by a perfectly conducting hemisphere buried in a spherically layered three-layer soil. Determine the grounding resistance of the hemisphere, assuming that the hemisphere leaks a stationary electric current into the soil. Let it be: $r_0 = 0.5$ m, $r_1 = 1$ m, $r_2 = 3$ m, $\kappa_1 = 0.1$ S/m, $\kappa_2 = 0.01$ S/m, $\kappa_3 = 0.005$ S/m.

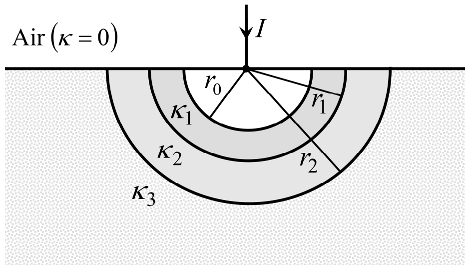


Figure 15.46. Perfectly conducting hemisphere buried in a three-layer soil

Solution:

Using the method of images, a perfectly conducting hemisphere on the surface of a spherically layered three-layer soil, which leaks an electric current *I* into the soil, is replaced by a solid perfectly conducting sphere in an unbounded spherically layered three-layer medium, which leaks an electric current 2·*I* into that medium.

In the spherical coordinate system, the electric field intensity vector in the i-th medium is described by the expression:

$$\vec{E}_{\mathrm{i}} = \frac{2 \cdot I}{4 \cdot \pi \cdot \kappa_{\mathrm{i}} \cdot r^2} \cdot \overline{\mathrm{e}}_{\mathrm{r}} \quad ; \quad \mathrm{i} = 1, 2, 3 \tag{15.259}$$

which can be obtained from the following expression:

$$\oint_S \vec{J}_{\mathrm{i}} \cdot d\vec{S} = \kappa_{\mathrm{i}} \cdot E_{\mathrm{i}} \cdot 4 \cdot \pi \cdot r^2 = 2 \cdot I \tag{15.260}$$

The electric scalar potential of the hemisphere (hemispherical grounding electrode) is calculated by integrating the electric field intensity from the hemisphere to infinity:

$$\varphi_{\mathrm{sphere}} = \int_{r_0}^{r_1} E_1 \cdot dr + \int_{r_1}^{r_2} E_2 \cdot dr + \int_{r_2}^{\infty} E_3 \cdot dr \tag{15.261}$$

which gives the following expression:

$$\varphi_{\mathrm{sphere}} = \frac{I}{2 \cdot \pi} \cdot \left[\frac{1}{\kappa_1} \cdot \left(\frac{1}{r_0} - \frac{1}{r_1} \right) + \frac{1}{\kappa_2} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{1}{\kappa_3} \cdot \frac{1}{r_2} \right] \tag{15.262}$$

and the grounding resistance of the hemisphere is::

$$R_{\mathrm{sphere}} = \frac{\varphi_{\mathrm{sphere}}}{I} = \frac{1}{2 \cdot \pi} \cdot \left[\frac{1}{\kappa_1} \cdot \left(\frac{1}{r_0} - \frac{1}{r_1} \right) + \frac{1}{\kappa_2} \cdot \left(\frac{1}{r_1} - \frac{1}{r_2} \right) + \frac{1}{\kappa_3} \cdot \frac{1}{r_2} \right] \tag{15.263}$$

If the given data are substituted, then it follows that:

$$R_{\mathrm{sphere}} = 22.8122085\ \Omega \tag{15.264}$$

Example 15.5. Determine the grounding resistance and electric scalar potential of the grounding system formed by two solid perfectly conducting hemispheres on the surface of a homogeneous soil with electrical conductivity κ = 0.01 S/m. Let it be: I =100 A, d =5 m, $r_1 = 0.2$ m, $r_2 = 0.4$ m. Assume that $d >> r_1, r_2$.

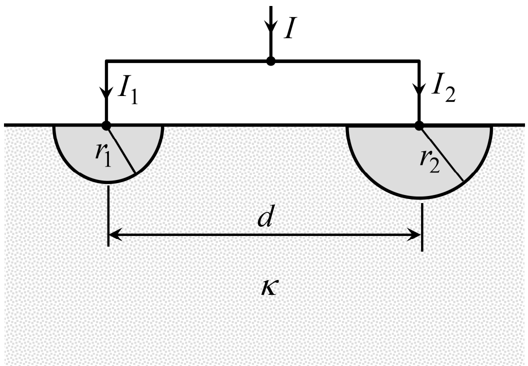


Figure 15.47. Grounding system formed by two solid perfectly conducting hemispheres

Solution:

According to subchapter 15.7, using the method of images, a solid perfectly conducting hemisphere on the surface of a homogeneous medium (soil) that leaks an electric current I into the soil is equivalent to a solid perfectly conducting sphere in an unbounded LIH medium with soil conductivity, which leaks a stationary electric current $2 \cdot I$ into that medium.
The system of linear equations for calculating the electric currents is:

$$\Phi_{\mathrm{gr}} = \Phi_1 = \Phi_2 = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \left[\frac{2 \cdot I_1}{r_1} + \frac{2 \cdot I_2}{d} \right] = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \left[\frac{2 \cdot I_1}{d} + \frac{2 \cdot I_2}{r_2} \right] \tag{15.265}$$

$$I_1 + I_2 = I \tag{15.266}$$

which, after simplification, takes the following form:

$$\left(\frac{1}{r_1} - \frac{1}{d} \right) \cdot I_1 - \left(\frac{1}{r_2} - \frac{1}{d} \right) \cdot I_2 = 0 \tag{15.267}$$

$$I_1 + I_2 = I \tag{15.268}$$

If the given data are substituted, the system of linear equations becomes:

$$4.8 \cdot I_1 - 2.3 \cdot I_2 = 0 \tag{15.269}$$

$$I_1 + I_2 = 100 \tag{15.270}$$

and its solutions are:

$$I_1 = \frac{2.3}{7.1} \cdot 100 = 32.3943662 \ \mathrm{A} \quad ; \quad I_2 = \frac{4.8}{7.1} \cdot 100 = 67.6056338 \ \mathrm{A} \tag{15.271}$$

According to expression (15.265), the electric scalar potential of the grounding system is:

$$\Phi_{\mathrm{gr}} = \frac{1}{2 \cdot \pi \cdot \kappa} \cdot \left[\frac{I_1}{r_1} + \frac{I_2}{d} \right] = 2.79305717 \ \mathrm{kV} \tag{15.272}$$

whereas the grounding system resistance is:

$$R_{\mathrm{gr}} = \frac{\Phi_{\mathrm{gr}}}{I} = 27.9305717 \ \Omega \tag{15.273}$$

Example 15.6. In an unbounded conducting LIH medium 2, a hole in the shape of an infinitely long hollow cylinder is drilled and filled with conducting LIH medium 1. A stationary electric current flows through both media, such that far from the cylinder, the electric field is homogeneous. Determine the electric voltage between points *A* and *B*. Let it be: $E_0 = 1$ V/m, $\kappa_1 = 0.01$ S/m, $\kappa_2 = 0.1$ S/m, $a = 0.1$ m.

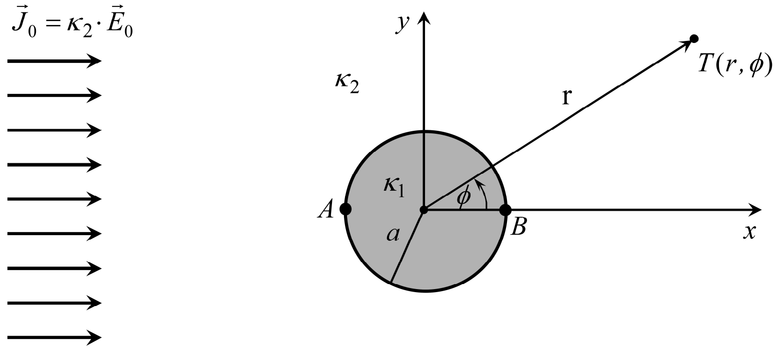


Figure 15.48. Solid conducting cylinder in a homogeneous stationary current field

Solution:

The general solutions of the Laplace differential equation in media 1 and 2, which can satisfy the given boundary conditions, are:

$$\varphi_1 = \sum_{n=1}^{\infty} \left(C_n \cdot r^n + D_n \cdot \frac{1}{r^n} \right) \cdot \cos(n \cdot \phi) \tag{15.274}$$

$$\varphi_2 = \sum_{n=1}^{\infty} \left(A_n \cdot r^n + B_n \cdot \frac{1}{r^n} \right) \cdot \cos(n \cdot \phi) \tag{15.275}$$

In subchapter 15.9, after satisfying the boundary conditions, expression (15.83) was obtained, which reads:

$$\varphi_1 = -\frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot r \cdot \cos\phi \; = \; -\frac{2 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot x \quad ; \quad r \le a \tag{15.276}$$

The electric voltage between points *A* and *B* is:

$$U_{\mathrm{AB}} = \varphi_{\mathrm{A}} - \varphi_{\mathrm{B}} = 2 \cdot \varphi_{\mathrm{A}} = 2 \cdot \varphi_1 \big|_{x=-a} = \frac{4 \cdot \kappa_2}{\kappa_1 + \kappa_2} \cdot E_0 \cdot a \tag{15.277}$$

If the given data are substituted, then it follows that:

$$U_{\mathrm{AB}} = 0.36363636 \;\; \mathrm{V} \tag{15.278}$$

Example 15.7. A thin, perfectly conducting circular plate is located on the surface of an LIH soil with electrical conductivity $\kappa = 0.01$ S/m, which leaks a stationary electric current of $I = 100$ A into the soil. From the requirement that the plate is equipotential, it follows that the surface density of the electric current (on one side of the plate) is described by the expression $J = k \cdot I / \sqrt{a^2 - r^2}$, where $a = 1$ m is the radius of the plate. Calculate: a) the constant *k*, b) the distribution of the electric scalar potential along the *z*-axis perpendicular to the plate, c) the electric scalar potential of the plate and the grounding resistance of the plate.

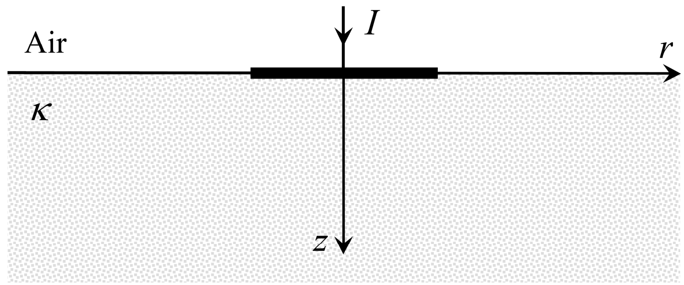


Figure 15.49. Perfectly conducting circular plate on the surface of an LIH soil

Solution:

a) Calculation of the constant k

The equipotential circular plate leaks a stationary electric current I into the soil from only one side, and the surface density of the electric current is described by the expression:

$$J = \frac{k \cdot I}{\sqrt{a^2 - r^2}} \tag{15.279}$$

It holds that:

$$I = \int_S J \cdot dS = k \cdot I \cdot \int_0^a \frac{2 \cdot \pi \cdot r \cdot dr}{\sqrt{a^2 - r^2}} = -k \cdot 2 \cdot \pi \cdot I \cdot \sqrt{a^2 - r^2}\Big|_0^a = k \cdot 2 \cdot \pi \cdot a \cdot I \tag{15.280}$$

and it follows that:

$$k = \frac{1}{2 \cdot \pi \cdot a} \quad ; \quad J = \frac{I}{2 \cdot \pi \cdot a \cdot \sqrt{a^2 - r^2}} \tag{15.281}$$

b) Distribution of the electric scalar potential along the z-axis

By the method of images, the equipotential plate that leaks an electric current I into the LIH soil can be replaced by an equipotential plate in an unbounded LIH medium with conductivity κ, which leaks an electric current $2 \cdot I$ into the surrounding medium, i.e., an electric current I from each of its sides.
It holds that:

$$\varphi(z) = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \int_S \frac{2 \cdot J \cdot dS}{R} = \frac{1}{2 \cdot \pi \cdot \kappa} \cdot \int_0^a \frac{J \cdot 2 \cdot \pi \cdot r \cdot dr}{\sqrt{r^2 + z^2}} \tag{15.282}$$

and it follows that:

$$\varphi(z) = \frac{I}{2 \cdot \pi \cdot \kappa \cdot a} \cdot \int_0^a \frac{r \cdot dr}{\sqrt{z^2 + r^2} \cdot \sqrt{a^2 - r^2}} \tag{15.283}$$

Since:

$$\begin{aligned} &\int_0^a \frac{r \cdot dr}{\sqrt{z^2 + r^2} \cdot \sqrt{a^2 - r^2}} = \begin{pmatrix} a^2 - r^2 = t^2 \\ -r \cdot dr = t \cdot dt \end{pmatrix} \\ &= \int_0^a \frac{dt}{\sqrt{z^2 + a^2 - t^2}} = \left(\arcsin \frac{t}{\sqrt{a^2 + z^2}} \right)\Bigg|_0^a = \arcsin \frac{a}{\sqrt{a^2 + z^2}} = \arctan \frac{a}{|z|} \end{aligned} \tag{15.284}$$

it follows that:

$$\varphi(z) = \frac{I}{2 \cdot \pi \cdot \kappa \cdot a} \cdot \arctan \frac{a}{|z|} \tag{15.285}$$

c) Electric scalar potential of the plate and the grounding resistance of the plate

The electric scalar potential of the plate can be obtained from the expression (15.285) for $z = 0$:

$$\Phi_{\text{pl}} = \varphi(0) = \frac{I}{4 \cdot \kappa \cdot a} = 2.5 \text{ kV} \tag{15.286}$$

and the grounding resistance of the plate is:

$$R_{\text{pl}} = \frac{\Phi_{\text{pl}}}{I} = \frac{1}{4 \cdot \kappa \cdot a} = 25 \ \Omega \tag{15.287}$$

Example 15.8. Two thin circular equipotential plates are located on the surface of an LIH soil with electrical conductivity κ = 0.01 S/m, and a stationary electric current of intensity I = 100 A flows between them. Let the radii of the plates be $r_1 = 20$ m and $r_2 = 5$ m and let the centers of the plates be separated by a distance of d = 100 m. Calculate the electrical resistance between the plates. Using the collocation point method to determine the total mutual resistances of the plates. Let the collocation points be at the centers of the plates.

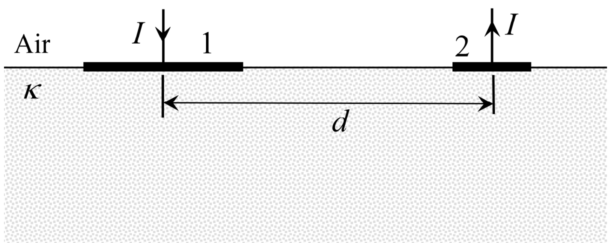


Figure 15.50. Two thin circular equipotential plates on the surface of an LIH soil

Solution:

Since plate 1 leaks an electric current I into the soil, and plate 2 leaks an electric current $-I$ into the soil, the electric scalar potentials of the plates are described by the expressions:

$$\Phi_1 = R_{1,1} \cdot I + R_{1,2} \cdot (-I) = R_{1,1} \cdot I - R_{1,2} \cdot I \tag{15.288}$$

$$\Phi_2 = R_{2,1} \cdot I + R_{2,2} \cdot (-I) = R_{2,1} \cdot I - R_{2,2} \cdot I \tag{15.289}$$

where the total self-resistances of the plates are described by the expressions:

$$R_{1,1} = \frac{1}{4 \cdot \kappa \cdot r_1} = 1.25 \ \Omega \quad ; \quad R_{2,2} = \frac{1}{4 \cdot \kappa \cdot r_2} = 5 \ \Omega \tag{15.290}$$

According to expression (15.232), the mutual total resistances of the plates are calculated using the point collocation method at the center point of each plate:

$$R_{1,2} = \frac{1}{2 \cdot \pi \cdot \kappa \cdot r_2} \cdot \arcsin \frac{r_2}{d} = \frac{2 \cdot R_{2,2}}{\pi} \cdot \arcsin \frac{r_2}{d} = 0.15922133 \ \Omega \tag{15.291}$$

$$R_{2,1} = \frac{1}{2 \cdot \pi \cdot \kappa \cdot r_1} \cdot \arcsin \frac{r_1}{d} = \frac{2 \cdot R_{2,1}}{\pi} \cdot \arcsin \frac{r_1}{d} = 0.16023554 \ \Omega \tag{15.292}$$

The electrical resistance between the plates is:

$$R = \frac{\Phi_1 - \Phi_2}{I} = R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1} = 5.93054313\ \Omega \tag{15.293}$$

Example 15.9. A thin, equipotential circular plate is buried in LIH soil with electrical conductivity $\kappa = 0.01$ S/m at a depth of $h = 0.8$ m. Let the radius of the plate be a =20 m. Calculate the grounding resistance of the plate, using the point collocation method to determine the mutual resistance between the plate and its image. Let the collocation point be at the center of the plate.

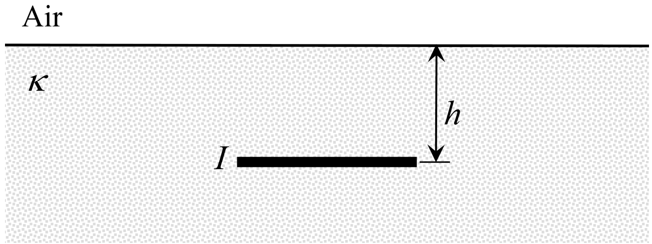


Figure 15.51. Equipotential circular plate buried in LIH soil

Solution:

By the method of images, the equipotential plate that leaks an electric current I into the soil, is mirrored with respect to the soil surface (Figure 15.52).

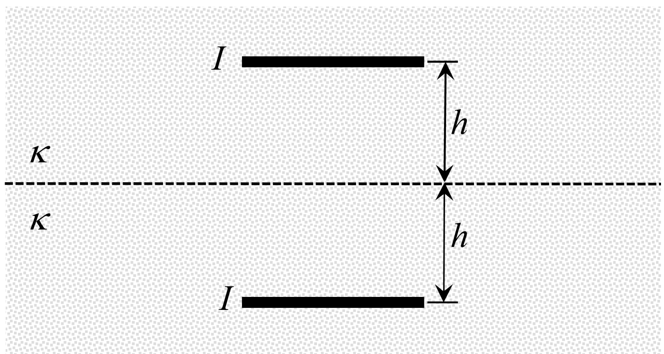


Figure 15.52. Real and image circular plate in an unbounded LIH soil

The electric scalar potential of the plate is described by the expression:

$$\Phi = I \cdot (R_{\text{self}} + R_{\text{mut}}) = I \cdot R_{\text{gr}} \tag{15.294}$$

where, according to expression (15.214), the self-resistance of the real plate is:

$$R_{\text{self}} = \frac{1}{8 \cdot a \cdot \kappa} = 0.625\ \Omega \tag{15.295}$$

whereas, according to expression (15.233), the mutual resistance between the real and image plates, calculated using the point collocation method, is described by the expression:

$$R_{\text{mut}} = \frac{1}{4 \cdot \pi \cdot a \cdot \kappa} \cdot \arctan \frac{a}{2 \cdot h} = 0.593236658\ \Omega \tag{15.296}$$

It follows that the grounding resistance of the equipotential plate is:

$$R_{\text{gr}} = R_{\text{self}} + R_{\text{mut}} = 1.218236658\ \Omega \tag{15.297}$$

Example 15.10. Two equipotential circular plates are buried parallel to the surface of an LIH soil at a depth of $h = 0.8$ m. Let the electrical resistivity of the soil be $\rho = 1000$ Ωm. Let the radii of the plates be $r_1 = r_2 = 10\ \text{m}$, and let the centers of the plates be spaced at a distance of $d = 100$ m. Calculate the electrical resistance between the plates, using the point collocation method to determine the mutual resistances of the plates and their images. The collocation points are located at the centers of the plates.

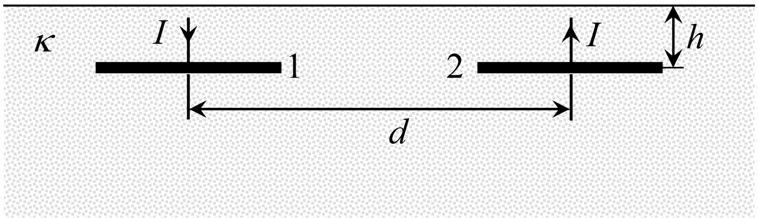


Figure 15.53. Two equipotential circular plates buried in LIH soil

Solution:

According to the method of image, the equipotential plates are mirrored with respect to the soil surface (Figure 15.54).

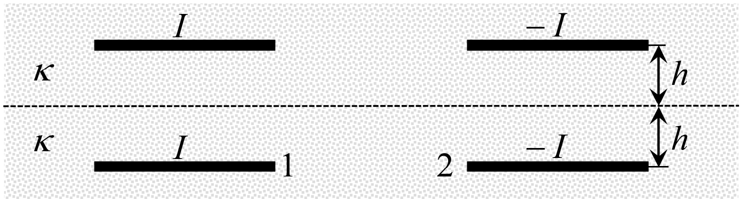


Figure 15.54. Real and image circular plates in an unbounded LIH soil

Since plate 1 leaks an electric current I into the soil, and plate 2 leaks an electric current $-I$ into the soil, the electric scalar potentials of the plates are described by the expressions:

$$\Phi_1 = R_{1,1} \cdot I + R_{1,2} \cdot (-I) = R_{1,1} \cdot I - R_{1,2} \cdot I \tag{15.298}$$

$$\Phi_2 = R_{2,1} \cdot I + R_{2,2} \cdot (-I) = R_{2,1} \cdot I - R_{2,2} \cdot I \tag{15.299}$$

and the electrical resistance between the equipotential plates is given by the following expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1} \tag{15.300}$$

For the total self and mutual resistances of the plates, the following expressions are valid:

$$R_{1,1} = R_{2,2} \quad ; \quad R_{1,2} = R_{2,1} \tag{15.301}$$

because the plates are at the same depth and have equal radii.
It follows that the electrical resistance between the plates is described by the following expression:

$$R = 2 \cdot (R_{1,1} - R_{1,2}) \tag{15.302}$$

where the total self-resistance of plate 1 is described by the following expression:

$$R_{1,1} = \frac{1}{8 \cdot r_1 \cdot \kappa} + \frac{1}{4 \cdot \pi \cdot r_1 \cdot \kappa} \cdot \arctan\frac{r_1}{2 \cdot h} = 23.73746154\ \Omega \tag{15.303}$$

It holds that:

$$\kappa = \frac{1}{\rho} = 0.001 \text{ S/m} \tag{15.304}$$

The total mutual resistance between the plates is described by the following expression:

$$R_{1,2} = \frac{1}{4 \cdot \pi \cdot r_2 \cdot \kappa} \cdot \left(\arcsin\frac{r_2}{d} + \arctan\frac{r_2}{\alpha_{1,2}} \right) = 1.594110635 \ \Omega \tag{15.305}$$

where:

$$\alpha_{1,2} = \sqrt{\frac{A + \sqrt{A^2 + 4 \cdot r_2^2 \cdot (2 \cdot h)^2}}{2}} = 99.51173726 \tag{15.306}$$

whereas:

$$A = d^2 + (2 \cdot h)^2 - r_2^2 = 9.90256 \times 10^3 \tag{15.307}$$

It follows that the electrical resistance between the plates is:

$$R = 2 \cdot (R_{1,1} - R_{1,2}) = 44.28670181 \ \Omega \tag{15.308}$$

Example 15.11. Two equipotential circular plates are buried parallel to the surface of an LIH soil at a depth of $h = 0.8$ m. Let the electrical resistivity of the soil be $\rho = 1000$ Ωm. Let the radii of the plates be $r_1 = 10$ m and $r_2 = 5$ m, and let the centers of the plates be spaced at a distance of $d = 100$ m. Calculate the electrical resistance between the plates, using the point collocation method to determine the mutual resistances of the plates and their images. The collocation points are located at the centers of the plates.

Solution:

This example differs from the previous one only in that the radius of plate 2 is half the size. Therefore, in this case, the radii of the plates are not equal, and the electrical resistance between the plates is described by the following expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1} = 65.622216435 \ \Omega \tag{15.309}$$

where the total self-resistances of the plates are:

$$R_{1,1} = \frac{1}{8 \cdot r_1 \cdot \kappa} + \frac{1}{4 \cdot \pi \cdot r_1 \cdot \kappa} \cdot \arctan\frac{r_1}{2 \cdot h} = 23.73746154 \, \Omega \tag{15.310}$$

$$R_{2,2} = \frac{1}{8 \cdot r_2 \cdot \kappa} + \frac{1}{4 \cdot \pi \cdot r_2 \cdot \kappa} \cdot \arctan\frac{r_2}{2 \cdot h} = 45.07092455 \, \Omega \tag{15.311}$$

The total mutual resistance $R_{1,2}$ is:

$$R_{1,2} = \frac{1}{4 \cdot \pi \cdot r_2 \cdot \kappa} \cdot \left(\arcsin\frac{r_2}{d} + \arctan\frac{r_2}{\alpha_{1,2}} \right) = 1.5921111 \ \Omega \tag{15.312}$$

where:

$$\alpha_{1,2} = \sqrt{\frac{A + \sqrt{A^2 + 4 \cdot r_2^2 \cdot (2 \cdot h)^2}}{2}} = 99.88776909 \tag{15.313}$$

and in this case, it is:

$$A = d^2 + (2 \cdot h)^2 - r_2^2 = 9.97756 \times 10^3 \tag{15.314}$$

The total mutual resistance $R_{2,1}$ is:

$$R_{2,1} = \frac{1}{4 \cdot \pi \cdot r_1 \cdot \kappa} \cdot \left(\arcsin \frac{r_1}{d} + \arctan \frac{r_1}{\alpha_{2,1}} \right) = 1.594110635\ \Omega \tag{15.315}$$

where:

$$\alpha_{2,1} = \sqrt{\frac{A + \sqrt{A^2 + 4 \cdot r_1^2 \cdot (2 \cdot h)^2}}{2}} = 99.51173726 \tag{15.316}$$

and in this case, it is:

$$A = d^2 + (2 \cdot h)^2 - r_2^2 = 9.90256 \times 10^3 \tag{15.317}$$

Example 15.12. For a straight thin-wire cylindrical conductor of length $\ell = 15$ m, which is buried parallel to the surface of an LIH soil at a depth h = 0.8, calculate the grounding resistance. Let the electrical resistivity of the soil be ρ =100 Ωm. Let the radius of the conductor be $r_0 = 5.5$ mm. Solve the problem using the average potential method (APM), assuming that the conductor is approximated in the numerical model by only one thin-wire cylindrical segment.

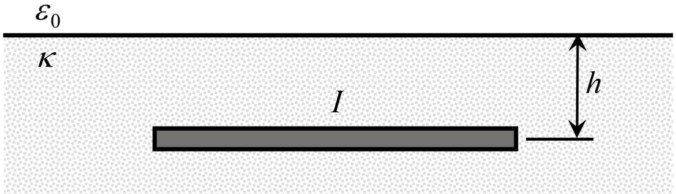


Figure 15.55. Cylindrical conductor in LIH soil

Solution:

The conductor is mirrored with respect to the soil surface (Figure 15.56).

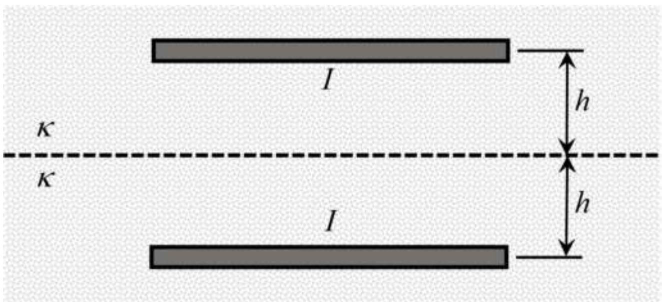


Figure 15.56. Real and image cylindrical conductor

The grounding resistance of the cylindrical conductor is given by the expression:

$$R_{gr} = P(\ell,\ r_0) + P(\ell, 2 \cdot h) \tag{15.318}$$

where, according to expression (15.169), it holds that:

$$P(\ell, v) = \frac{1}{2 \cdot \pi \cdot \kappa \cdot \ell^2} \cdot \left[\ell \cdot \ln \frac{\sqrt{\ell^2 + v^2} + \ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2}\ +\ v} \right] \tag{15.319}$$

If the given data are substituted, then it follows that:

$$R_{gr} = 8.06870065 + 2.15922329 = 10.22792394\ \Omega \tag{15.320}$$

Using the UZEM* software package, the grounding resistance of the given conductor was calculated by dividing the conductor into n mutually equal segments. The procedure has numerically converged if further increasing the total number of equal segments does not affect the final result. Therefore, the result obtained with a sufficiently large number of segments can be considered a numerically accurate solution.

Numerical results show that for n = 100 mutually equal segments, the result can be considered accurate. If the conductor is approximated by a single segment, the error relative to the exact result is 6.52%. If the conductor is divided into 3 equal segments, the error compared to the exact result is 6.31%.

Example 15.13. In LIH soil with electrical resistivity ρ = 100 Ωm, two identical thin-wire cylindrical conductors are buried to a depth of $\ell = 1$ m, perpendicular to the soil surface. Let the radius of the conductor be $r_0 = 0.01$ m. Let a stationary electric current of intensity I = 100 A flow between the conductors. Using the average potential method (APM), calculate the electric scalar potential of each conductor and the electrical resistance between the conductors. The distance between the axes of the conductors is d = 20 m. Approximate the part of the conductor buried in the soil as a single thin-wire segment of length ℓ.

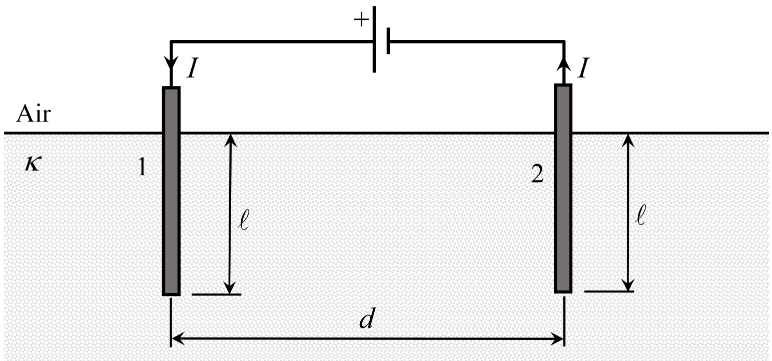


Figure 15.57. Two thin-wire cylindrical conductors

Solution:

According to the method of images, a segment of a thin cylindrical conductor buried in LIH soil with electrical conductivity κ is replaced by two conductor segments in an unbounded LIH medium with the same electrical conductivity κ (Figure 15.58).

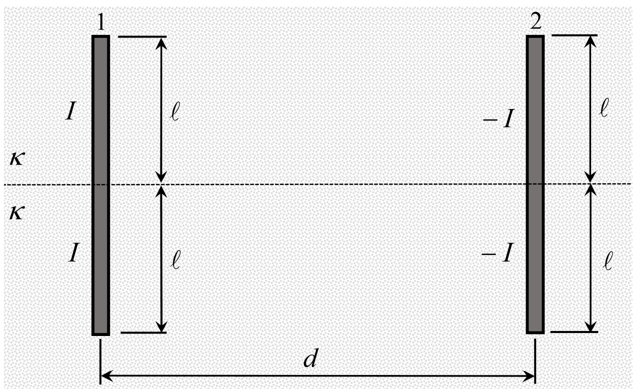


Figure 15.58. Real and image segments of thin-wire cylindrical conductors

* The UZEM software package was developed by the author of this textbook in his doctoral dissertation

In this case, the conductor segment and its image can be considered as a single unified conductor segment of length $2 \cdot \ell,$ which leaks a stationary electric current $\pm 2 \cdot I$ into an unbounded LIH medium. The electric scalar potentials of the unified thin-wire conductors are described by the expressions:

$$\Phi_1 = R_{1,1} \cdot 2 \cdot I + R_{1,2} \cdot (-2 \cdot I) = 2 \cdot R_{1,1} \cdot I - 2 \cdot R_{1,2} \cdot I \tag{15.321}$$

$$\Phi_2 = R_{2,1} \cdot 2 \cdot I + R_{2,2} \cdot (-2 \cdot I) = 2 \cdot R_{2,1} \cdot I - 2 \cdot R_{2,2} \cdot I \tag{15.322}$$

and the electrical resistance between the thin-wire conductors is described by the expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = 2 \cdot \left(R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1}\right) \tag{15.323}$$

For this specific case, the following expressions hold:

$$R_{1,1} = R_{2,2} \quad ; \quad R_{1,2} = R_{2,1} \quad ; \quad \Phi_1 = -\Phi_2 \tag{15.324}$$

because the conductors are identical and have equal radii.

According to the APM, the electrical resistance between the conductors is described by the expression:

$$R = 4 \cdot \left(R_{1,1} - R_{1,2}\right) = 4 \cdot \left[P(2 \cdot \ell, r_0) - P(2 \cdot \ell, d)\right] \tag{15.325}$$

where the function $P(\ell, v)$ is described by the expression (15.169) and also by the expression (15.319). If the given data are substituted, then it follows that:

$$R_{1,1} = P(2 \cdot \ell, r_0) = 39.760551797 \ \Omega \tag{15.326}$$

$$R_{1,2} = P(2 \cdot \ell, d) = 0.397556775 \ \Omega \tag{15.327}$$

$$R = 157.4519801 \ \Omega \tag{15.328}$$

The electric scalar potentials of the conductors are:

$$\Phi_1 = -\Phi_2 = \frac{R}{2} \cdot I = 7.872599004 \text{ kV} \tag{15.329}$$

Example 15.14. In LIH soil with electrical resistivity ρ = 100 Ωm, two thin-wire cylindrical conductors are buried to a depth of $\ell = 1$ m, perpendicular to the soil surface. Let the radii of the conductors be $r_{01} = 0.01$ m and $r_{02} = 0.008$ m. Let a stationary electric current of intensity I = 100 A flow between the conductors. Using the average potential method (APM), calculate the electric scalar potential of each conductor and the electrical resistance between the conductors. The distance between the axes of the conductors is d = 20 m. Approximate the part of the conductor buried in the soil as a single thin-wire segment of length ℓ. Let the conductors be shown in Figure 15.57.

Solution:

This example differs from the previous one only in that the radii of the conductors are different. In this case, the electrical resistance between the conductors is described by the following expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = 2 \cdot \left(R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1}\right) \tag{15.330}$$

and the electrical resistance between the conductors is described by the expression:

$$R = 2 \cdot \left[P(2 \cdot \ell, r_{01}) + P(2 \cdot \ell, r_{02}) - 2 \cdot P(2 \cdot \ell, d)\right] \tag{15.331}$$

where the function $P(\ell, v)$ is described by the expression (15.169) and also by the expression (15.319). If the given data are substituted, then it follows that:

$$R_{1,1} = P(2\cdot\ell, r_{01}) = 39.760551797\ \Omega \tag{15.332}$$

$$R_{2,2} = P(2\cdot\ell, r_{02}) = 41.528331995\ \Omega \tag{15.333}$$

$$R_{1,2} = R_{2,1} = P(2\cdot\ell, d) = 0.397556775\ \Omega \tag{15.334}$$

$$R = 160.9875405\Omega \tag{15.335}$$

The electric scalar potentials of the conductors are:

$$\Phi_1 = 2\cdot\left(R_{1,1} - R_{1,2}\right)\cdot I = 7.872599004\ \text{kV} \tag{15.336}$$

$$\Phi_2 = 2\cdot\left(R_{2,1} - R_{2,2}\right)\cdot I = -8.226155044\ \text{kV} \tag{15.337}$$

Example 15.15. In aLIH soil with electrical resistivity ρ = 100 Ωm, two thin-wire cylindrical conductors are buried to depths $\ell_1 = 1$ m and $\ell_2 = 0.8$ m, perpendicular to the soil surface. Let the radii of the conductors be $r_{01} = 0.01$ m and $r_{02} = 0.008$ m. Let a stationary electric current of intensity I = 100 A flow between the conductors. Calculate the electric scalar potential of each conductor and the electrical resistance between the conductors. The distance between the axes of the conductors is d = 20 m. The part of the conductor buried in the soil can be approximated as a single thin-wire segment. Using the average potential method (APM), calculate the total self-resistances of the conductor segments, and calculate the total mutual resistances of the conductor segments using the point collocation method (PCM). Let the collocation points T_1 and T_2 be located at the center of the buried part of the conductors.

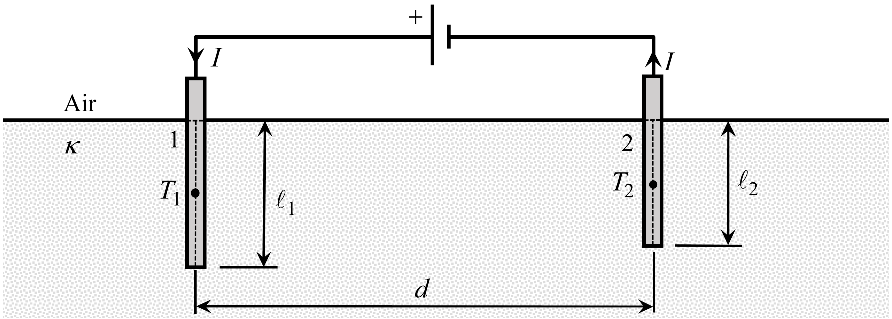


Figure 15.59. Two thin-wire cylindrical conductors of different lengths

Solution:

According to the method of images, a segment of a thin cylindrical conductor buried in LIH soil with electrical conductivity κ is replaced by two conductor segments in an unbounded LIH medium with the same electrical conductivity κ (Figure 15.60). In this case, the conductor segment and its image can be considered as a single unified conductor segment of double length that leaks a stationary electric current $\pm 2\cdot \mathrm{I}$ into an unbounded LIH medium.

The electric scalar potentials of the unified thin-wire conductors are described by the expressions:

$$\Phi_1 = R_{1,1}\cdot 2\cdot I + R_{1,2}\cdot(-2\cdot I) = 2\cdot R_{1,1}\cdot I - 2\cdot R_{1,2}\cdot I \tag{15.338}$$

$$\Phi_2 = R_{2,1}\cdot 2\cdot I + R_{2,2}\cdot(-2\cdot I) = 2\cdot R_{2,1}\cdot I - 2\cdot R_{2,2}\cdot I \tag{15.339}$$

and the electrical resistance between the thin-wire conductors is described by the expression:

$$R = \frac{\Phi_1 - \Phi_2}{I} = 2\cdot\left(R_{1,1} + R_{2,2} - R_{1,2} - R_{2,1}\right) \tag{15.340}$$

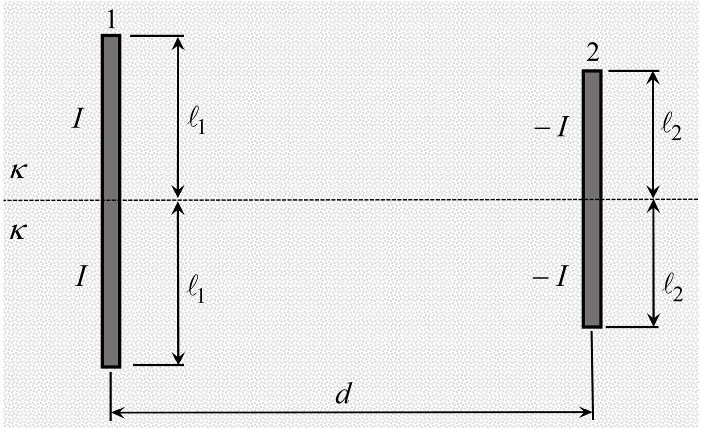


Figure 15.60. Real and image segments of thin-wire cylindrical conductors

The total self-resistances of the segments, according to the APM, are described by the expressions:

$$R_{1,1} = P(2 \cdot \ell_1, r_{01}) \quad ; \quad R_{2,2} = P(2 \cdot \ell_2, r_{02}) \tag{15.341}$$

where the function $P(\ell, v)$ is described by the expression (15.169) and also by the expression (15.319). The total mutual resistances of the segments, according to the PCM, are described by expressions:

$$R_{1,2} = H(2 \cdot \ell_2, \ell_1/2, d) \quad ; \quad R_{2,1} = H(2 \cdot \ell_1, \ell_2/2, d) \tag{15.342}$$

where, according to expression (15.165), the function $H(\ell, u, v)$ is described by the expression:

$$H(\ell, u, v) = \frac{1}{4 \cdot \pi \cdot \kappa \cdot \ell} \cdot \ln \frac{\sqrt{v^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{v^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{15.343}$$

If the given data are substituted, then it follows that:

$$R_{1,1} = P(2 \cdot \ell_1, r_{01}) = 39.760551797\ \Omega \tag{15.344}$$

$$R_{2,2} = P(2 \cdot \ell_2, r_{02}) = 49.700689746\ \Omega \tag{15.345}$$

$$R_{1,2} = H(2 \cdot \ell_2, \ell_1/2, d) = 0.3976573466\ \Omega \tag{15.346}$$

$$R_{2,1} = H(2 \cdot \ell_1, \ell_2/2, d) = 0.3976425012\ \Omega \tag{15.347}$$

$$R = 177.3318834\ \Omega \tag{15.348}$$

The electric scalar potentials of the conductors are:

$$\Phi_1 = 2 \cdot \left(R_{1,1} - R_{1,2}\right) \cdot I = 7.87257899\ \text{kV} \tag{15.349}$$

$$\Phi_2 = 2 \cdot \left(R_{2,1} - R_{2,2}\right) \cdot I = -9.860609449\ \text{kV} \tag{15.350}$$

The total mutual resistance $R_{1,2}$ is equal to the electric scalar potential at the collocation point T_1 due to the unit electric current of the conductor segment 2 (and its image), whereas the total mutual resistance $R_{2,1}$ is equal to the electric scalar potential at the collocation point T_2 due to the unit electric current of the conductor segment 1 (and its image).

16. MAGNETOSTATIC FIELD

The static magnetic field, or magnetostatic field, is a field generated by stationary electric currents and permanent magnets. Stationary electric currents flowing through good conductors produce a stationary current field and a magnetostatic field, which, to a very good approximation, can be considered independent of each other.

The two fundamental cases of the magnetostatic field are:

- Magnetostatic field in a current-free region ($\vec{J}_{\text{tot}} = 0$),
- Magnetostatic field in a current-carrying region ($\vec{J}_{\text{tot}} \neq 0$).

A current-carrying region is a region in which independent sources of the magnetostatic field exist, namely, a stationary electric current.

The magnetostatic field in a linear and isotropic current-free region is analogous to the electrostatic field and the stationary current field in a linear and isotropic source-free region.

16.1. Maxwell's Equations of the Magnetostatic Field

Maxwell's differential equations of the magnetostatic field in a current-carrying region are given as:

$$\nabla \times \vec{H} = \vec{J}_s \equiv \vec{J} \tag{16.1}$$

$$\nabla \cdot \vec{B} = 0 \tag{16.2}$$

where the vector of the surface density of source (impressed) electric current $\vec{J}$ is a known function. The medium in which the field is considered may be either a perfect dielectric or a conducting medium. If the magnetostatic field is considered in a conducting medium, then the vector of the surface density of source (impressed) electric current $\vec{J}$ that creates the magnetostatic field is isolated from the medium itself. Therefore, the only property of the medium that is relevant to the magnetostatic field is its magnetic permeability μ.

Maxwell's differential equations of the magnetostatic field in a current-free region are given as:

$$\nabla \times \vec{H} = 0 \tag{16.3}$$

$$\nabla \cdot \vec{B} = 0 \tag{16.4}$$

Thus, in the general case, Maxwell's differential equations of the magnetostatic field are given as:

$$\nabla \times \vec{H} = \vec{J} \tag{16.5}$$

$$\nabla \cdot \vec{B} = 0 \tag{16.6}$$

In the general case, Maxwell's integral equations of the magnetostatic field are as follows:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J} \cdot d\vec{S} = I_{\text{enc}} \tag{16.7}$$

$$\oint_S \vec{B} \cdot d\vec{S} = 0 \quad ; \quad S \equiv \partial V \tag{16.8}$$

where I_{enc} is the stationary electric current flowing through the oriented surface S enclosed by the oriented curve C (Figure 1.4). In other words, it is the electric current enclosed by the oriented closed curve C, considering the right-hand rule. Maxwell's integral equation (16.7) is Ampère's law.

16.2. Magnetic Properties of Materials

According to expression (3.3), the constitutive equation holds:

$$\vec{B} = \mu_0 \cdot \left(\vec{H} + \vec{M}\right) \tag{16.9}$$

where $\vec{M}$ is the magnetization vector.
In the case of an isotropic material, the constitutive equation is:

$$B = \mu_0 \cdot (1 + \chi_m) \cdot H = \mu_0 \cdot \mu_r \cdot H = \mu \cdot H \tag{16.10}$$

where μ_r is the relative magnetic permeability of the material, whereas χ_m is the magnetic susceptibility of the material.

In an LIH material, the magnetic permeability μ is a scalar constant, whereas in a nonlinear isotropic material, μ is a scalar function dependent on the magnitude of the magnetic field intensity.

The magnetic properties of a material can be explained by the interaction between an external magnetic field and the microscopic magnetic dipoles of atoms and molecules, which are generated by the motion of bound electrons within the atom. The resulting change, manifested in the microscopic magnetic dipole moments within the material, is referred to as the *magnetization* of the material.

With respect to magnetic properties, there are five basic types of materials:

- Diamagnetic materials ($\mu_r < 1, \mu_r \approx 1$) - e.g., C, Cu, Ag, Zn, water,
- Paramagnetic materials ($\mu_r > 1, \mu_r \approx 1$) - e.g., Al, Pb,
- Ferromagnetic materials ($\mu_r >> 1$) - e.g., Fe, Ni, Co, Gd,
- Antiferromagnetic materials ($\mu_r = 1$) - e.g., MnO,
- Ferrimagnetic materials ($\mu_r >> 1$) - e.g., Fe_3O_4, ferrites.

The influence of diamagnetic and paramagnetic materials on the magnetic field can, in most cases, be neglected, i.e., their relative magnetic permeability can be assumed to be equal to 1.

Only diamagnetic materials tend to move out of a magnetic field because their magnetic permeability is lower than the magnetic permeability of vacuum (air).

Antiferromagnetic materials (e.g., manganese fluoride, manganese dioxide, manganese oxide, nickel fluoride) do not respond to a magnetic field at low temperatures, but above the Néel temperature they become paramagnetic. Below the Néel temperature, neighboring microscopic magnetic dipoles within these materials are oriented in opposite directions due to the influence of the external magnetic field.

The difference between ferromagnetic and ferrimagnetic materials is that the molecular magnetic dipoles within each domain of a ferromagnetic material are parallel to each other, whereas strong interatomic bonds in ferrimagnetic materials force some molecular magnetic dipoles in each domain to be antiparallel to the main direction of magnetization in the magnetic domain (Figure 16.1).

A subset of ferrimagnetic materials, known as *ferrites*, is particularly important in engineering because these materials have significantly lower electrical conductivity than ferromagnetic materials (they are semiconductors in terms of electrical conductivity), whereas their magnetic properties are somewhat inferior to those of ferromagnetic materials. Eddy current losses in ferrites are much lower than in ferromagnetic materials, making ferrites suitable for use at high frequencies.

Ferromagnetic materials include iron (Fe), cobalt (Co), nickel (Ni), gadolinium (Gd), dysprosium (Dy), terbium (Tb), holmium (Ho), erbium (Er), and their alloys. Among these chemical elements, the most pronounced ferromagnetic properties are found in Fe, Co, Ni, and Gd, which are referred to as the primary ferromagnetic materials. Ferromagnetic properties are also found in alloys of manganese, copper, and aluminum, as well as alloys of manganese, silver, and aluminum.

In ferromagnetic materials, there are microscopic regions of spontaneous magnetization called magnetic domains. These domains rotate under the influence of an external magnetic field, similar to permanent magnets.

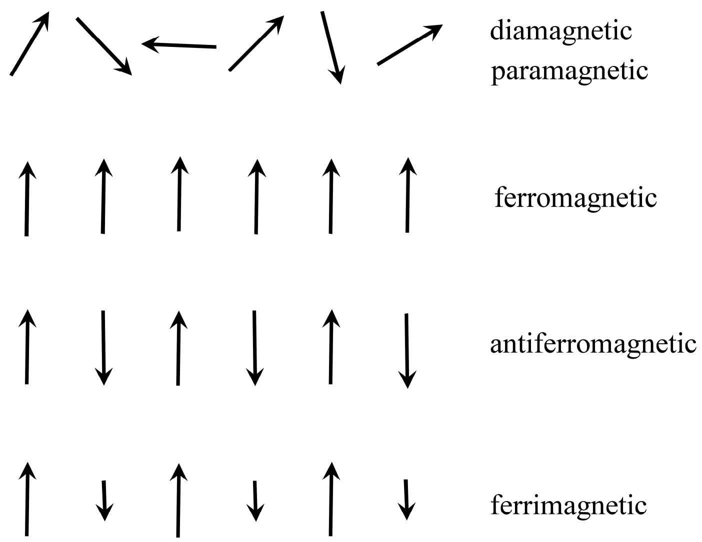


Figure 16.1. Magnetic dipoles in magnetic domains of different materials

16.3. Natural Magnet and Electromagnet

Magnetism was discovered in China and Europe around 800 years BCE. A natural magnet is a permanent magnet found in nature, whereas artificial permanent magnets can be obtained by magnetizing magnetically hard materials. A permanent magnet has two poles:

- North magnetic pole (N),
- South magnetic pole (S).

The English physicist and royal physician William Gilbert determined that like magnetic poles repel each other, whereas unlike poles attract. He published this discovery in 1600 in his book *De Magnete, Magneticisque Corporibus, et de Magno Magnete Tellure* (*On the Magnet and Magnetic Bodies, and on That Great Magnet the Earth*), also known simply as *De Magnete*. The publication of this book is considered by many to mark the beginning of the scientific study of electricity and magnetism. The book describes more than 600 experiments, many of which Gilbert conducted using a metal model of the Earth called a '*terrella*'. He concluded that the Earth itself is a natural magnet, which explains the orientation of the magnetic compass needle. He confirmed the results of Peregrinus's experiment, which showed that isolated magnetic poles or magnetic charges do not exist - when a magnet is divided, its poles cannot be separated. Since it was already known in ancient Greece that amber (electron in Greek), when rubbed with cloth, attracts small objects such as strands of wool, Gilbert introduced the term *electricus* in *De Magnete*, meaning '*like amber*'. From this term, the words electricity and electrical engineering later originated. He called substances that exhibit similar properties to amber 'electrical materials'. Among other contributions, he introduced the concepts of electric force and magnetic pole.

The magnetic field is visually represented by magnetic field lines, where the magnetic flux density vector is tangent to each line. These lines are closed curves that emerge from the north magnetic pole of a magnet and enter the south magnetic pole (Figure 16.2).

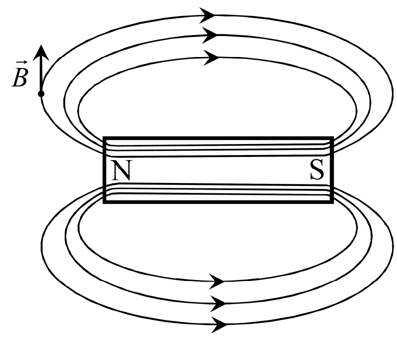


Figure 16.2. Magnetic field lines of a straight permanent magnet

If a small test permanent magnet, such as a compass needle, is placed along a magnetic field line (Figure 16.3), then the north magnetic pole of the magnet repels the north magnetic pole of the compass and attracts its south magnetic pole. Thus, a magnetic field line indicates the direction of the magnetic force acting on the north magnetic pole of the test magnet. Ferromagnetic materials are attracted by both poles of a magnet because a medium with higher permeability tends to occupy the space where a medium with lower permeability, such as air, is present.

The Earth is a large natural magnet (Figure 16.3). The Earth's south magnetic pole is located near its geographic north pole. However, the Earth's south magnetic pole is also referred to as the north geomagnetic pole, which can sometimes cause unnecessary confusion. Today, many scientists claim that the magnetic poles switch places and that the last such reversal occurred about 780,000 years ago. This conclusion is based on the fact that certain rocks contain information about the intensity and direction of the magnetic field at the time of their formation. The Earth's magnetic field reduces the harmful effects of cosmic radiation and the solar wind. Therefore, during each pole reversal, some species go extinct and new ones arise due to mutations. According to one theory, the Earth's magnetic field is generated by electric currents in the molten iron core, which does not rotate uniformly (due to turbulences in its flow), resulting in an irregular magnetic field that is not aligned with the Earth's axis of rotation. For accurate use of a compass, it is therefore necessary to know the so-called magnetic declination for each location on Earth, which is the angle between the geographic meridian and the magnetic field line (the direction in which the compass needle points). Curves that connect locations with the same magnetic declination are called isogons.

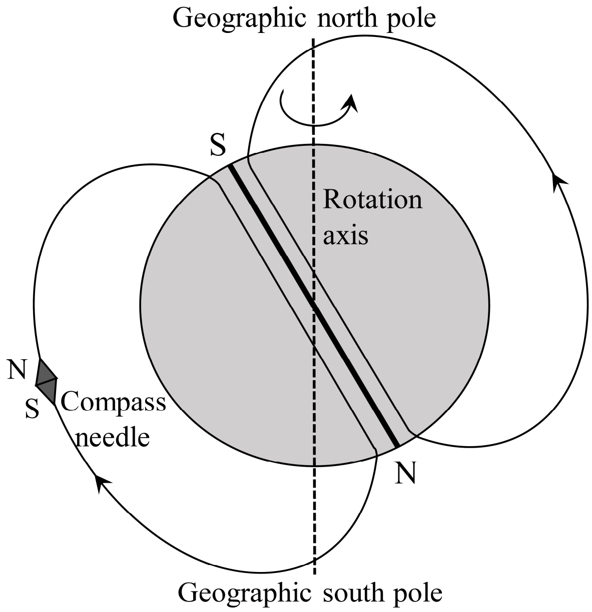


Figure 16.3. Earth's magnetic field

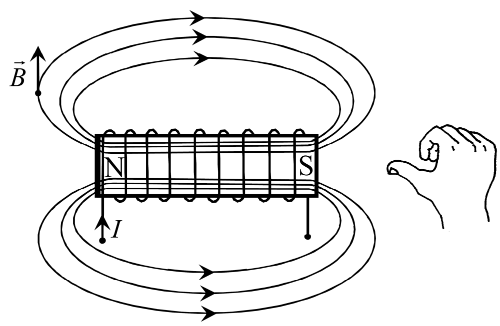


Figure 16.4. Magnetic field lines of a straight electromagnet

An electromagnet is formed when a coil is wound around a ferromagnetic core and a stationary electric current is passed through it. The direction of the magnetic field, i.e., the position of the north and south magnetic poles of the magnet, is determined by the right-hand rule (Figure 16.4). The curled fingers indicate the direction of the stationary electric current, whereas the extended thumb shows the direction of the magnetic field line through the body of the electromagnet.

16.4. Ferromagnetism

In ferromagnetic materials, microscopic regions of spontaneous magnetization, called magnetic domains, rotate under the influence of an external magnetic field, similar to permanent magnets. In an unmagnetized material, magnetic domains are oriented randomly (Figure 16.5).

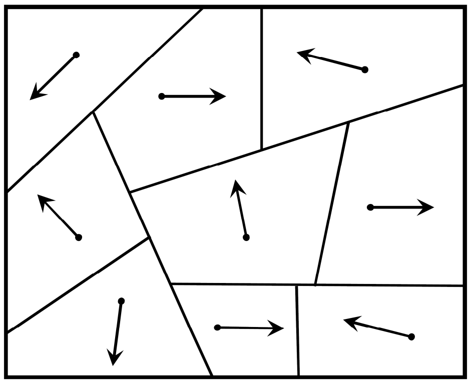


Figure 16.5. Orientation of magnetic domains in an unmagnetized ferromagnetic material

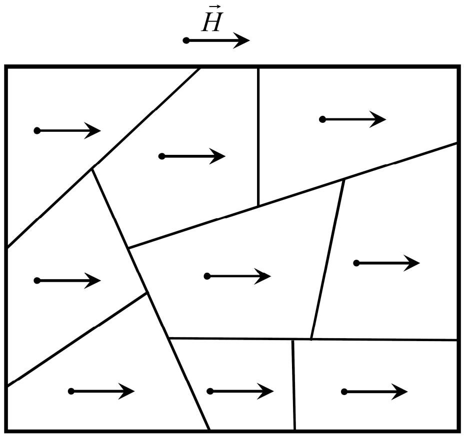


Figure 16.6. Orientation of magnetic domains in a magnetically saturated ferromagnetic material

Under the influence of an external magnetic field, magnetic domains in a ferromagnetic material align in the direction of the external field. When the external magnetic field is strong enough, all magnetic domains become aligned with the field direction (Figure 16.6). This state is referred to as magnetic saturation. Further increase in excitation, i.e., the intensity of the external magnetic field, results in only a negligible increase in magnetic flux density in the ferromagnetic material. The alignment of magnetic domains in a ferromagnetic material under the influence of an external field occurs because like magnetic poles repel each other, whereas unlike poles attract (Figure 16.7).

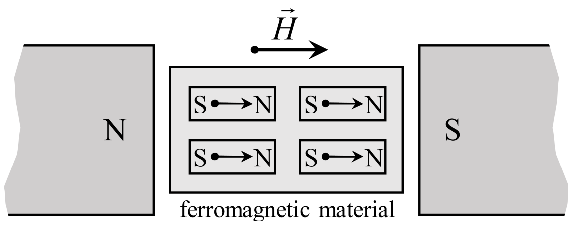


Figure 16.7. Effect of an external magnetic field on magnetic domains

16.5. Magnetic Hysteresis

Hysteresis is a phenomenon that causes the magnetic flux density (B) to lag behind the magnetic field intensity (H), so that the magnetization curve for increasing and decreasing fields is not the same. The loop that represents the magnetization curve is called the hysteresis loop (Figure 16.8). The existence of magnetic hysteresis means that the current state of magnetization of a material depends on its magnetic history.

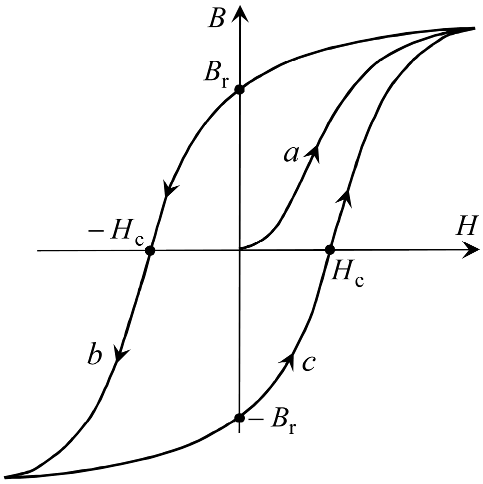


Figure 16.8. Graphical representation of the magnetic hysteresis loop B(H)

Curve a in Figure 16.8 represents the initial magnetization curve, whereas the closed curve is the hysteresis loop that forms during magnetization with alternating current. The quantity B_r is known as remanent magnetism or remanence, whereas the quantity H_c has several names, including coercivity, magnetic coercivity, coercive field, and coercive force.

If a ferromagnetic material is magnetized in one direction under the influence of an external excitation and that excitation is then removed, the material will not become completely demagnetized. Remanent magnetism is the magnetic induction remaining in the ferromagnetic material after the external magnetic field has been removed. Coercivity is the strength of a magnetic field in the opposite direction of the remanent magnetism that is required to demagnetize the ferromagnetic material.

If a ferromagnetic material is magnetized - meaning its magnetic domains are predominantly aligned in one direction - then a certain amount of energy is required to demagnetize it. This property of ferromagnetic materials is the basis for the creation of magnetic memory.

Magnetic materials with relatively high remanent magnetization and coercivity are called hard magnetic materials, whereas those with low remanent magnetization and coercivity are called soft magnetic materials. The cores of transformers and the cores of inductors used in electronics are made of soft magnetic materials. The area of the hysteresis loop is proportional to the amount of thermal energy loss in the ferromagnetic material due to hysteresis.

In physics, a hysteresis loop is represented as the dependence of magnetization (M) on the intensity of the external magnetic field (H), which provides greater insight into the magnetization of the material. However, this representation is not commonly used in electrical engineering practice (Figure 16.9). The quantity M_r is known as remanent magnetization.

According to expression (16.9), the following holds:

$$\vec{B} = \mu_0 \cdot \vec{H} + \vec{P}_m \quad ; \quad \vec{P}_m = \mu_0 \cdot \vec{M} \tag{16.11}$$

where $\vec{P}_m$ is the magnetic polarization vector.

The hysteresis loop $M = M(H)$ is equivalent to the magnetic hysteresis loop $P_m = P_m(H)$, and for high-permeability materials, it is approximately equal to the magnetic hysteresis loop $B(H)$.

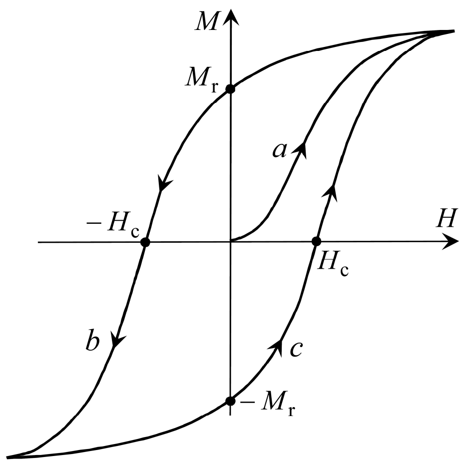


Figure 16.9. Graphical representation of the magnetic hysteresis loop M(H)

16.6. Magnetostatic Field in a Current-Free Region

In a linear and isotropic medium without field sources, there is an analogy among the magnetostatic field, the electrostatic field, and the stationary current field (Table 15.2). In the case of a magnetostatic field, the region without field sources is called a current-free region, which is a region where the surface density of source (impressed) electric current is zero. In the current-free region, instead of the magnetic vector potential, a magnetic scalar potential can be used.

The fundamental equations of the magnetostatic field in a linear and isotropic current-free region are:

$$\nabla\times\vec{H}=0 \;\;;\;\; \nabla\cdot\vec{B}=0 \;\;;\;\; \vec{B}=\mu\cdot\vec{H} \;\;;\;\; \Delta\varphi_{\mathrm{m}}=0 \;\;;\;\; \vec{H}=-\nabla\varphi_{\mathrm{m}} \qquad (16.12)$$

where φ_{m} is the magnetic scalar potential.

Analogous to the electrostatic field, the following holds:

$$V_{\mathrm{AB}}=\int_{\mathrm{A}}^{\mathrm{B}}\vec{H}\cdot d\vec{\ell}=\varphi_{\mathrm{mA}}-\varphi_{\mathrm{mB}} \qquad (16.13)$$

where:

V_{AB} - the magnetic voltage between point *A* and point *B*,

φ_{mA} - the magnetic scalar potential of a point *A*,

φ_{mB} - the magnetic scalar potential of a point *B*.

For magnetic circuits, analogous to electrical circuits, Kirchhoff's laws apply:

$$\sum_{\mathrm{i=1}}^{\mathrm{n}}\Phi_{\mathrm{i}}=0 \qquad (16.14)$$

$$\sum_{\mathrm{k=1}}^{\mathrm{N}}\Theta_{\mathrm{k}}=\sum_{\mathrm{j=1}}^{\mathrm{M}}V_{\mathrm{j}}=\sum_{\mathrm{j=1}}^{\mathrm{M}}\Phi_{\mathrm{j}}\cdot R_{\mathrm{mj}} \quad ; \quad R_{\mathrm{mj}}=\frac{\ell_{\mathrm{j}}}{\mu_{\mathrm{j}}\cdot S_{\mathrm{j}}} \qquad (16.15)$$

where:

$\Phi_{\mathrm{i}}, \Phi_{\mathrm{j}}$ - the magnetic flux,

Θ_{k} - the magnetomotive force,

R_{mj} - the magnetic reluctance,

V_{j} - the magnetic voltage.

Expression (16.14) states that the sum of the magnetic fluxes entering the considered node of the magnetic circuit is zero, whereas expression (16.17) states that along a closed curve (contour), the sum of the magnetomotive forces is equal to the sum of the magnetic voltages.

The second Kirchhoff's law (16.15) can also be written in the more commonly used form:

$$\sum_{k=1}^{N} \Theta_k = \sum_{j=1}^{m} H_j \cdot \ell_j \tag{16.16}$$

and this is, in fact, Ampère's law.

Unlike the electric scalar potential, the magnetic scalar potential is not always well-defined, meaning that it can have two different values at the same point. To illustrate this, consider the magnetic scalar potential around an isolated, infinitely long straight conductor carrying a stationary electric current I (Figure 16.10). Let the infinitely long straight conductor be located along the z-axis of a cylindrical coordinate system (r, ϕ, z). Figure 16.10 shows the field lines and equipotential lines in a current-free region. The field lines are concentric circles centered on the conductor's axis, whereas the equipotential lines are radial rays that are perpendicular to the field lines.

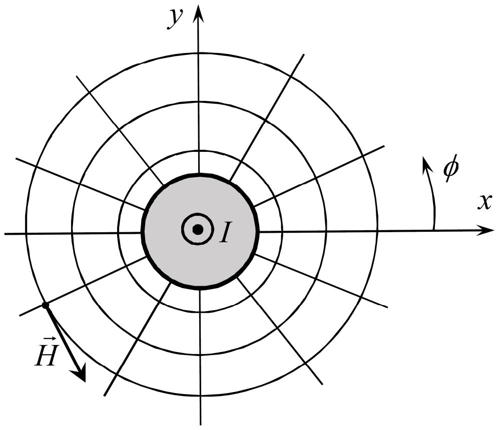


Figure 16.10. Equipotential lines and magnetic field lines around an infinitely long straight conductor carrying a stationary electric current

In the cylindrical coordinate system (r, ϕ, z), the magnetic scalar potential in a current-free region depends only on the angle ϕ, so it holds that:

$$\Delta\varphi_m = \frac{1}{r^2} \cdot \frac{\partial^2 \varphi_m}{\partial \phi^2} = 0 \tag{16.17}$$

from which it follows that:

$$\varphi_m = C_1 \cdot \phi + C_2 \tag{16.18}$$

and the unknown constants C_1 and C_2 can be determined from the boundary conditions:

$$\varphi_m\big|_{\phi=0} = I \quad ; \quad \varphi_m\big|_{\phi=2\cdot\pi} = 0 \tag{16.19}$$

and it follows that the distribution of the magnetic scalar potential is described by the expression:

$$\varphi_m = I \cdot \left(1 - \frac{\phi}{2\cdot\pi}\right) \tag{16.20}$$

The magnetic field intensity is described by the expression:

$$H = H_\phi = -\frac{1}{r} \cdot \frac{\partial \varphi_m}{\partial \phi} = \frac{I}{2\cdot r\cdot\pi} \tag{16.21}$$

16.7. Biot-Savart Law

The fundamental equations of the magnetostatic field in a current-carrying region, within which independent sources of the magnetostatic field are present, are as follows:

$$\nabla\times\vec{H}=\vec{J} \quad ; \quad \nabla\cdot\vec{B}=0 \quad ; \quad \vec{B}=\nabla\times\vec{A} \quad ; \quad \Delta\vec{A}=-\mu\cdot\vec{J} \tag{16.22}$$

whereas in a linear and isotropic medium, the distribution of the magnetic vector potential is described by Poisson's differential equation, which reads:

$$\Delta\vec{A}=-\mu\cdot\vec{J} \tag{16.23}$$

where $\vec{J}$ is the surface density vector of a stationary electric current, which represents an independent source of the magnetostatic field in the considered region.

The particular solution (the solution in a homogeneous and unbounded medium) of Poisson's differential equation (16.23) is given by:

$$\vec{A}=\frac{\mu}{4\cdot\pi}\cdot\int_V\frac{\vec{J}\cdot dV}{r} \tag{16.24}$$

where r is the distance between the source point and the field point.
The following holds:

$$\vec{B}=\nabla\times\vec{A}=\frac{\mu}{4\cdot\pi}\cdot\int_V\left(\nabla\times\frac{\vec{J}}{r}\right)\cdot dV=\frac{\mu}{4\cdot\pi}\cdot\int_V\left[\nabla\left(\frac{1}{r}\right)\right]\times\vec{J}\cdot dV \tag{16.25}$$

because it is:

$$\nabla\times\vec{J}=0 \tag{16.26}$$

Furthermore, it is:

$$\nabla\left(\frac{1}{r}\right)=-\frac{\vec{r}_0}{r^2}=-\frac{\vec{r}}{r^3} \tag{16.27}$$

and after substituting the expression (16.27) into the expression (16.25), the following expression is obtained, as explained in Figure 16.11:

$$\vec{B}=\frac{\mu}{4\cdot\pi}\cdot\int_V\frac{\vec{J}\times\vec{r}}{r^3}\cdot dV \tag{16.28}$$

or equivalently written:

$$\vec{H}=\frac{1}{4\cdot\pi}\cdot\int_V\frac{\vec{J}\times\vec{r}}{r^3}\cdot dV \tag{16.29}$$

These two expressions are known as the Biot-Savart law. This is the integral form of the Biot-Savart law, whereas the differential form of the Biot-Savart law is given by:

$$d\vec{B}=\frac{\mu}{4\cdot\pi}\cdot\frac{\left(\vec{J}\times\vec{r}\right)\cdot dV}{r^3} \quad ; \quad d\vec{H}=\frac{1}{4\cdot\pi}\cdot\frac{\left(\vec{J}\times\vec{r}\right)\cdot dV}{r^3} \tag{16.30}$$

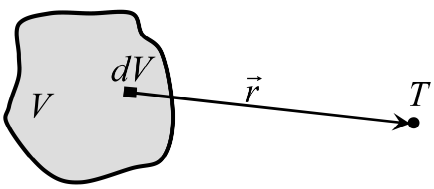


Figure 16.11. Figure associated with the Biot-Savart law

16.8. Magnetic Field of a Thin-Wire Conductor

The thin-wire approximation of a conductor implies that in the numerical model, the electric current flows along the axis of the conductor. In other words, a thin-wire conductor is an idealized conductor represented by a geometric line. Let a stationary electric current of intensity *I* flow through a thin-wire conductor, which is, in the general case, curvilinear (Figure 16.12).

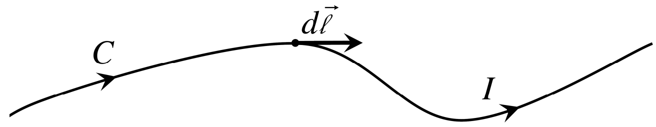


Figure 16.12. Thin-wire conductor carrying a stationary electric current of intensity I

For a thin-wire conductor, it holds that:

$$\vec{J} \cdot dV = I \cdot d\vec{\ell} \tag{16.31}$$

and in an unbounded LIH medium, it holds that:

$$\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \int_C \frac{d\vec{\ell}}{r} \quad ; \quad d\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \frac{d\vec{\ell}}{r} \tag{16.32}$$

whereas the Biot-Savart law is described by the following expressions (Figure 16.13):

$$\vec{H} = \frac{I}{4 \cdot \pi} \cdot \int_C \frac{d\vec{\ell} \times \vec{r}}{r^3} \quad ; \quad d\vec{H} = \frac{I}{4 \cdot \pi} \cdot \frac{d\vec{\ell} \times \vec{r}}{r^3} \tag{16.33}$$

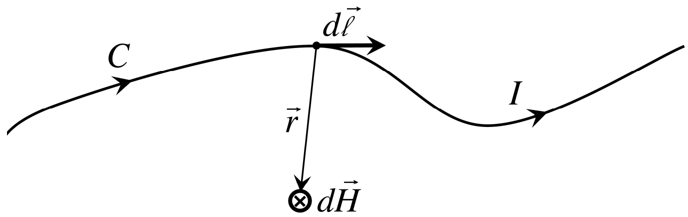


Figure 16.13. Vector of the infinitesimal magnetic field intensity

If necessary, the distance between the source point and the field point can be denoted by *R* instead of *r*, and the following expressions hold:

$$\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \int_C \frac{d\vec{\ell}}{R} \quad ; \quad d\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \frac{d\vec{\ell}}{R} \tag{16.34}$$

$$\vec{H} = \frac{I}{4 \cdot \pi} \cdot \int_C \frac{d\vec{\ell} \times \vec{R}}{R^3} \quad ; \quad d\vec{H} = \frac{I}{4 \cdot \pi} \cdot \frac{d\vec{\ell} \times \vec{R}}{R^3} \tag{16.35}$$

16.8.1. Magnetic Field Intensity of a Straight Thin-Wire Conductor

Let a stationary electric current of intensity *I* flow through a short, straight thin-wire conductor of length ℓ (Figure 16.14-a). Let the conductor be located in an unbounded LIH medium with magnetic permeability μ. The goal is to derive an expression for the distribution of the magnetic field intensity around the conductor, starting from the Biot-Savart law.

Since the distribution of the magnetic field intensity is axially symmetric with respect to the axis along which the thin-wire conductor lies, the solution is sought in the cylindrical coordinate system (r, ϕ, z). The distribution of the magnetic field intensity does not depend on the angle ϕ, and therefore, it only has a ϕ-component. Integration along the axis of the thin-wire conductor is performed as shown in Figure 16.14-b.

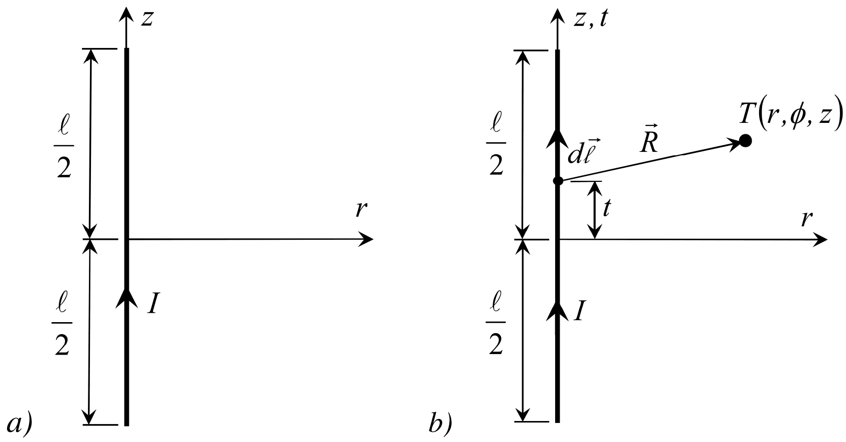


Figure 16.14. Short, straight thin-wire conductor and integration along the conductor's axis

It follows from expression (16.35) that:

$$\vec{H} = \vec{e}_{\phi} \cdot \frac{I}{4 \cdot \pi} \cdot \int_C \frac{\sin \vartheta \cdot R \cdot d\ell}{R^3} \quad ; \quad \vartheta = \angle\left(\vec{R}, d\vec{\ell}\right) \tag{16.36}$$

and it follows that:

$$\vec{H} = \vec{e}_{\phi} \cdot \frac{I}{4 \cdot \pi} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{r \cdot dt}{R^3} = \vec{e}_{\phi} \cdot \frac{I \cdot r}{4 \cdot \pi} \cdot \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{dt}{\left(\sqrt{(t-z)^2 + r^2}\right)^3} \tag{16.37}$$

where:

$$d\ell = dt \quad ; \quad \sin \vartheta = \frac{r}{R} \quad ; \quad R = \sqrt{(z-t)^2 + r^2} \tag{16.38}$$

After performing the analytical integration, it follows from expression (16.37) that:

$$\vec{H} = \vec{e}_{\phi} \cdot \frac{I \cdot r}{4 \cdot \pi} \cdot \left. \frac{t - z}{r^2 \cdot \sqrt{(t-z)^2 + r^2}} \right|_{-\frac{\ell}{2}}^{\frac{\ell}{2}} = \vec{e}_{\phi} \cdot \frac{I}{4 \cdot \pi \cdot r} \cdot \left(\sin\alpha_1 + \sin\alpha_2\right) \tag{16.39}$$

where the angles α_1 and α_2 are shown in Figure 16.15.

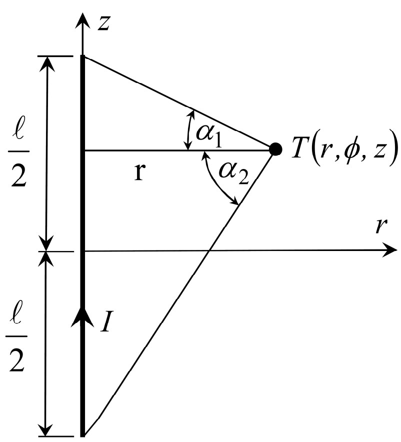


Figure 16.15. Graphical representation of the angles α_1 and α_2

According to Figure 16.15, it holds that:

$$\sin\alpha_1 = \frac{\frac{\ell}{2} - z}{\sqrt{\left(\frac{\ell}{2} - z\right)^2 + r^2}} \quad ; \quad \sin\alpha_2 = \frac{\frac{\ell}{2} + z}{\sqrt{\left(\frac{\ell}{2} + z\right)^2 + r^2}} \tag{16.40}$$

In the special case when $\alpha_1 = \alpha, \alpha_2 = 0$ (Figure 16.16-a), it holds that:

$$H = \frac{I}{4 \cdot \pi \cdot r} \cdot \sin\alpha \tag{16.41}$$

In the special case when $\alpha_1 = \alpha_2 = \alpha$ (Figure 16.16-b), it holds that:

$$H = \frac{2 \cdot I}{4 \cdot \pi \cdot r} \cdot \sin\alpha = \frac{I}{2 \cdot \pi \cdot r} \cdot \sin\alpha \tag{16.42}$$

For an infinitely long straight conductor, it holds that:

$$\alpha_1, \alpha_2 \to \frac{\pi}{2} \quad \Rightarrow \quad \vec{H} = \frac{I}{2 \cdot \pi \cdot r} \cdot \vec{e}_\phi \tag{16.43}$$

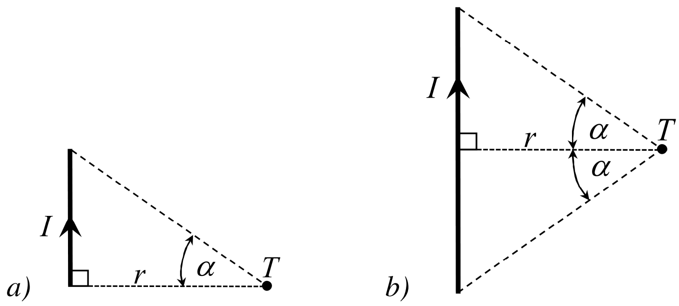


Figure 16.16. Special cases of the straight thin-wire conductor

16.8.2. Magnetic Field Intensity Along the Axis of a Circular Thin-Wire Loop

Let a stationary electric current of intensity I flow through a circular thin-wire loop (current loop) of radius a (Figure 16.17). Let the thin-wire circular loop be located in an unbounded LIH medium with magnetic permeability μ. The goal is to derive an expression for the distribution of the magnetic field intensity along the axis of the thin-wire circular loop, starting from the Biot-Savart law.

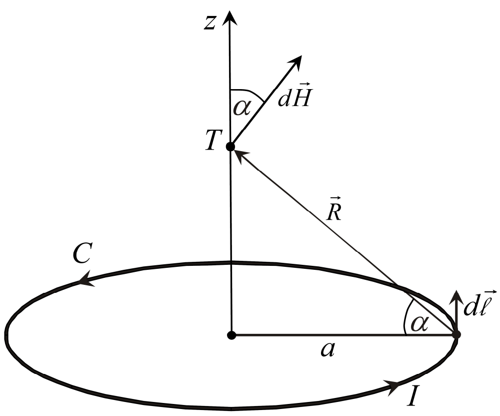


Figure 16.17. Calculation of the magnetic field intensity along the axis of a circular thin-wire loop

Since the distribution of the magnetic field intensity is axially symmetric, the solution is sought in the cylindrical coordinate system (r, ϕ, z). Along the z-axis of the cylindrical coordinate system, the magnetic field intensity has only a z-component. Integration along the thin-wire conductor is performed as shown in Figure 16.17, and to calculate the magnetic field intensity along the z-axis, only the z-component dH_z of the infinitesimal magnetic field intensity dH needs to be integrated.
It follows from expression (16.35) that:

$$dH = \frac{I}{4\cdot\pi}\cdot\frac{d\ell}{R^2} \quad ; \quad \angle\left(d\vec{\ell},\ \vec{R}\right) = \frac{\pi}{2} \tag{16.44}$$

and it follows that, according to Figure 16.17:

$$dH_z = \cos\alpha\cdot dH \ = \frac{I\cdot\cos\alpha}{4\cdot\pi}\cdot\frac{d\ell}{R^2} \tag{16.45}$$

With the substitutions:

$$\cos\alpha = \frac{a}{R} = \frac{a}{\sqrt{z^2+a^2}} \quad ; \quad R = \sqrt{z^2+a^2} \tag{16.46}$$

it follows from expression (16.45) that:

$$dH_z = \frac{I\cdot a}{4\cdot\pi}\cdot\frac{d\ell}{\sqrt{\left(z^2+a^2\right)^3}} \tag{16.47}$$

and it follows that:

$$H = H_z = \frac{I\cdot a}{4\cdot\pi\cdot\sqrt{\left(z^2+a^2\right)^3}}\cdot\oint_C d\ell \ = \frac{I\cdot a^2}{2\cdot\sqrt{\left(z^2+a^2\right)^3}} \tag{16.48}$$

In the special case, at the center of the circular thin-wire loop, the magnetic field intensity is described by the expression:

$$H\Big|_{z=0} = \frac{I}{2\cdot a} \tag{16.49}$$

The expression for the magnetic field intensity at the center of the circular thin-wire loop (Figure 16.18) can be more easily obtained by substituting $R = a$ into expression (16.44), so that:

$$dH = dH_z = \frac{I\cdot d\ell}{4\cdot\pi\cdot a^2} \tag{16.50}$$

and it follows that:

$$H = H_z = \frac{I}{4\cdot\pi\cdot a^2}\cdot\oint_C d\ell \ = \frac{I}{2\cdot a} \tag{16.51}$$

because, in this special case, each infinitesimal segment of the thin-wire loop contributes to the magnetic field intensity in the direction of the z-axis.

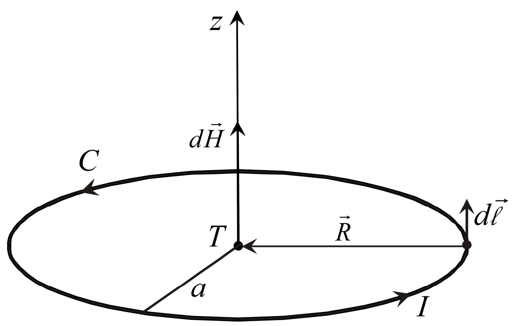


Figure 16.18. Calculation of the magnetic field intensity at the center of the circular thin-wire loop

16.9. Method of Images for a Current-Carrying Segment of a Thin-Wire Conductor

16.9.1. Infinitesimal Segment of a Thin-Wire Conductor Parallel to the Boundary Plane

Let an infinitesimal (infinitely small) segment of a thin-wire conductor, of length $d\ell$, carrying a stationary electric current of intensity I, be positioned at a distance a in front of a homogeneous half-space. Assume that this infinitesimal segment lies in the half-space with magnetic permeability μ_1, whereas the remaining part of the space has magnetic permeability μ_2 (Figure 16.19). For simplicity, let the segment lie along the x-axis of a Cartesian coordinate system (x, y, z), and let the stationary electric current through it flow in the direction of the y-axis.

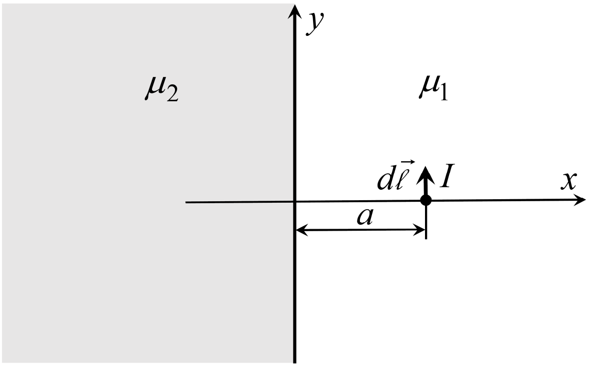


Figure 16.19. Infinitesimal segment of a thin-wire conductor in front of a homogeneous half-space, parallel to the boundary plane

Expressions for the distribution of the infinitesimal magnetic field intensity in both half-spaces can be determined using the method of images. The positions of the image infinitesimal segments of the thin-wire conductor are coherently chosen, and their magnitudes are determined based on the satisfaction of the boundary conditions (Figures 16.20 and 16.21).

According to Figure 16.20, the magnetic field intensity in medium 1, which is considered unbounded, is created by the real electric current I and the image electric current I_2. According to Figure 16.21, the magnetic field intensity in medium 2, which is considered unbounded, is created by the image electric current I_1, which is located at the position of the real electric current.

From expression (16.33), it follows that the distribution of the infinitesimal magnetic field intensity in medium 1 $(x \geq 0)$, according to Figure 16.20, is described by the following expressions:

$$dH_{1x} = \frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot r_{1z}}{r_1^3}+\frac{I_2\cdot r_{2z}}{r_2^3}\right)=\frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot z}{r_1^3}+\frac{I_2\cdot z}{r_2^3}\right) \tag{16.52}$$

$$dH_{1z} = -\frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot r_{1x}}{r_1^3}+\frac{I_2\cdot r_{2x}}{r_2^3}\right)=-\frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot(x-a)}{r_1^3}+\frac{I_2\cdot(x+a)}{r_2^3}\right) \tag{16.53}$$

whereas the distribution of the infinitesimal magnetic field intensity in medium 2 $(x \leq 0)$, according to Figure 16.21, is described by the following expressions:

$$dH_{2x} = \frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot r_{1z}}{r_1^3}=\frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot z}{r_1^3} \tag{16.54}$$

$$dH_{2z} = -\frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot r_{1x}}{r_1^3}=-\frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot(x-a)}{r_1^3} \tag{16.55}$$

where:

$$\vec{r}_1=(x-a)\cdot\vec{\mathrm{i}}+\mathrm{y}\cdot\vec{\mathrm{j}}+z\cdot\vec{\mathrm{k}}=\{r_{1x},r_{1y},r_{1z}\} \quad ; \quad r_1=\sqrt{(x-a)^2+y^2+z^2} \tag{16.56}$$

$$\vec{r}_2 = (x+a)\cdot\vec{\mathrm{i}} + \mathrm{y}\cdot\vec{\mathrm{j}} + z\cdot\vec{\mathrm{k}} = \{r_{2\mathrm{x}}, r_{2\mathrm{y}}, r_{2\mathrm{z}}\} \quad ; \quad r_2 = \sqrt{(x+a)^2 + y^2 + z^2} \tag{16.57}$$

$$d\vec{\ell}\times\vec{r}_1 = d\ell\cdot\left(\vec{\mathrm{j}}\times\vec{r}_1\right) = r_{1\mathrm{z}}\cdot\vec{\mathrm{i}} - r_{1\mathrm{x}}\cdot\vec{\mathrm{k}} = z\cdot\vec{\mathrm{i}} - (x-a)\cdot\vec{\mathrm{k}} \tag{16.58}$$

$$d\vec{\ell}\times\vec{r}_2 = d\ell\cdot\left(\vec{\mathrm{j}}\times\vec{r}_2\right) = r_{2\mathrm{z}}\cdot\vec{\mathrm{i}} - r_{2\mathrm{x}}\cdot\vec{\mathrm{k}} = z\cdot\vec{\mathrm{i}} - (x+a)\cdot\vec{\mathrm{k}} \tag{16.59}$$

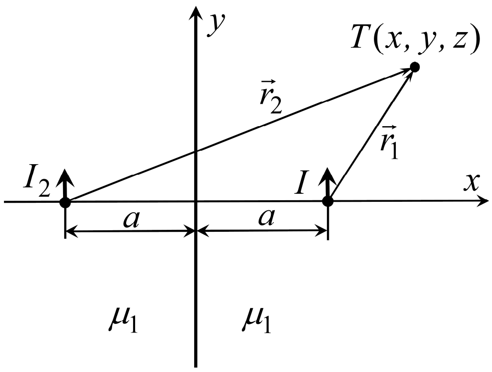


Figure 16.20. Infinitesimal segment of a thin-wire conductor and its image when the magnetic field intensity is calculated in medium 1, which is considered unbounded

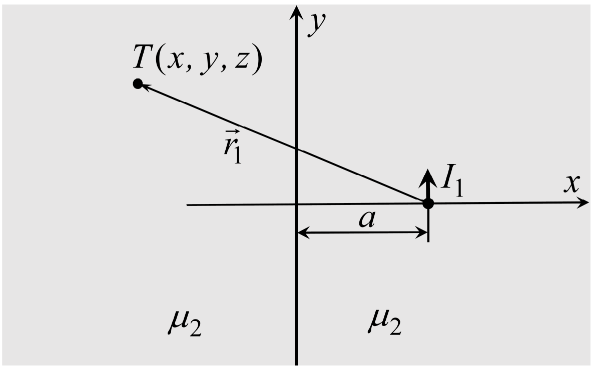


Figure 16.21. Image of the infinitesimal segment of a thin-wire conductor when the magnetic field intensity is calculated in medium 2, which is considered unbounded

Expressions (16.52) - (16.55) must satisfy the following boundary conditions:

$$dH_{1\mathrm{z}}\big|_{x=0} = dH_{2\mathrm{z}}\big|_{x=0} \tag{16.60}$$

$$\mu_1\cdot dH_{1\mathrm{x}}\big|_{x=0} = \mu_2\cdot dH_{2\mathrm{x}}\big|_{x=0} \tag{16.61}$$

If expressions (16.53) and (16.55) are substituted into boundary condition (16.60), taking into account expressions (16.56) and (16.57), the following expression is obtained:

$$I - I_2 = I_1 \tag{16.62}$$

If expressions (16.52) and (16.54) are substituted into boundary condition (16.61), taking into account expressions (16.56) and (16.57), the following expression is obtained:

$$\mu_1\cdot(I + I_2) = \mu_2\cdot I_1 \tag{16.63}$$

From the system of two linear equations (16.62) and (16.63), it follows that:

$$I_2 = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \cdot I = k_{\mathrm{R}} \cdot I \quad ; \quad k_{\mathrm{R}} = \frac{\mu_2 - \mu_1}{\mu_1 + \mu_2} \tag{16.64}$$

$$I_1 = \frac{2 \cdot \mu_1}{\mu_1 + \mu_2} \cdot I = k_{\mathrm{T}} \cdot I \quad ; \quad k_{\mathrm{T}} = \frac{2 \cdot \mu_1}{\mu_1 + \mu_2} \tag{16.65}$$

where k_{R} is the reflection factor, whereas k_{T} is the transmission factor.

It is valid that:

$$k_{\mathrm{R}} + k_{\mathrm{T}} = 1 \tag{16.66}$$

In the special case when $\mu_2 \to \infty$, it is valid that:

$$\lim_{\mu_2 \to \infty} k_{\mathrm{R}} = 1 \quad \Rightarrow \quad \lim_{\mu_2 \to \infty} I_2 = I \tag{16.67}$$

$$\lim_{\mu_2 \to \infty} k_{\mathrm{T}} = 0 \quad \Rightarrow \quad \lim_{\mu_2 \to \infty} I_1 = 0 \tag{16.68}$$

and it follows that the magnetic field intensity vector is perpendicular to the boundary surface of the infinitely permeable medium, and that the magnetic field intensity in the infinitely permeable medium is equal to zero.

According to expression (16.33) and Figure 16.20, the distribution of the infinitesimal magnetic field intensity in medium 1 $(x \geq 0)$ is described by the following expression:

$$d\vec{H}_1 = \frac{I}{4 \cdot \pi} \cdot \left(\frac{d\vec{\ell} \times \vec{r}_1}{r_1^3} + k_{\mathrm{R}} \cdot \frac{d\vec{\ell} \times \vec{r}_2}{r_2^3} \right) \tag{16.69}$$

whereas, according to expression (16.33) and Figure 16.21, the distribution of the infinitesimal magnetic field intensity in medium 2 $(x \leq 0)$ is described by the following expression:

$$d\vec{H}_2 = \frac{k_{\mathrm{T}} \cdot I}{4 \cdot \pi} \cdot \frac{d\vec{\ell} \times \vec{r}_1}{r_1^3} \tag{16.70}$$

Expressions (16.69) and (16.70) are valid if the infinitesimal segment of the thin-wire conductor lies in the plane x = const., and then it holds that:

$$d\vec{\ell} = d\ell_{\mathrm{y}} \cdot \vec{\mathrm{j}} + d\ell_{\mathrm{z}} \cdot \vec{\mathrm{k}} \tag{16.71}$$

16.9.2. Infinitesimal Segment of a Thin-Wire Conductor Perpendicular to the Boundary Plane

Let an infinitesimal segment of a thin-wire conductor, of length $d\ell$, carrying a stationary electric current of intensity I, be positioned at a distance a in front of a homogeneous half-space. Assume that this infinitesimal segment lies in the half-space with magnetic permeability μ_1, whereas the remaining part of the space has magnetic permeability μ_2 (Figure 16.22). For simplicity, let the segment lie along the x-axis of a Cartesian coordinate system (x, y, z), and let the stationary electric current through it flow in the direction of the x-axis (Figure 16.22).

Expressions for the distribution of the infinitesimal magnetic field intensity in both half-spaces can be determined using the method of images. The positions of the image infinitesimal segments of the thin-wire conductor are coherently chosen, and their magnitudes are determined based on the satisfaction of boundary conditions (Figures 16.23 and 16.24).

The distribution of the magnetic field intensity is axisymmetric with respect to the x-axis and has no x-component, which means that the normal component of the magnetic field intensity at the boundary surface is zero. Therefore, it is sufficient to satisfy the boundary condition:

$$dH_{1y}\Big|_{x=0} = dH_{2y}\Big|_{x=0} \quad \text{or} \quad dH_{1z}\Big|_{x=0} = dH_{2z}\Big|_{x=0} \tag{16.72}$$

whereas the boundary condition (16.61) for the normal component of the magnetic flux density is automatically satisfied.

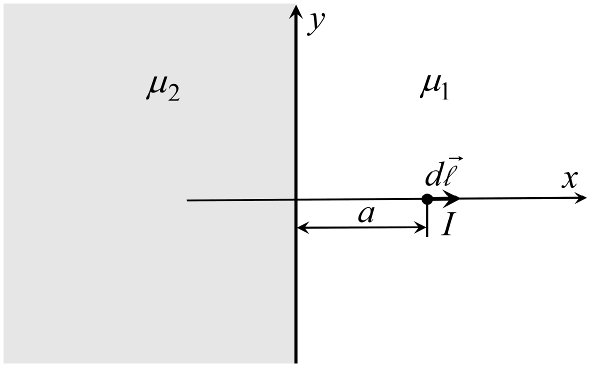


Figure 16.22. An infinitesimal segment of a thin-wire conductor in front of a homogeneous half-space perpendicular to the boundary plane

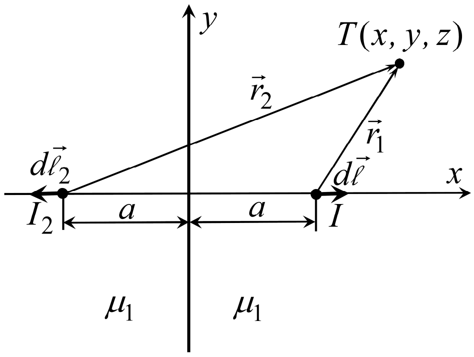


Figure 16.23. Infinitesimal segment of a thin-wire conductor and its image when the magnetic field intensity is calculated in medium 1, which is considered unbounded

According to Figure 16.23, the magnetic field intensity in medium 1, which is considered unbounded, is created by the real electric current I and the image electric current I_2. According to Figure 16.24, the magnetic field intensity in medium 2, which is considered unbounded, is created by the image electric current I_1, which is located at the position of the real electric current.

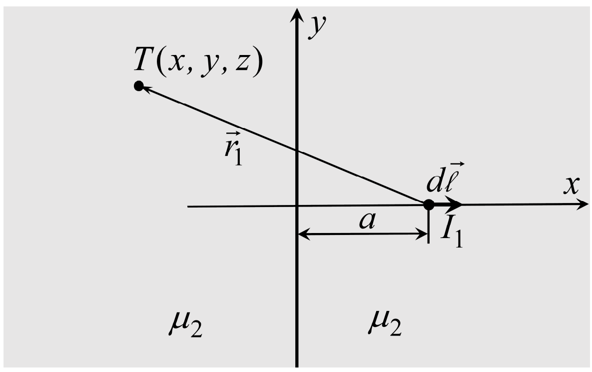


Figure 16.24. Image of the infinitesimal segment of a thin-wire conductor when the magnetic field intensity is calculated in medium 2, which is considered to be unbounded

According to expression (16.33) and Figure 16.23, the distribution of the infinitesimal magnetic field intensity in medium 1 $(x \geq 0)$ is described by the following expression:

$$dH_{1y} = -\frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot r_{1z}}{r_1^3} - \frac{I_2\cdot r_{2z}}{r_2^3}\right) = -\frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot z}{r_1^3} - \frac{I_2\cdot z}{r_2^3}\right) \tag{16.73}$$

$$dH_{1z} = \frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot r_{1y}}{r_1^3} - \frac{I_2\cdot r_{2y}}{r_2^3}\right) = \frac{d\ell}{4\cdot\pi}\cdot\left(\frac{I\cdot y}{r_1^3} - \frac{I_2\cdot y}{r_2^3}\right) \tag{16.74}$$

whereas, according to expression (16.33) and Figure 16.24, the distribution of the infinitesimal magnetic field intensity in medium 2 $(x \le 0)$ is described by the following expression:

$$dH_{2y} = -\frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot r_{1z}}{r_1^3} = -\frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot z}{r_1^3} \tag{16.75}$$

$$dH_{2z} = \frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot r_{1y}}{r_1^3} = \frac{d\ell}{4\cdot\pi}\cdot\frac{I_1\cdot y}{r_1^3} \tag{16.76}$$

where:

$$d\vec{\ell}\times\vec{r}_1 = d\ell\cdot\left(\vec{\mathrm{i}}\times\vec{r}_1\right) = -\,r_{1z}\cdot\vec{\mathrm{j}} + r_{1x}\cdot\vec{\mathrm{k}} = -\,z\cdot\vec{\mathrm{j}} + y\cdot\vec{\mathrm{k}} \tag{16.77}$$

$$d\vec{\ell}_2\times\vec{r}_2 = -\,d\vec{\ell}\times\vec{r}_2 = -\,d\ell\cdot\left(\vec{\mathrm{i}}\times\vec{r}_2\right) = \ r_{2z}\cdot\vec{\mathrm{j}} - r_{2x}\cdot\vec{\mathrm{k}} = z\cdot\vec{\mathrm{j}} - y\cdot\vec{\mathrm{k}} \tag{16.78}$$

whereas the vectors $\vec{r}_1$ and $\vec{r}_2$, as well as their magnitudes, are described by (16.56) and (16.57)

Expressions (16.73) - (16.76) must satisfy the boundary conditions (16.72), from which expression (16.62) is obtained. This means that for the infinitesimal segment of the conductor perpendicular to the boundary plane between two media, the reflection factor (16.64) and transmission factor (16.65) hold, which were derived for the infinitesimal segment of the conductor parallel to the boundary plane between the two media.

According to expression (16.33), the distribution of the infinitesimal magnetic field intensity in medium 1 $(x \ge 0)$ is described by the following expression:

$$d\vec{H}_1 = \frac{I}{4\cdot\pi}\cdot\left(\frac{d\vec{\ell}\times\vec{r}_1}{r_1^3} + k_{\mathrm{R}}\cdot\frac{d\vec{\ell}_2\times\vec{r}_2}{r_2^3}\right) \tag{16.79}$$

whereas the distribution of the infinitesimal magnetic field intensity in medium 2 $(x \le 0)$ is described by the following expression:

$$d\vec{H}_2 = \frac{k_{\mathrm{T}}\cdot I}{4\cdot\pi}\cdot\frac{d\vec{\ell}\times\vec{r}_1}{r_1^3} \tag{16.80}$$

16.9.3. Method of Images for a Thin-Wire Conductor of Arbitrary Shape

Let us consider a thin-wire conductor of arbitrary shape and length ℓ, carrying a stationary electric current of intensity I, placed in front of a homogeneous half-space. The conductor is located in a half-space with magnetic permeability μ_1, whereas the remaining part of the space has magnetic permeability μ_2 (Figure 16.25). For the sake of visual clarity in the graphical representation, assume that the conductor lies in the plane $z = 0$ of the Cartesian coordinate system (x, y, z). The stationary electric current flows through the conductor in the indicated direction (Figure 16.25).

According to Figure 16.26, the magnetic field intensity in medium 1, which is considered to be unbounded, is generated by the real electric current I and an image electric current I_2, whose position is the mirror image of the real current. According to Figure 16.27, the magnetic field intensity in medium 2, which is also considered to be unbounded, is generated by an electric current I_1 located at the position of the real electric current. The reflection factor k_{R} is given by expression (16.64), whereas the transmission factor k_{T} is given by expression (16.65).

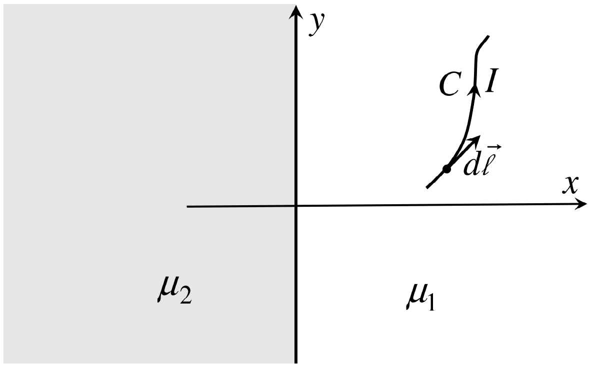


Figure 16.25. Thin-wire conductor in front of a homogeneous half-space

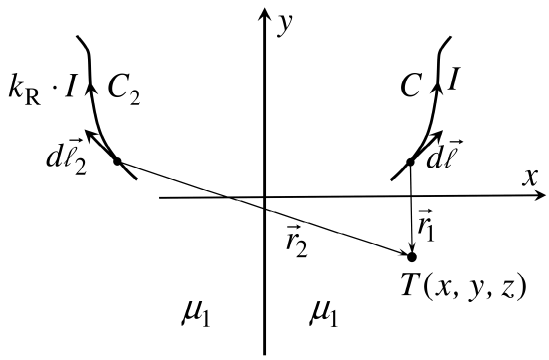


Figure 16.26. Thin-wire conductor and its mirror image when the magnetic field intensity is calculated in medium 1, which is considered unbounded

According to expression (16.33), the distribution of the magnetic field intensity in medium 1 $(x \geq 0)$ is described by the following expression:

$$\vec{H}_1 = \frac{I}{4 \cdot \pi} \cdot \left(\int_C \frac{d\vec{\ell} \times \vec{r}_1}{r_1^3} + k_R \cdot \int_{C_2} \frac{d\vec{\ell}_2 \times \vec{r}_2}{r_2^3} \right) \tag{16.81}$$

whereas the distribution of the magnetic field intensity in medium 2 $(x \leq 0)$ is described by the following expression:

$$\vec{H}_2 = \frac{k_T \cdot I}{4 \cdot \pi} \cdot \int_C \frac{d\vec{\ell} \times \vec{r}_1}{r_1^3} \tag{16.82}$$

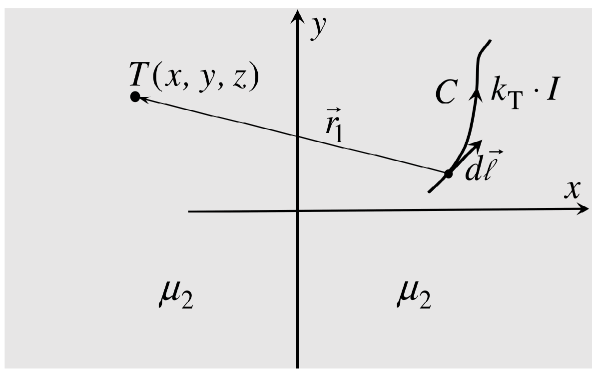


Figure 16.27. The image of the thin-wire conductor when the magnetic field intensity is calculated in medium 2, which is considered unbounded

Since the boundary surface between the two media is $x = 0$, it follows that:

$$d\vec{\ell} = d\ell_{\mathrm{x}} \cdot \vec{\mathrm{i}} + d\ell_{\mathrm{y}} \cdot \vec{\mathrm{j}} + d\ell_{\mathrm{z}} \cdot \vec{\mathrm{k}} \quad ; \quad d\vec{\ell}_2 = -\, d\ell_{\mathrm{x}} \cdot \vec{\mathrm{i}} + d\ell_{\mathrm{y}} \cdot \vec{\mathrm{j}} + d\ell_{\mathrm{z}} \cdot \vec{\mathrm{k}} \tag{16.83}$$

16.10. Energy Stored in the Magnetostatic Field

According to expression (12.3), the energy stored in the magnetic field can be calculated by integrating over the observed volume, such that:

$$W_{\mathrm{m}} = \int_0^t \int_V \vec{H} \cdot \frac{\partial \vec{B}}{\partial t} \cdot dV \cdot dt = \int_V \int_0^B \vec{H} \cdot \delta\vec{B} \cdot dV \quad ; \quad \delta\vec{B} = \frac{\partial \vec{B}}{\partial t} \cdot dt \tag{16.84}$$

If the medium is linear, then the previous expression takes the following form:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V \vec{H} \cdot \vec{B} \cdot dV \tag{16.85}$$

whereas for a linear and isotropic medium, the stored magnetostatic energy is described by the following expression:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V H \cdot B \cdot dV = \frac{1}{2} \cdot \int_V \mu \cdot H^2 \cdot dV \tag{16.86}$$

In the case of an unbounded medium, according to expression (12.12), the total stored magnetostatic energy can be calculated by integrating over the source, such that:

$$W_{\mathrm{m}} = \int_V \int_0^A \vec{J} \cdot \delta\vec{A} \cdot dV \tag{16.87}$$

where for a linear medium, the total stored magnetostatic energy is described by the following expression:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V \vec{J} \cdot \vec{A} \cdot dV \tag{16.88}$$

where the surface density of electric current $\vec{J}$ is isolated from the medium if the medium is conducting. In this case, expressions (16.87) and (16.88) hold for both a perfect dielectric and a conducting medium, with the only significant characteristic of the medium being its magnetic permeability.

16.10.1. Calculation of Inductance from the Energy Stored in the Magnetostatic Field

The inductance of the system can be calculated by first determining the energy stored in the magnetostatic field of the system. For the magnetostatic field in a linear and isotropic medium, the following holds:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \int_V B \cdot H \cdot dV = \frac{1}{2} \cdot \int_V \mu \cdot H^2 \cdot dV = \frac{1}{2} \cdot L \cdot I^2 = \frac{\Psi^2}{2 \cdot L} \tag{16.89}$$

where Ψ is the magnetic flux linkage, and L is the inductance of the system.

From expression (16.89), it follows that the inductance of the system is given by the following expression:

$$L = \frac{2 \cdot W_{\mathrm{m}}}{I^2} = \frac{1}{I^2} \cdot \int_V B \cdot H \cdot dV = \frac{1}{I^2} \cdot \int_V \mu \cdot H^2 \cdot dV \tag{16.90}$$

16.10.2. Self-Inductances and Mutual Inductances of a System of Thin-Wire Loops

Let there be n thin-wire loops (current loops) in a linear medium, each carrying a stationary electric current. The magnetic energy stored in such a system can be described by the following expression:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \sum_{\mathrm{j}=1}^{\mathrm{n}} L_{\mathrm{j}} \cdot I_{\mathrm{j}}^2 + \frac{1}{2} \cdot \sum_{\mathrm{j}=1}^{\mathrm{n}} \sum_{\substack{\mathrm{k}=1 \\ \mathrm{k} \neq \mathrm{j}}}^{\mathrm{n}} M_{\mathrm{j,k}} \cdot I_{\mathrm{j}} \cdot I_{\mathrm{k}} \tag{16.91}$$

where:

L_{j} - the inductance (self-inductance) of the j-th thin-wire loop,

$M_{\mathrm{j,k}} = M_{\mathrm{k,j}}$ - the mutual inductance between the j-th and k-th thin-wire loops.

16.11. Magnetic Dipole

A magnetic dipole is a thin-wire loop (current loop) of arbitrary shape carrying a stationary electric current, for which the dimensions of the loop are much smaller than the distance from the field point to the loop, i.e., $r >> a$ (Figure 16.28).

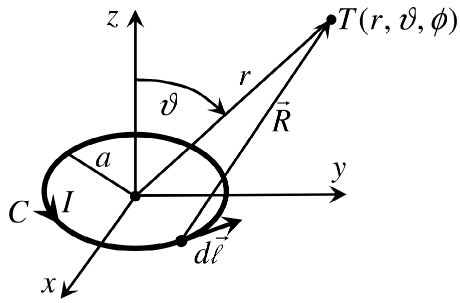


Figure 16.28. Magnetic dipole in a spherical coordinate system

For the sake of simplicity, let the thin-wire loop be circular, with radius a, and let it carry a stationary electric current of intensity I. In the mathematical derivation, for simplicity, let the field point T lie in the plane x =. The magnetic dipole is assumed to be located in an unbounded LIH medium with permeability μ.

According to expression (16.34), the magnetic vector potential of the thin-wire circular loop is described by the following expression:

$$\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \oint_C \frac{d\vec{\ell}}{R} \quad ; \quad d\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \frac{d\vec{\ell}}{R} \tag{16.92}$$

Due to the axial symmetry with respect to the z-axis, in the spherical coordinate system (r, ϑ, ϕ), the following holds (Figure 16.29):

$$A = A_{\phi} \quad ; \quad dA_{\phi} = \sin\phi \cdot dA \tag{16.93}$$

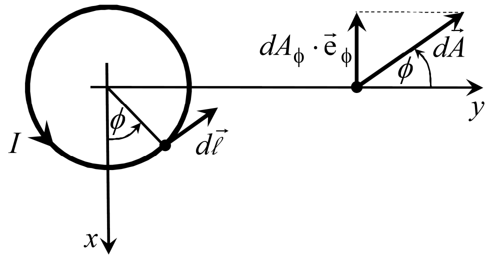


Figure 16.29. Graphical explanation of expression (16.93) – top view in the z = 0 plane

It follows that:

$$\vec{A} = \vec{e}_{\phi} \cdot \frac{\mu \cdot I}{4 \cdot \pi} \cdot \oint_C \frac{\sin\phi \cdot d\ell}{R} = \vec{e}_{\phi} \cdot \frac{\mu \cdot I \cdot a}{4 \cdot \pi} \cdot \int_0^{2\cdot\pi} \frac{\sin\phi \cdot d\phi}{R} \quad ; \quad d\ell = a \cdot d\phi \tag{16.94}$$

If two opposite points on the thin-wire circular loop are connected (Figure 16.30), it follows that:

$$\vec{A} = \vec{e}_{\phi} \cdot \frac{\mu \cdot I \cdot a}{4 \cdot \pi} \cdot \int_0^{\pi} \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \cdot \sin\phi \cdot d\phi \tag{16.95}$$

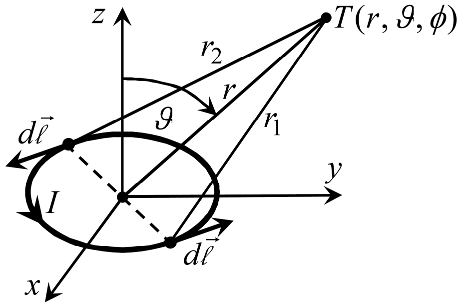


Figure 16.30. Connecting two opposite points on the thin-wire circular loop

Similar to the electric dipole described in subchapter 14.10, for $r >> a$ (Figure 16.31-a), the following holds:

$$r_1 \approx r - a \cdot \cos\alpha \quad ; \quad r_2 \approx r + a \cdot \cos\alpha \tag{16.96}$$

and it follows that:

$$r_2 - r_1 \approx 2 \cdot a \cdot \cos\alpha \quad ; \quad r_1 \cdot r_2 \approx r^2 - a^2 \cdot \cos^2\alpha \approx r^2 \tag{16.97}$$

from which it follows that:

$$\frac{1}{r_1} - \frac{1}{r_2} = \frac{r_2 - r_1}{r_1 \cdot r_2} \approx \frac{2 \cdot a \cdot \cos\alpha}{r^2} \tag{16.98}$$

If approximation (16.98) is substituted into expression (16.95), the following expression is obtained:

$$\vec{A} = \vec{e}_{\phi} \cdot \frac{\mu \cdot I \cdot a^2}{2 \cdot \pi \cdot r^2} \cdot \int_0^{\pi} \cos\alpha \cdot \sin\phi \cdot d\phi \tag{16.99}$$

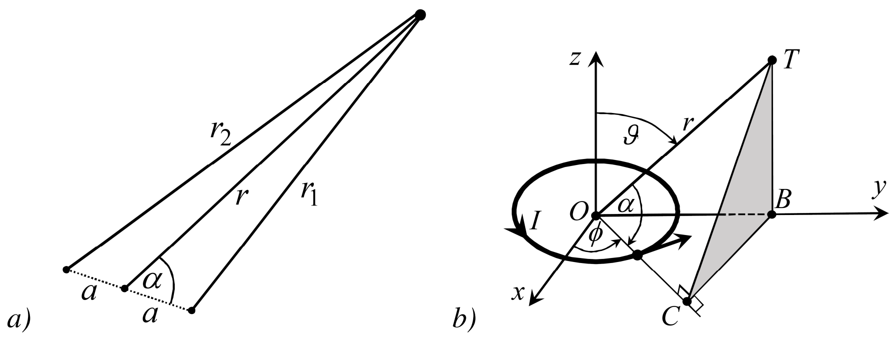


Figure 16.31. Definition of the angle α

According to Figure 16.31-b, for the point T that lies in the plane $x = 0$, the following is true:

$$\overline{OC} = r \cdot \cos\alpha = \overline{OB} \cdot \cos(\pi/2 - \phi) = \overline{OB} \cdot \sin\phi \quad ; \quad \overline{OB} = r \cdot \sin\vartheta \tag{16.100}$$

from which it follows that:

$$\cos\alpha = \sin\vartheta \cdot \sin\phi \tag{16.101}$$

If the expression (16.101) is substituted into the expression (16.99), the following expression is obtained:

$$\vec{A} = \vec{\mathrm{e}}_{\phi} \cdot \frac{\mu \cdot I \cdot a^2 \cdot \sin\vartheta}{2 \cdot \pi \cdot r^2} \cdot \int_0^{\pi} \sin^2\phi \cdot d\phi = \vec{\mathrm{e}}_{\phi} \cdot \frac{\mu \cdot I \cdot a^2 \cdot \sin\vartheta}{2 \cdot \pi \cdot r^2} \cdot \frac{\pi}{2} \tag{16.102}$$

from which it follows that for $r >> a$, the magnetic vector potential of a magnetic dipole can be approximated using the following expression:

$$\vec{A} = \frac{\mu \cdot \vec{m} \times \vec{r}}{4 \cdot \pi \cdot r^3} = \frac{\mu \cdot m \cdot \sin\vartheta}{4 \cdot \pi \cdot r^2} \cdot \vec{e}_{\phi} \tag{16.103}$$

where the magnetic dipole moment is (Figure 16.32):

$$\vec{m} = I \cdot S \cdot \vec{n} = I \cdot a^2 \cdot \pi \cdot \vec{\mathrm{k}} \quad ; \quad m = I \cdot S = I \cdot a^2 \cdot \pi \tag{16.104}$$

where S is the area of a circular thin-wire loop.

The unit normal vector $\vec{n}$ to the surface enclosed by the loop is oriented according to the right-hand rule with respect to the direction of the electric current (Figure 16.32).

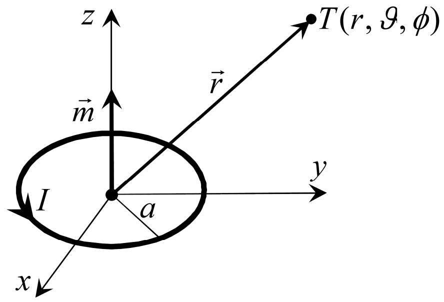


Figure 16.32. Graphical representation of the magnetic dipole moment $\vec{m}$

If the thin-wire loop has some other arbitrary shape, such as rectangular or square, then the expression for the magnetic dipole moment holds[*]:

$$\vec{m} = I \cdot S \cdot \vec{n} \tag{16.105}$$

where, in this general case, S is the area of the considered thin-wire loop, whereas $\vec{n}$ is the unit normal vector to the surface enclosed by the loop, oriented according to the right-hand rule.

It holds that:

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r^2 \cdot \sin\vartheta} \cdot \begin{vmatrix} \vec{e}_{\mathrm{r}} & r \cdot \vec{e}_{\vartheta} & r \cdot \sin\vartheta \cdot \vec{e}_{\phi} \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \vartheta} & 0 \\ 0 & 0 & r \cdot \sin\vartheta \cdot A \end{vmatrix} \tag{16.106}$$

[*] The definition and proof for an arbitrary thin-wire loop are given in [2] (pp. 63–64), whereas the proof for a square thin-wire loop is presented in [6] (pp. 160–161).

According to expression (16.108), the components of the magnetic flux density vector of the magnetic dipole are:

$$B_{\mathrm{r}} = \frac{\mu \cdot m \cdot \cos\vartheta}{2 \cdot \pi \cdot r^3} \quad ; \quad B_{\vartheta} = \frac{\mu \cdot m \cdot \sin\vartheta}{4 \cdot \pi \cdot r^3} \quad ; \quad B_{\phi} = 0 \tag{16.107}$$

and the components of the magnetic field intensity vector of the magnetic dipole are:

$$H_{\mathrm{r}} = \frac{m \cdot \cos\vartheta}{2 \cdot \pi \cdot r^3} \quad ; \quad H_{\vartheta} = \frac{m \cdot \sin\vartheta}{4 \cdot \pi \cdot r^3} \quad ; \quad H_{\phi} = 0 \tag{16.108}$$

Based on the comparison of expression (16.108) with the expressions for the components of the electric field intensity of the electric dipole (14.87) - (14.89), taking into account the expression for the electric scalar potential of the electric dipole (14.84), it is easy to conclude that the magnetic scalar potential of the magnetic dipole can be described by the following expression:

$$\varphi_{\mathrm{m}} = \frac{\vec{m} \cdot \vec{r}}{4 \cdot \pi \cdot r^3} = \frac{m \cdot \cos\vartheta}{4 \cdot \pi \cdot r^2} \tag{16.109}$$

and it follows that:

$$\vec{H} = -\nabla \varphi_{\mathrm{m}} = -\frac{\partial \varphi_{\mathrm{m}}}{\partial r} \cdot \vec{e}_{\mathrm{r}} - \frac{1}{r} \cdot \frac{\partial \varphi_{\mathrm{m}}}{\partial \vartheta} \cdot \vec{e}_{\vartheta} - \frac{1}{r \cdot \sin\vartheta} \cdot \frac{\partial \varphi_{\mathrm{m}}}{\partial \phi} \cdot \vec{e}_{\phi} \tag{16.110}$$

from which the expressions for the components of the magnetic field intensity vector of the magnetic dipole can be obtained:

$$H_{\mathrm{r}} = -\frac{\partial \varphi_{\mathrm{m}}}{\partial r} = \frac{m \cdot \cos\vartheta}{2 \cdot \pi \cdot r^3} \quad ; \quad H_{\vartheta} = -\frac{1}{r} \cdot \frac{\partial \varphi_{\mathrm{m}}}{\partial \vartheta} = \frac{m \cdot \sin\vartheta}{4 \cdot \pi \cdot r^3} \quad ; \quad H_{\phi} = 0 \tag{16.111}$$

which are, as expected, identical to the expressions (16.108).

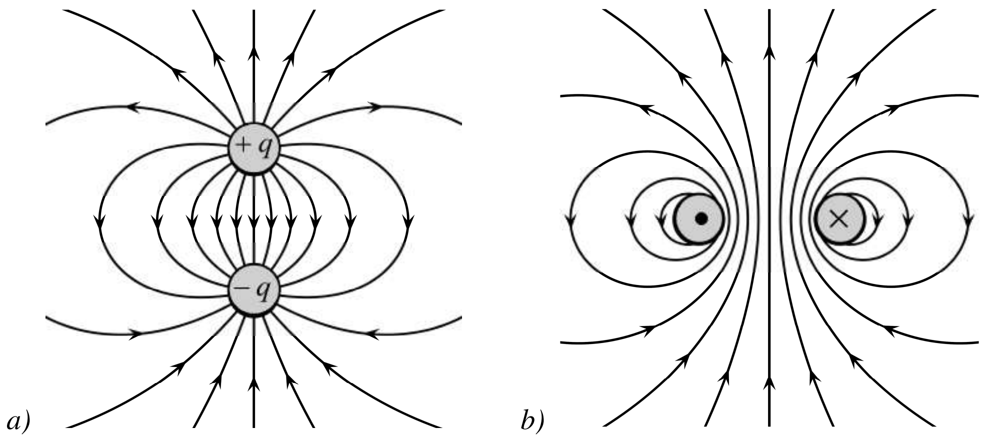


Figure 16.33. a) Electric field lines of two opposite point charges of equal magnitude, b) Magnetic field lines of a circular thin-wire loop

In Figure 16.33-a, the electric field lines of two opposite point charges of equal magnitude are shown, whereas in Figure 16.33-b, the magnetic field lines of a circular thin-wire loop through which a stationary electric current flows are shown.

The approximate expressions valid for the electric dipole and magnetic dipole provide a good approximation only for points that are sufficiently distant from the field sources. The field lines for these are shown in Figure 16.34.

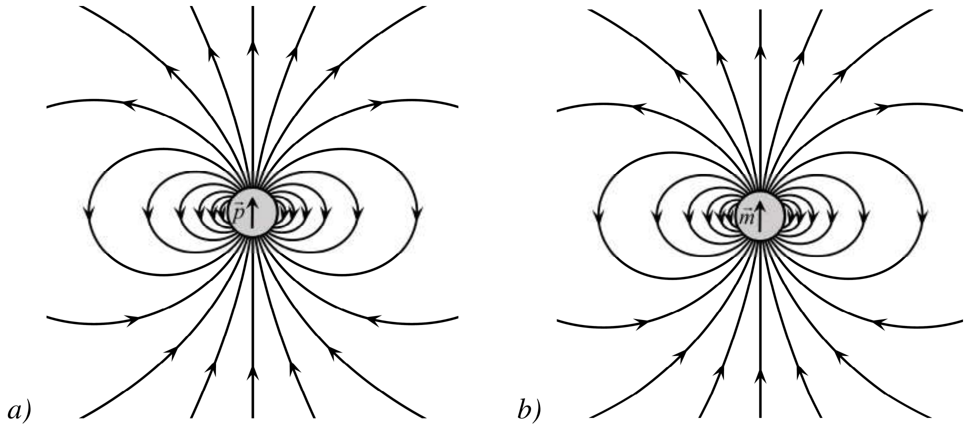


Figure 16.34. a) Electric field lines of an electric dipole, b) Magnetic field lines of a magnetic dipole

16.12. Cylindrical Medium in a Homogeneous Magnetostatic Field

Let a solid infinitely long cylinder (medium 1), with radius a and magnetic permeability μ_1, be placed in a magnetostatic field that is homogeneous far from the cylinder, whereas around the cylinder is an unbounded material (medium 2) with magnetic permeability μ_2. For solving this problem, a cylindrical coordinate system with the origin on the axis of the cylinder (Figure 16.35) is suitable, as well as the use of the magnetic scalar potential.

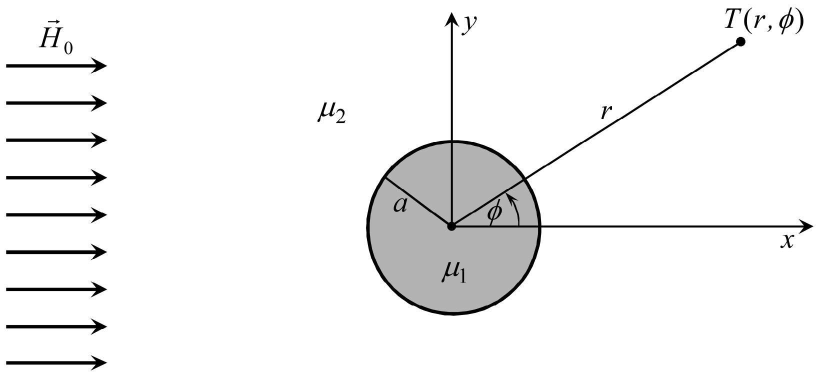


Figure 16.35. Cylindrical LIH medium in a homogeneous magnetostatic field

The boundary condition at large distances from the cylinder is that there is no distortion of the field, i.e., it remains homogeneous. The general solution of the Laplace differential equation in medium 2, which must be adapted to satisfy this requirement, is given by expression (14.183):

$$\varphi_{\mathrm{m}}(r,\phi)=\sum_{\mathrm{n}=1}^{\infty}\left(C_{\mathrm{n}}\cdot r^{n}+D_{\mathrm{n}}\cdot\frac{1}{r^{n}}\right)\cdot\left[F_{\mathrm{n}}\cdot\sin(n\cdot\phi)+G_{\mathrm{n}}\cdot\cos(n\cdot\phi)\right] \tag{16.112}$$

The distribution of the magnetic scalar potential is axisymmetric with respect to the x-axis, i.e., it is an even function with respect to the angle ϕ. Therefore:

$$F_{\mathrm{n}}=0 \quad ; \quad \forall \mathrm{n} \tag{16.113}$$

and the general solution of the Laplace differential equation (16.112) takes the following new form:

$$\varphi_{\mathrm{m}}(r,\phi)=\sum_{\mathrm{n}=1}^{\infty}\left(C_{\mathrm{n}}\cdot r^{n}+D_{\mathrm{n}}\cdot\frac{1}{r^{n}}\right)\cdot\cos(n\cdot\phi) \tag{16.114}$$

The further procedure for deriving the expression for the distribution of the magnetic scalar potential is analogous to the procedure for deriving the expression for the distribution of the electric scalar potential in the case of a conducting cylinder in a homogeneous stationary current field.

From the general solution of the Laplace differential equation (16.114), it follows that the initial expression for the distribution of the magnetic scalar potential in medium 2 is:

$$\varphi_{\mathrm{m2}} = \sum_{\mathrm{n}=1}^{\infty} \left(C_{\mathrm{n}} \cdot r^{n} + D_{\mathrm{n}} \cdot \frac{1}{r^{n}} \right) \cdot \cos(n \cdot \phi) \tag{16.115}$$

It is sufficient to adjust the radial component of the magnetic field intensity at infinity, which, according to Figure 16.36, is described by the following expression:

$$H_0 \cdot \cos \phi = - \left. \frac{\partial \varphi_{\mathrm{m2}}}{\partial r} \right|_{r \to \infty} \tag{16.116}$$

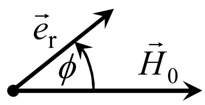


Figure 16.36. Radial component of the magnetic field intensity far from the cylinder

If expression (16.115) is substituted into expression (16.116), it follows that:

$$H_0 \cdot \cos \phi = - \left\{ \sum_{\mathrm{n}=1}^{\infty} \left(C_{\mathrm{n}} \cdot n \cdot r^{n-1} - D_{\mathrm{n}} \cdot \frac{n}{r^{n+1}} \right) \cdot \cos(n \cdot \phi) \right\}_{r \to \infty} \tag{16.117}$$

From expression (16.117), it follows that $n = 1$, which means that all coefficients C_{n} and D_{n} are equal to zero, except for the coefficients C_1 and D_1. Therefore, expression (16.117) becomes:

$$H_0 \cdot \cos\phi = - \left(C_1 - D_1 \cdot \frac{1}{r^2} \right)_{r \to \infty} \cdot \cos\phi \; = \; - C_1 \cdot \cos\phi \tag{16.118}$$

from which it follows that:

$$C_1 = H_0 \tag{16.119}$$

whereas the coefficient D_1 is, for now, unknown, and the term with D_1 tends to zero as r approaches infinity.

Thus, the distribution of the magnetic scalar potential in medium 2 is described by the expression:

$$\varphi_{\mathrm{m2}} = \left(- H_0 \cdot r + D_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{16.120}$$

and, to satisfy the boundary conditions between the two media, the magnetic scalar potential inside the cylinder must be described by the expression:

$$\varphi_{\mathrm{m1}} = \left(F_1 \cdot r + G_1 \cdot \frac{1}{r} \right) \cdot \cos\phi \tag{16.121}$$

From the condition that the magnetic scalar potential at the axis of the cylinder must be finite, it follows that the coefficient:

$$G_1 = 0 \tag{16.122}$$

and it follows that:

$$\varphi_{\mathrm{m1}} = F_1 \cdot r \cdot \cos\phi \tag{16.123}$$

Therefore, the distribution of the magnetic scalar potential in both media is described by expressions (16.120) and (16.123), where two coefficients are unknown, and they can be determined based on the satisfaction of the boundary conditions.

$$\varphi_{m1}\Big|_{r=a} = \varphi_{m2}\Big|_{r=a} \tag{16.124}$$

$$-\mu_1 \cdot \frac{\partial \varphi_{m1}}{\partial r}\Bigg|_{r=a} = -\mu_2 \cdot \frac{\partial \varphi_{m2}}{\partial r}\Bigg|_{r=a} \tag{1 6.125}$$

from which the following system of linear equations follows:

$$\frac{1}{a} \cdot D_1 - a \cdot F_1 = H_0 \cdot a \quad ; \quad \frac{\mu_2}{a^2} \cdot D_1 + \mu_1 \cdot F_1 = -\mu_2 \cdot H_0 \tag{16.126}$$

and it follows that:

$$D_1 = H_0 \cdot a^2 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \quad ; \quad F_1 = -H_0 \cdot \frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \tag{16.127}$$

After satisfying all boundary conditions, according to expressions (16.120), (16.123), and (16.127), the distribution of the magnetic scalar potential in both media is described by the expressions:

$$\varphi_{m1} = -\frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot r \cdot \cos\phi = -\frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot x \quad ; \quad r \le a \tag{16.128}$$

$$\varphi_{m2} = -H_0 \cdot r \cdot \cos\phi + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \left(\frac{a}{r}\right)^2 \cdot H_0 \cdot r \cdot \cos\phi \quad ; \quad r \ge a \tag{16.129}$$

and it follows that the magnetic scalar potential is equal to zero on the plane passing through the center of the cylinder, where $\phi = \pm \pi / 2$, which corresponds to the plane $x = 0$.

From the expression (16.128) that describes the distribution of the magnetic scalar potential in medium 1, the expressions for the nonzero components of the magnetic field intensity in medium 1 can be easily derived, which in the cylindrical coordinate system are as follows:

$$\vec{H}_1 = H_{1r} \cdot \vec{e}_r + H_{1\phi} \cdot \vec{e}_\phi \tag{16.130}$$

$$H_{1r} = -\frac{\partial \varphi_{m1}}{\partial r} = \frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot \cos\phi \tag{16.131}$$

$$H_{1\phi} = -\frac{1}{r \cdot} \frac{\partial \varphi_{m1}}{\partial \phi} = -\frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot \sin\phi \tag{16.132}$$

whereas in the Cartesian coordinate system, it is:

$$\vec{H}_1 = H_{1x} \cdot \vec{i} = \frac{2 \cdot \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot \vec{i} \tag{16.133}$$

from which it follows that, as expected, the magnetic field intensity in the cylinder changed in magnitude, but the magnetic field remained homogeneous.

From the expression (16.129) that describes the distribution of the magnetic scalar potential in medium 2, the expressions for the nonzero components of the magnetic field intensity in medium 2 can be easily derived, which in the cylindrical coordinate system are as follows:

$$\vec{H}_2 = H_{2r} \cdot \vec{e}_r + H_{2\phi} \cdot \vec{e}_\phi \tag{16.134}$$

$$H_{2r} = -\frac{\partial \varphi_{m2}}{\partial r} = H_0 \cdot \cos\phi \cdot \left(1 + \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \left(\frac{a}{r}\right)^2\right) \tag{16.135}$$

$$H_{2\phi} = -\frac{1}{r} \cdot \frac{\partial \varphi_{m2}}{\partial \phi} = -H_0 \cdot \sin\phi \cdot \left(1 - \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot \left(\frac{a}{r}\right)^2\right) \tag{16.136}$$

It can be imagined that medium 2 is present everywhere, and the influence of medium 1 can be replaced by a sinusoidal surface current (Figure 16.37). The obtained solution is valid only in medium 2. It holds that:

$$\mu_1 \cdot \vec{H}_1 = \mu_2 \cdot \left(\vec{H}_1 + \vec{M}\right) \tag{16.137}$$

from which it follows that the magnetization vector, considering the expression (16.133), is:

$$\vec{M} = \frac{\mu_1 - \mu_2}{\mu_2} \cdot \vec{H}_1 = 2 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot H_0 \cdot \vec{i} \tag{16.138}$$

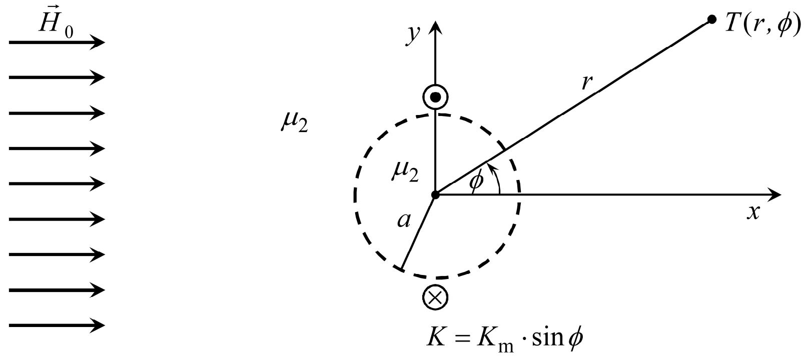


Figure 16.37. The linear density of the equivalent surface current for $\mu_1 > \mu_2$

The linear density of the equivalent surface current is described by the expression:

$$\vec{K} = \vec{M} \times \vec{n} = \vec{M} \times \vec{n} = M \cdot \left(\vec{i} \times \vec{e}_r\right) = M \cdot \sin\phi \cdot \vec{e}_z = K_m \cdot \sin\phi \cdot \vec{e}_z \tag{16.139}$$

from which it follows that:

$$K_m = M = \frac{\mu_1 - \mu_2}{\mu_2} \cdot H_1 = 2 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + \mu_2} \cdot H_0 \tag{16.140}$$

and this is the maximum value of the linear density of the equivalent surface current for $\mu_1 > \mu_2$, and the minimum value of the linear density of the equivalent surface current for $\mu_1 < \mu_2$.

16.13. Solid Sphere in a Homogeneous Magnetostatic Field

Let a solid sphere (LIH medium 1 with permeability μ_1, radius a) be placed in a magnetostatic field that is homogeneous far from the sphere, whereas around the sphere is an unbounded LIH medium (medium 2) with permeability μ_2. For solving this problem, a spherical coordinate system with the origin at the center of the sphere (Figure 16.38) is suitable, as well as the use of the magnetic scalar potential.

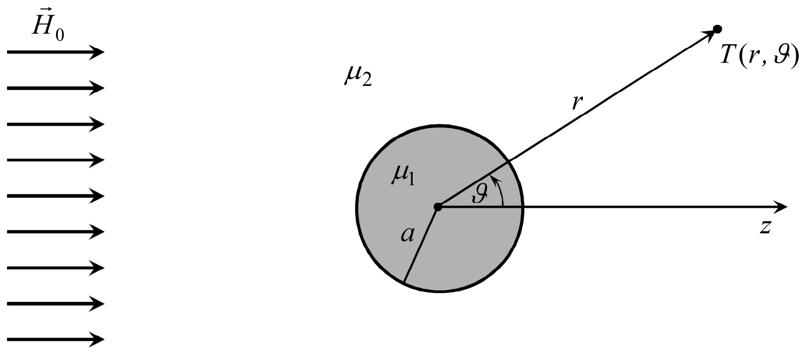


Figure 16.38. A solid sphere in a homogeneous magnetostatic field

The expressions for the distribution of the magnetic scalar potential are analogous to the expressions for the distribution of the electric scalar potential that were obtained for a solid dielectric sphere in a homogeneous electric field. According to the analogous expressions (14.217) and (14.218), after applying the boundary conditions to the general solution of the Laplace differential equation, it follows that:

$$\varphi_{\mathrm{m1}} = -\frac{3\cdot\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot H_0\cdot r\cdot\cos\vartheta = -\frac{3\cdot\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot H_0\cdot z \quad ; \quad r \le a \tag{16.141}$$

$$\varphi_{\mathrm{m2}} = -H_0\cdot r\cdot\cos\vartheta + \frac{\mu_1-\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot\left(\frac{a}{r}\right)^3\cdot H_0\cdot r\cdot\cos\vartheta \quad ; \quad r \ge a \tag{16.142}$$

From the expression (16.141) that describes the distribution of the scalar magnetic potential in medium 1, expressions for the nonzero components of the magnetic field intensity in medium 1 can easily be derived, which in the spherical coordinate system are given as:

$$\vec{H}_1 = H_{1\mathrm{r}}\cdot\vec{\mathrm{e}}_{\mathrm{r}} + H_{1\vartheta}\cdot\vec{\mathrm{e}}_{\vartheta} \tag{16.143}$$

$$H_{1\mathrm{r}} = -\frac{\partial\varphi_{\mathrm{m1}}}{\partial r} = \frac{3\cdot\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot H_0\cdot\cos\vartheta \tag{16.144}$$

$$H_{1\vartheta} = -\frac{1}{r\cdot}\frac{\partial\varphi_{\mathrm{m1}}}{\partial\vartheta} = -\frac{3\cdot\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot H_0\cdot\sin\vartheta \tag{16.145}$$

whereas in the Cartesian coordinate system, it is:

$$\vec{H}_1 = H_{1z}\cdot\vec{\mathrm{k}} = \frac{3\cdot\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot H_0\cdot\vec{\mathrm{k}} \tag{16.146}$$

From the expression (16.142) that describes the distribution of the scalar magnetic potential in medium 2, expressions for the nonzero components of the magnetic field intensity in medium 2 can be easily derived, which in the spherical coordinate system are given as:

$$\vec{H}_2 = H_{2\mathrm{r}}\cdot\vec{\mathrm{e}}_{\mathrm{r}} + H_{2\vartheta}\cdot\vec{\mathrm{e}}_{\vartheta} \tag{16.147}$$

$$H_{2\mathrm{r}} = -\frac{\partial\varphi_{\mathrm{m2}}}{\partial r} = H_0\cdot\cos\vartheta\cdot\left(1 + 2\cdot\frac{\mu_1-\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot\left(\frac{a}{r}\right)^3\right) \tag{16.148}$$

$$H_{2\vartheta} = -\frac{1}{r\cdot}\frac{\partial\varphi_{\mathrm{m2}}}{\partial\vartheta} = -H_0\cdot\sin\vartheta\cdot\left(1 - \frac{\mu_1-\mu_2}{\mu_1 + 2\cdot\mu_2}\cdot\left(\frac{a}{r}\right)^3\right) \tag{16.149}$$

It can be imagined that medium 2 is present everywhere, and the influence of medium 1 can be replaced by a sinusoidal surface current (Figure 16.39). The obtained solution is valid only in medium 2. It holds that:

$$\mu_1 \cdot \vec{H}_1 = \mu_2 \cdot \left(\vec{H}_1 + \vec{M}\right) \tag{16.150}$$

from which it follows that the magnetization vector, considering the expression (16.146), is:

$$\vec{M} = \frac{\mu_1 - \mu_2}{\mu_2} \cdot \vec{H}_1 = 3 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + 2 \cdot \mu_2} \cdot H_0 \cdot \vec{\mathrm{k}} \tag{16.151}$$

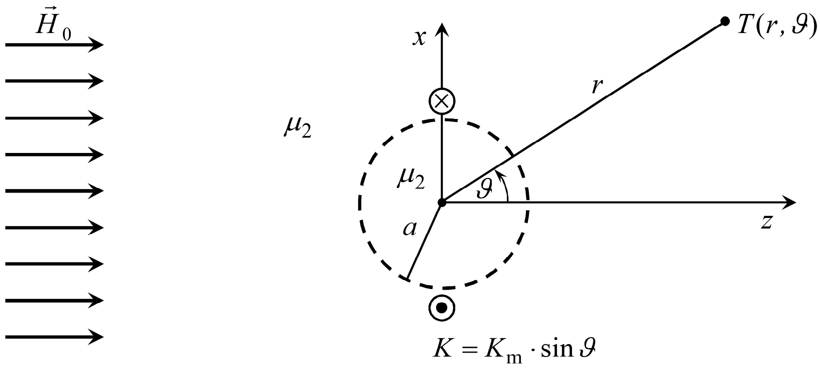


Figure 16.39. The linear density of the equivalent surface current for $\mu_1 < \mu_2$

The linear density of the equivalent surface current is described by the expression:

$$\vec{K} = \vec{M} \times \vec{n} = \vec{M} \times \vec{\mathrm{n}} = M \cdot \left(\vec{\mathrm{k}} \times \vec{\mathrm{e}}_{\mathrm{r}}\right) = M \cdot \sin\vartheta \cdot \vec{\mathrm{e}}_{\phi} = K_{\mathrm{m}} \cdot \sin\vartheta \cdot \vec{\mathrm{e}}_{\phi} \tag{16.152}$$

from which it follows that:

$$K_{\mathrm{m}} = M = \frac{\mu_1 - \mu_2}{\mu_2} \cdot H_1 = 3 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + 2 \cdot \mu_2} \cdot H_0 \tag{16.153}$$

and this is the maximum value of the linear density of the equivalent surface current for $\mu_1 > \mu_2$, and the minimum value of the linear density of the equivalent surface current for $\mu_1 < \mu_2$.

It is important to note that the unit vector $\vec{\mathrm{e}}_{\phi}$, which appears in expression (16.152), is the unit vector of the cylindrical coordinate system (r, ϕ, z).

Another way to determine the expression (16.151), which describes the magnetization vector of the solid sphere, is based on expressions (16.109) and (16.142). Based on these two expressions, the distribution of the magnetic field intensity in medium 2 can be described using the following expression:

$$\varphi_{\mathrm{m}2} = -\vec{H}_0 \cdot \vec{r} + \frac{\vec{m} \cdot \vec{r}}{4 \cdot \pi \cdot r^3} \tag{16.154}$$

where the magnetic dipole moment of the solid sphere is:

$$\vec{m} = 4 \cdot \pi \cdot a^3 \cdot \frac{\mu_1 - \mu_2}{\mu_1 + 2 \cdot \mu_2} \cdot \vec{H}_0 = 3 \cdot V_{\mathrm{sphere}} \cdot \frac{\mu_1 - \mu_2}{\mu_1 + 2 \cdot \mu_2} \cdot \vec{H}_0 = \vec{M} \cdot V_{\mathrm{sphere}} \tag{16.155}$$

from which the expression (16.151) can be easily derived.

Analogous to electric field lines, the magnetic field lines for $\mu_1 > \mu_2$ are shown in Figure 14.36, and the magnetic field lines for $\mu_1 < \mu_2$ are shown in Figure 14.37.

16.14. Magnetic Force on a Thin-Wire Current-Carrying Conductor

Let a stationary electric current of intensity I flow through a thin-wire conductor. From Lorentz's force law (8.7), it follows that the magnetic force acting on the conductor, through which an electric current flows and which is located in a magnetic field, is described by the expression:

$$\vec{F}_{\mathrm{m}} = \int_V \left(\vec{J} \times \vec{B}\right) \cdot dV \tag{16.156}$$

where $\vec{J}$ is the vector of the surface density of electric current flowing through the thin-wire conductor. For a thin-wire current-carrying conductor, it holds that:

$$\vec{J} \cdot dV = I \cdot d\vec{\ell} \tag{16.157}$$

and the expression for the magnetic force on a thin-wire conductor (16.156) takes on a new form:

$$\vec{F}_{\mathrm{m}} = I \cdot \int_C \left(d\vec{\ell} \times \vec{B}\right) \tag{16.158}$$

whereas the magnetic force on an infinitesimal segment of the thin-wire conductor (Figure 16.40) is described by the following expression:

$$d\vec{F}_{\mathrm{m}} = I \cdot \left(d\vec{\ell} \times \vec{B}\right) \tag{16.159}$$

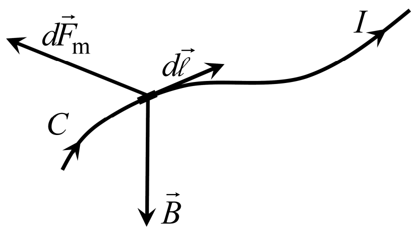


Figure 16.40. Magnetic force on an infinitesimal segment of the thin-wire conductor

In the special case when the magnetic field is homogeneous, and the segment of the thin-wire conductor is straight and of finite length ℓ, the expression for the magnetic force is:

$$\vec{F}_{\mathrm{m}} = I \cdot \left(\vec{\ell} \times \vec{B}\right) \tag{16.160}$$

where the vector $\vec{\ell}$ takes the direction of the stationary electric current.

If, under the previously mentioned assumptions, it is also true that $\vec{\ell} \perp \vec{B}$, then the expression for the magnetic force (16.160) takes on a new form:

$$F_{\mathrm{m}} = B \cdot I \cdot \ell \tag{16.161}$$

16.15. Magnetic Force Between Two Infinitely Long, Parallel Conductors

Let two straight, infinitely long, parallel thin-wire conductors be located in an unbounded LIH medium with magnetic permeability μ. Let the conductors be separated by a distance d and let stationary electric currents flow through them (Figure 16.41).

A straight conductor does not act on itself. According to the law of action and reaction, it holds that:

$$\vec{F}_{1,2} = -\,\vec{F}_{2,1} \tag{16.162}$$

where:

$\vec{F}_{1,2}$ - the force on conductor 1 due to the stationary current flowing through conductor 2,

$\vec{F}_{2,1}$ - the force on conductor 2 due to the stationary current flowing through conductor 1.

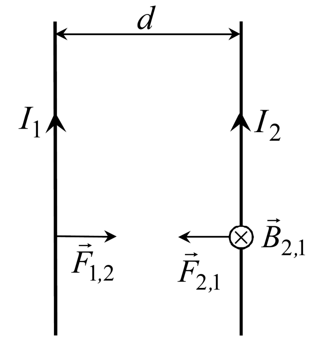


Figure 16.41. Two parallel thin-wire conductors

It holds that:

$$\vec{F}_{2,1} = I_2 \cdot \left(\vec{\ell}_2 \times \vec{B}_{2,1}\right) \tag{16.163}$$

where ℓ_2 is the length of conductor 2 on which the magnetic force acts, whereas the magnetic induction at the location of conductor 2 due to the electric current in conductor 1 is described by the expression:

$$B_{2,1} = \mu \cdot \frac{I_1}{2 \cdot \pi \cdot d} \tag{16.164}$$

and thus the magnitude of the magnetic force acting on a unit length of conductor 2 is:

$$\frac{F_{2,1}}{\ell_2} = I_2 \cdot B_{2,1} = \mu \cdot \frac{I_1 \cdot I_2}{2 \cdot \pi \cdot d} \quad (\text{N/m}) \tag{16.165}$$

Therefore, the attractive magnetic force per unit length of the infinitely long thin-wire conductors is described by the expression:

$$F_{\text{pul}} = \mu \cdot \frac{I_1 \cdot I_2}{2 \cdot \pi \cdot d} \tag{16.166}$$

In Figure 16.42, a graphical explanation is provided for why the force between parallel thin-wire conductors is attractive when their currents flow in the same direction. If the electric currents flow in the same direction, the magnetic field intensities between the conductors cancel each other out, and thus the magnetic force between the conductors is attractive. The reason is the tendency for the magnetic field to evenly distribute itself throughout the entire space. If the current directions are opposite, the magnetic field intensities between the conductors add up, and therefore the magnetic force between the conductors is repulsive.

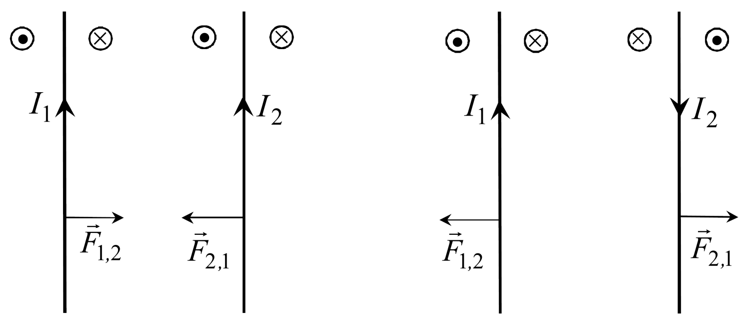


Figure 16.42. Directions of magnetic field intensity depending on the directions of electric currents

Therefore, if a stationary electric current flows through a circular closed loop, the magnetic forces tend to expand the loop. In this case, all magnetic field lines pass through the loop (Figure 16.33-b), and they tend to be evenly distributed throughout the entire space.

16.16. Force in the Magnetostatic Field of an Isolated System

The force can be calculated from the energy of the system, i.e., from the inductance of the system. An isolated system is one that does not exchange energy with its surroundings, meaning it is a system in which the time-invariant magnetic flux linkages are specified. Such a system tends toward a state of minimum energy, and in it, the force is described by the following expression:

$$\vec{F} = -\nabla W_{\mathrm{m}} \tag{16.167}$$

where W_{m} is the energy stored in the magnetostatic field.

In an isolated system, the force in the direction of vector $\vec{s}$ is described by the following expression:

$$\vec{F} = -\frac{\partial W_{\mathrm{m}}}{\partial s} \cdot \vec{s}_0 = -\vec{s}_0 \cdot \left.\frac{\partial W_{\mathrm{m}}}{\partial s}\right|_{\Psi = \mathrm{const.}} \tag{16.168}$$

where $\vec{s}_0$ is a unit vector.

In an isolated system, the force in the direction of vector $\vec{s}$ is also described by the following expression:

$$\vec{F} = -\vec{s}_0 \cdot \left.\frac{\partial W_{\mathrm{m}}}{\partial s}\right|_{\Psi = \mathrm{const.}} = -\vec{s}_0 \cdot \frac{\Psi^2}{2} \cdot \frac{\partial}{\partial s}\left(\frac{1}{L}\right) = \vec{s}_0 \cdot \frac{1}{2} \cdot \left(\frac{\Psi}{L}\right)^2 \cdot \frac{\partial L}{\partial s} \tag{16.169}$$

where Ψ and L are the magnetic flux linkage and the inductance of the isolated system, respectively.

16.17. Force in the Magnetostatic Field of a Non-Isolated System

A non-isolated system is a system that exchanges energy with its surroundings, meaning it is a system in which the magnitudes of stationary electric currents are specified. Such a system tends toward a state of maximum energy, and in it, the force is described by the following expression:

$$\vec{F} = \nabla W_{\mathrm{m}} \tag{16.170}$$

where W_{m} is the energy stored in the magnetostatic field.

In a non-isolated system, the force in the direction of vector $\vec{s}$ is described by the following expression:

$$\vec{F} = \frac{\partial W_{\mathrm{m}}}{\partial s} \cdot \vec{s}_0 = \vec{s}_0 \cdot \left.\frac{\partial W_{\mathrm{m}}}{\partial s}\right|_{I = \mathrm{const.}} \tag{16.171}$$

where $\vec{s}_0$ is a unit vector.

In a non-isolated system, the force in the direction of vector $\vec{s}$ is also described by the following expression:

$$\vec{F} = \vec{s}_0 \cdot \left.\frac{\partial W_{\mathrm{m}}}{\partial s}\right|_{I = \mathrm{const.}} = \vec{s}_0 \cdot \frac{I^2}{2} \cdot \frac{\partial L}{\partial s} \tag{16.172}$$

where I and L are the electric current and the inductance of the non-isolated system, respectively.

The force between two electric current circuits, in a non-isolated system, in the direction of the vector $\vec{s}$ is described by the following expression:

$$\vec{F}_{\mathrm{j,k}} = \vec{s}_0 \cdot \left.\frac{\partial W_{\mathrm{m}}}{\partial s}\right|_{I_{\mathrm{j}}, I_{\mathrm{k}} = \mathrm{const.}} = \vec{s}_0 \cdot \frac{I_{\mathrm{j}} \cdot I_{\mathrm{k}}}{2} \cdot \frac{\partial M_{\mathrm{j,k}}}{\partial s} \tag{16.173}$$

where $M_{\mathrm{j,k}}$ is the mutual inductance of the j-th and k-th electric current circuits.

16.18. Forces and Stresses in the Magnetostatic Field

The volume V is enclosed by a surface $S \equiv \partial V$ (Figure 16.43), where $\vec{n}$ is the vector of the outer unit normal to the surface S.

The magnetic force acting on the volume V can be calculated by integrating over the closed surface S:

$$\vec{F} = \int_V \vec{J} \times \vec{B} \cdot dV = \oint_S \vec{t}_{\mathrm{m}} \cdot dS \tag{16.174}$$

where $\vec{t}_{\mathrm{m}}$ is the *magnetic stress vector.*

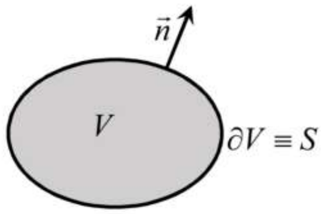


Figure 16.43. Volume V enclosed by a surface S

In a magnetostatic field, the magnetic stress vector in an LIH medium is described by the expression:

$$\vec{t}_{\mathrm{m}} = \mu \cdot \left[\vec{H} \cdot \left(\vec{H} \cdot \vec{n} \right) - \frac{1}{2} \cdot H^2 \cdot \vec{n} \right] \tag{16.175}$$

The surface integral described by expression (16.174) can be expressed as a vector sum of three surface integrals of the first kind:

$$\vec{F} = \oint_S \vec{t}_{\mathrm{m}} \cdot dS = \vec{\mathrm{i}} \cdot \oint_S t_{\mathrm{mx}} \cdot dS + \vec{\mathrm{j}} \cdot \oint_S t_{\mathrm{my}} \cdot dS + \vec{\mathrm{k}} \cdot \oint_S t_{\mathrm{mz}} \cdot dS \tag{16.176}$$

where the magnetic stress vector in the Cartesian coordinate system can be written as:

$$\vec{t}_{\mathrm{m}} = t_{\mathrm{mx}} \cdot \vec{\mathrm{i}} + t_{\mathrm{my}} \cdot \vec{\mathrm{j}} + t_{\mathrm{mz}} \cdot \vec{\mathrm{k}} \tag{16.177}$$

The magnetic field intensity vector $\vec{H}$ splits in half the angle between the external unit normal vector $\vec{n}$ and the magnetic stress vector $\vec{t}_{\mathrm{m}}$ (Figure 16.44):

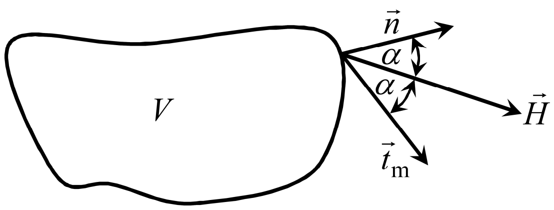


Figure 16.44. The relative position of the three vectors

In the special case where the magnetic field intensity vector is collinear with the vector of the outer unit normal, the following holds:

$$\vec{H} = \pm H \cdot \vec{n} \quad \Rightarrow \quad \vec{t}_{\mathrm{m}} = \frac{1}{2} \cdot \mu \cdot H^2 \cdot \vec{n} \tag{16.178}$$

whereas in the second special case, when the magnetic field intensity vector is perpendicular to the vector of the outer unit normal, the following holds:

$$\vec{H} \perp \vec{n} \quad \Rightarrow \quad \vec{t}_{\mathrm{m}} = -\frac{1}{2} \cdot \mu \cdot H^2 \cdot \vec{n} \tag{16.179}$$

Based on the magnetic stress vector at the boundary between two LIH media with magnetic permeabilities μ_1 and μ_2, an expression for the magnetostatic pressure at the boundary of these two media can be easily derived as:

$$t_{\mathrm{n}}^{\mathrm{m}} = \frac{|\mu_1 - \mu_2|}{2} \cdot \left(\frac{B_{\mathrm{n}}^2}{\mu_1 \cdot \mu_2} + H_{\mathrm{t}}^2 \right) \tag{16.180}$$

where:

B_{n} - the normal component of the magnetic flux density vector,

H_{t} - the tangential component of the magnetic field intensity vector.

The magnetic force at the boundary between two media is described by a surface integral over the boundary surface of the two LIH media, S_{b}:

$$\vec{F}_{\mathrm{b}} = \int_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot \vec{n} \cdot dS = \int_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot d\vec{S} \tag{16.181}$$

where $\vec{n}$ is the vector of the outer unit normal to the boundary surface between the two LIH media, which is oriented from the medium with higher magnetic permeability towards the medium with lower magnetic permeability.

The surface integral described by expression (16.181) can be expressed as a vector sum of three surface integrals of the first kind:

$$\vec{F}_{\mathrm{b}} = \int_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot \vec{n} \cdot dS = \vec{\mathrm{i}} \cdot \oint_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot n_{\mathrm{x}} \cdot dS + \vec{\mathrm{j}} \cdot \oint_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot n_{\mathrm{y}} \cdot dS + \vec{\mathrm{k}} \cdot \oint_{S_{\mathrm{b}}} t_{\mathrm{n}}^{\mathrm{m}} \cdot n_{\mathrm{z}} \cdot dS \tag{16.182}$$

where the vector of the outer unit normal to the boundary surface between the two LIH media in a Cartesian coordinate system is described by the expression:

$$\vec{n} = n_{\mathrm{x}} \cdot \vec{\mathrm{i}} + n_{\mathrm{y}} \cdot \vec{\mathrm{j}} + n_{\mathrm{z}} \cdot \vec{\mathrm{k}} = \cos\alpha \cdot \vec{\mathrm{i}} + \cos\beta \cdot \vec{\mathrm{j}} + \cos\gamma \cdot \vec{\mathrm{k}} \tag{16.183}$$

where α, β, and γ are angles between the unit normal vector and Cartesian coordinate axes.

16.19. Magnetic Vector Potential of a Thin-Wire Conductor Segment

Let there be a thin-wire conductor segment of length ℓ and radius r_0, along the axis of which a stationary electric current of intensity I flows. The thin-wire segment is located in an unbounded LIH medium with magnetic permeability μ. If $r_0 << \ell$, then the thin-wire approximation can be used, which assumes that the stationary electric current flows along the axis of the conductor segment. The task is to find the distribution of the vector magnetic potential in points outside the conductor segment, including the surface of the conductor segment.

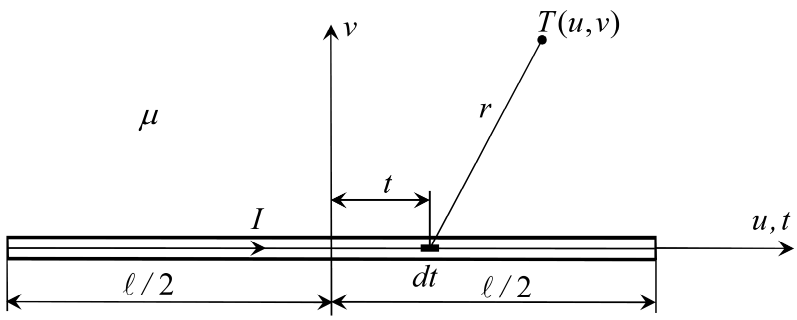


Figure 16.45. A segment of a cylindrical conductor in the local coordinate system (u, v)

For simplicity, the segment is given in the local 2D coordinate system (u, v), in which the axis of the conductor segment and the field point $T(u, v)$ lie. The position of the segment in the (x, y, z) coordinate system is arbitrary. According to Figure 16.45, the point $T(u, v)$, where the magnetic vector potential is calculated, can be in either the first or second quadrant of the local coordinate system (u, v).

In an unbounded LIH medium with magnetic permeability μ, the infinitesimal magnetic vector potential of a thin-wire conductor segment is described by the expression (16.32), which reads:

$$d\vec{A} = \frac{\mu \cdot I}{4 \cdot \pi} \cdot \frac{d\vec{\ell}}{r} \tag{16.184}$$

and it is:

$$\vec{A} = \vec{e}_{\mathrm{u}} \cdot \frac{\mu \cdot I}{4 \cdot \pi} \cdot \int\limits_{-\ell/2}^{\ell/2} \frac{dt}{\sqrt{v^2 + (t-u)^2}} \tag{16.185}$$

where $\vec{e}_{\mathrm{u}}$ is the unit vector of the local u-axis, and the direction of this unit vector is, in fact, the direction of the stationary electric current of intensity I.

After integration, from expression (16.185), the following expressions for the distribution of the magnetic vector potential around the thin-wire conductor segment are obtained:

$$\vec{A} = \vec{e}_{\mathrm{u}} \cdot \frac{\mu \cdot I}{4 \cdot \pi} \cdot \left(\operatorname{arsinh} \frac{u + \frac{\ell}{2}}{v} - \operatorname{arsinh} \frac{u - \frac{\ell}{2}}{v} \right) \tag{16.186}$$

$$\vec{A} = \vec{e}_{\mathrm{u}} \cdot \frac{\mu \cdot I}{4 \cdot \pi} \cdot \ln \frac{\sqrt{v^2 + \left(u + \frac{\ell}{2}\right)^2} + u + \frac{\ell}{2}}{\sqrt{v^2 + \left(u - \frac{\ell}{2}\right)^2} + u - \frac{\ell}{2}} \tag{16.187}$$

Expressions (16.186) and (16.187) can be easily derived based on the analogy with the expressions for the distribution of the electric scalar potential around a segment of a cylindrical conductor in the electrostatic case, or in the case of a stationary electric current field.

16.20. Neumann's Formula

Let two thin-wire loops be given (Figure 16.46). Let loop 1 carry a stationary electric current I_1, and loop 2 carry a stationary electric current I_2.. Assume that the thin-wire loops are located in an unbounded LIH medium with magnetic permeability μ. The goal is to derive expressions for the self-inductance and mutual inductance of these two thin-wire loops.

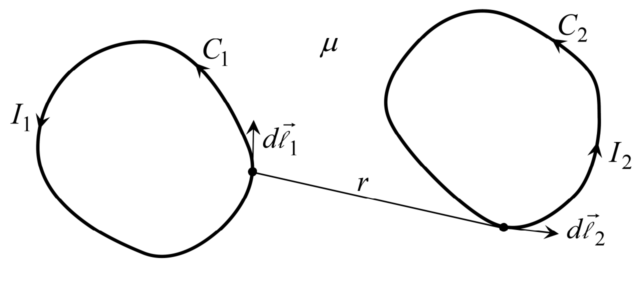


Figure 16.46. Two thin-wire loops

According to expression (16.32), the magnetic vector potential along the axis of loop 2 due to the electric current in loop 1 is given by the expression:

$$\vec{A}_{2,1} = \frac{\mu \cdot I_1}{4 \cdot \pi} \cdot \int_{C_1} \frac{d\vec{\ell}_1}{r} \tag{16.188}$$

The magnetic flux linkage, caused by the current in loop 1 and linking loop 2, is described by the expression:

$$\Psi_{2,1} = \int_{S_2} \vec{B}_{2,1} \cdot d\vec{S} = \int_{S_2} \left(\nabla \times \vec{A}_{2,1}\right) \cdot d\vec{S} = \oint_{C_2} \vec{A}_{2,1} \cdot d\vec{\ell}_2 \tag{16.189}$$

where S_2 is the oriented surface bounded by the curve C_2, that is, the oriented surface enclosed by thin-wire loop 2, following the right-hand rule.

If expression (16.188) is substituted into expression (16.189), it follows that:

$$\Psi_{2,1} = \oint_{C_2} \vec{A}_{2,1} \cdot d\vec{\ell}_2 = \frac{\mu \cdot I_1}{4 \cdot \pi} \cdot \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{r} \tag{16.190}$$

from which it follows that the mutual inductance M of the thin-wire loops is described by the expression:

$$M = M_{2,1} = M_{1,2} = \frac{\mu}{4 \cdot \pi} \cdot \oint_{C_1} \oint_{C_2} \frac{d\vec{\ell}_1 \cdot d\vec{\ell}_2}{r} \tag{16.191}$$

which is known as *Neumann's formula*.

Using Neumann's formula, the self-inductance of the thin-wire loop can also be calculated (Figure 16.47). In this case, the curve C is on the axis of the thin-wire loop, and the curve C′ is on the inner edge of the thin-wire loop.

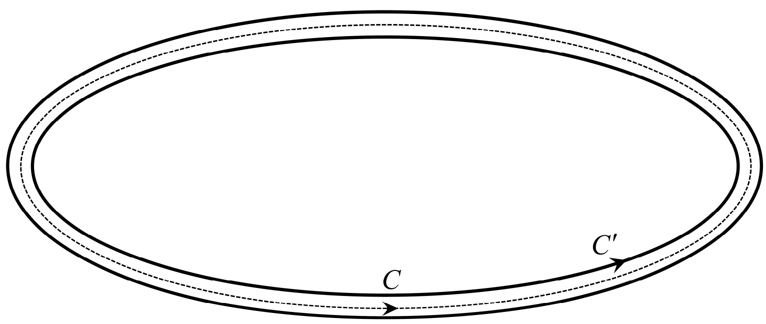


Figure 16.47. Integration curves for calculating the self-inductance of a thin-wire loop

From expression (16.191), it follows that the self-inductance of the thin-wire loop is described by the following expressions:

$$L = L_{\text{int}} + L_{\text{ext}} \quad ; \quad L_{\text{ext}} = \frac{\mu}{4 \cdot \pi} \cdot \oint_{C} \oint_{C'} \frac{d\vec{\ell} \cdot d\vec{\ell}'}{r} \tag{16.192}$$

where L_{ext} is the external inductance of the thin-wire loop, wheresa L_{int} is the internal inductance of the thin-wire loop, which in the case of a thin-wire conductor with a circular cross-section can be approximated by the following expression:

$$L_{\text{int}} = L_{\text{int}}^{1} \cdot \ell_{\text{loop}} = \frac{\mu_{\text{cond}}}{8 \cdot \pi} \cdot \ell_{\text{loop}} \tag{16.193}$$

where:

$$L_{\text{int}}^{1} = \frac{\mu_{\text{cond}}}{8 \cdot \pi} \tag{16.194}$$

is the unit internal inductance of the loop, or the unit internal inductance of an infinitely long straight conductor with a circular cross-section through which a stationary electric current flows.

16.21. Inductances of Segments of Thin-Wire Cylindrical Conductors

The mutual inductance of two thin-wire loops, described by Neumann's formula (16.191), can be calculated by approximating each of the thin-wire loops as a set of straight segments of a thin cylindrical conductor. Let thin-wire loop 1 have *m* segments of the conductor, and thin-wire loop 2 have *n* segments of the conductor.

Under the given assumptions, from expression (16.191), the mutual inductance of the two thin-wire loops shown in Figure 16.48 is approximated by the following expression:

$$M = \frac{\mu}{4 \cdot \pi} \cdot \sum_{i=1}^{m} \sum_{k=1}^{n} \int_{C_{1i}} \int_{C_{2k}} \frac{d\vec{\ell}_{1i} \cdot d\vec{\ell}_{2k}}{r} = \sum_{i=1}^{m} \sum_{k=1}^{n} L_{i,k} \tag{16.195}$$

where $L_{i,k}$ is the mutual inductance of the i-th conductor segment of thin-wire loop 1 and the k-th conductor segment of thin-wire loop 2. The mutual inductances of the conductor segments are also referred to as *partial inductances*.

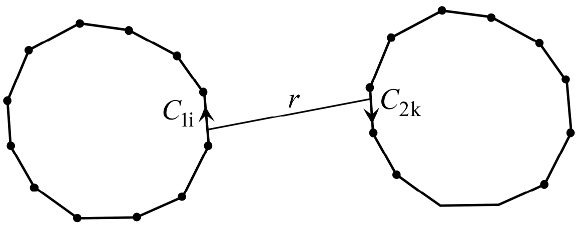


Figure 16.48. Two thin-wire loops approximated as a set of straight segments of cylindrical conductors

From expression (16.195), it follows that the mutual inductances of the two segments of thin-wire conductors are described by the following expression:

$$L_{i,k} = \frac{\mu}{4 \cdot \pi} \cdot \int_{C_{1i}} \int_{C_{2k}} \frac{d\vec{\ell}_{1i} \cdot d\vec{\ell}_{2k}}{r} \tag{16.196}$$

and can also be described by the expression:

$$L_{i,k} = \cos\beta_{i,k} \cdot \frac{\mu}{4 \cdot \pi} \cdot \int_{C_{1i}} \int_{C_{2k}} \frac{d\ell_{1i} \cdot d\ell_{2k}}{r} \tag{16.197}$$

where $\beta_{i,k}$ is the angle between the differential arc elements of the oriented thin-wire segments:

$$\beta_{i,k} = \angle\, d\vec{\ell}_{1i}, d\vec{\ell}_{2k} \tag{16.198}$$

It holds that:

$$L_{i,k} = \cos\beta_{i,k} \cdot \mu \cdot \varepsilon \cdot \ell_{1i} \cdot \ell_{2k} \cdot \alpha_{i,k} = \cos\beta_{i,k} \cdot \mu \cdot \kappa \cdot \ell_{1i} \cdot \ell_{2k} \cdot R_{i,k} \tag{16.199}$$

where $\alpha_{i,k}$ are the potential coefficients of the two cylindrical conductor segments in a perfect dielectric, which were derived using the average potential method in subchapter 14.24, whereas $R_{i,k}$ are the mutual resistances of the two cylindrical conductor segments in a conducting medium, which were obtained using the average potential method.

If the thin-wire loop is approximated as a set of *n* straight segments of a thin cylindrical conductor, then from expression (16.192), the following expression for the self-inductance of the thin-wire loop is obtained:

$$L = L_{\text{int}} + \sum_{i=1}^{n} L_{i,i} + 2 \cdot \sum_{i=1}^{n} \sum_{k=i+1}^{n} L_{i,k} \quad ; \quad L_{\text{int}} = \frac{\mu_{\text{cond}}}{8 \cdot \pi} \cdot \sum_{i=1}^{n} \ell_i \tag{16.200}$$

where the mutual inductance of the i-th and k-th segments of the thin-wire loop is described by the expression:

$$L_{i,k} = \frac{\mu}{4 \cdot \pi} \cdot \int_{C_i} \int_{C_k} \frac{d\vec{\ell}_i \cdot d\vec{\ell}_k}{r} \tag{16.201}$$

whereas the external inductance of the i-th segment of the thin-wire loop is described by the expression:

$$L_{i,i} = \frac{\mu}{4 \cdot \pi} \cdot \int_{C_i} \int_{C_i'} \frac{d\vec{\ell}_i \cdot d\vec{\ell}_i'}{r} \tag{16.202}$$

The integration curves C_i and C_k lie on the axis of the corresponding segment of the thin-wire conductor, whereas the integration curve C_i' lies on the surface of the i-th segment of the thin-wire conductor (Figure 16.49).

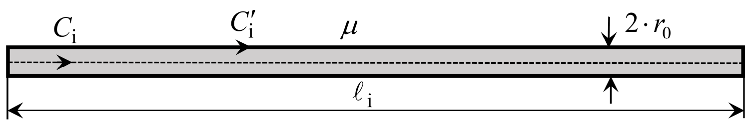


Figure 16.49. Straight segment of a cylindrical conductor in an unbounded LIH medium with magnetic permeability μ

Based on the analogy with the potential coefficients and resistances described by expression (16.199), it is easy to conclude that the external inductance of the i-th segment of a cylindrical conductor, with length ℓ_i and radius r_0, in an unbounded LIH medium with permeability μ, is described by the following expression:

$$L_{i,i} = \frac{\mu}{2 \cdot \pi} \cdot \left[\ell_i \cdot \ln \frac{\sqrt{\ell_i^2 + r_0^2} + \ell_i}{r_0} - \frac{\ell_i^2}{\sqrt{\ell_i^2 + r_0^2} + r_0} \right] = F(\ell_i, r_0) \tag{16.203}$$

where the auxiliary function is:

$$F(\ell, v) = \frac{\mu}{2 \cdot \pi} \cdot \left[\ell \cdot \ln \frac{\sqrt{\ell^2 + v^2} + \ell}{v} - \frac{\ell^2}{\sqrt{\ell^2 + v^2} + v} \right] \tag{16.204}$$

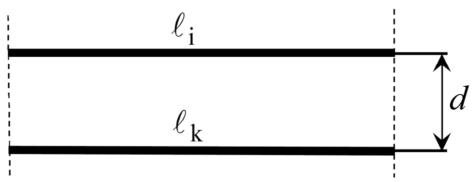


Figure 16.50. Special case of parallelism between two conductor segments

If the two segments are located in an unbounded LIH medium with magnetic permeability μ, and are parallel to each other as shown in Figure 16.50 and separated by a distance d, then their mutual inductance is described by the following expression:

$$L_{i,k} = L_{k,i} = \pm F(\ell_i, d) = \pm \frac{\mu}{2 \cdot \pi} \cdot \left[\ell_i \cdot \ln \frac{\sqrt{\ell_i^2 + d^2} + \ell_i}{d} - \frac{\ell_i^2}{\sqrt{\ell_i^2 + d^2} + d} \right] \tag{16.205}$$

where the sign depends on the orientation of the segments. These two segments may belong to the same thin-wire loop, or they may belong to two different thin-wire loops.

In the general case of parallelism between two conductor segments (Figure 16.51), the k-th conductor segment is observed in the local coordinate system (u, v) of the i-th conductor segment. In this local coordinate system, the endpoints of the k-th segment are denoted as $T_1(u_1, v_k)$ and $T_2(u_2, v_k)$.

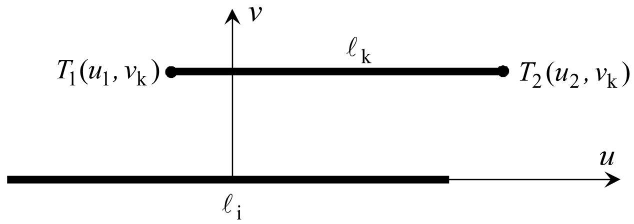


Figure 16.51. General case of parallelism between two conductor segments

If the two conductor segments are located in an unbounded LIH medium with magnetic permeability μ, and are mutually parallel as shown in Figure 16.51, then their mutual inductance can be described by the following expression:

$$L_{i,k} = \pm \frac{\mu}{4 \cdot \pi} \cdot \left(C_1 + C_2 - C_3 - C_4\right) \tag{16.206}$$

where the sign depends on the orientation of the conductor segments, whereas the auxiliary functions C_m are described by the following expressions:

$$C_m = w_m \cdot \ln\left(\sqrt{w_m^2 + v_k^2} + w_m\right) - \sqrt{w_m^2 + v_k^2} \quad ; \quad m = 1, 2, 3, 4 \tag{16.207}$$

$$w_1 = u_2 + \frac{\ell_i}{2} \quad ; \quad w_2 = u_1 - \frac{\ell_i}{2} \quad ; \quad w_3 = u_1 + \frac{\ell_i}{2} \quad ; \quad w_4 = u_2 - \frac{\ell_i}{2} \tag{16.208}$$

Mutually non-parallel (oblique and perpendicular) conductor segments lie in the same plane or in mutually parallel planes. Thus, if non-parallel conductor segments do not lie in the same plane, then there are two and only two mutually parallel planes in which they lie (Figure 16.52). If they lie in the same plane, this is a special case of two parallel planes that have overlapped. Let the conductor segments be oriented from point P to point K. This means that the angle $\alpha \in (0, \pi)$.

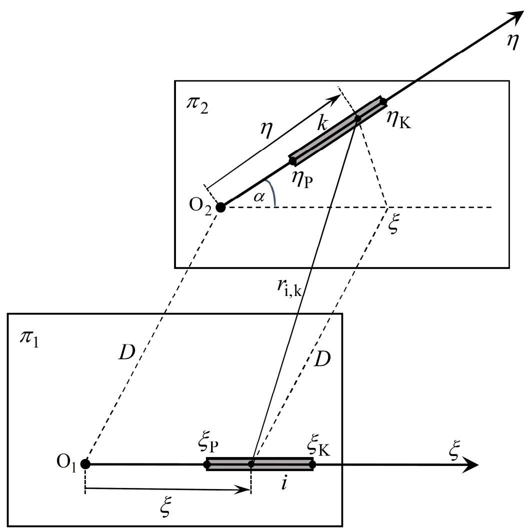


Figure 16.52. General case of non-parallelism between two conductor segments

From expression (16.197), it follows that the mutual inductance of two non-parallel segments of a thin-wire conductor in an unbounded LIH medium with magnetic permeability μ is defined by the expression:

$$L_{\mathrm{i,k}} = \pm\cos\alpha\cdot\frac{\mu}{4\cdot\pi}\cdot\int_{\xi_{\mathrm{P}}}^{\xi_{\mathrm{K}}}\int_{\eta_{\mathrm{P}}}^{\eta_{\mathrm{K}}}\frac{d\xi\cdot d\eta}{r_{\mathrm{i,k}}} \quad ; \quad \cos\beta_{\mathrm{i,k}} = \pm\cos\alpha \tag{16.209}$$

where:

$$r_{\mathrm{i,k}} = \sqrt{\xi^2+\eta^2+D^2-2\cdot\xi\cdot\eta\cdot\cos\alpha} \tag{16.210}$$

The parameters $\xi_{\mathrm{P}}, \xi_{\mathrm{K}}, \eta_{\mathrm{P}}, \eta_{\mathrm{K}}, \alpha$, and D can be easily computed from the global coordinates of the starting and ending points of the conductor segments.

The solution to the integral (16.209) is known as Tseitlin's formula*, which can be written as:

$$L_{\mathrm{i,k}} = \pm\cos\alpha\cdot\frac{\mu}{4\cdot\pi}\cdot\left[A(\xi_{\mathrm{P}},\eta_{\mathrm{P}})+A(\xi_{\mathrm{K}},\eta_{\mathrm{K}})-A(\xi_{\mathrm{P}},\eta_{\mathrm{K}})-A(\xi_{\mathrm{K}},\eta_{\mathrm{P}})\right] \tag{16.211}$$

where:

$$\begin{aligned} A(\xi,\eta) = {} & \xi\cdot\ln\left(\eta-\xi\cdot\cos\alpha+r_{\mathrm{i,k}}\right) + \eta\cdot\ln\left(\xi-\eta\cdot\cos\alpha+r_{\mathrm{i,k}}\right) \\ & +\frac{2\cdot D}{\sin\alpha}\cdot\arctan\left(\frac{\xi+\eta+r_{\mathrm{i,k}}}{D}\cdot\tan\frac{\alpha}{2}\right) \end{aligned} \tag{16.212}$$

The sign in expressions (16.209) and (16.211) depends on the orientation of the arc differential vectors of the i-th and k-th conductor segments. If both arc length differential vectors are oriented from point *P* to point *K*, or from point *K* to point *P*, then the sign is positive. Otherwise, the sign is negative.

16.22. Relationship Between Magnetic Field Lines and the Magnetic Vector Potential

In the 2D computation of the magnetic field using the magnetic vector potential, it is very easy to draw the magnetic field lines because the lines where A = const. are also the field lines. The proof of this statement follows below.

Since the magnetic flux density vector is tangential to the magnetic field line (Figure 16.53), it holds that the vectors:

$$\vec{B} = B_{\mathrm{x}}\cdot\vec{\mathrm{i}}+B_{\mathrm{y}}\cdot\vec{\mathrm{j}} \quad ; \quad d\vec{\ell} = dx\cdot\vec{\mathrm{i}}+dy\cdot\vec{\mathrm{j}} \tag{16.213}$$

are mutually collinear, and the following expression holds:

$$\frac{dy}{dx} = \frac{B_{\mathrm{y}}}{B_{\mathrm{x}}} \tag{16.214}$$

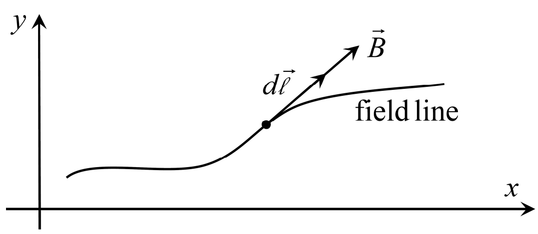


Figure 16.53. Magnetic field line in the (x, y) coordinate system

* The formula is named after the Russian scientist L. A. Tseitlin (rus. Цейтлин)

Since the magnetic induction lies in the (x, y) plane, the magnetic vector potential is directed along the z-axis, i.e., the following holds:

$$\vec{A} = A \cdot \vec{\mathrm{k}} \tag{16.215}$$

and it is:

$$\vec{B} = \nabla \times \vec{A} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & 0 \\ 0 & 0 & A \end{vmatrix} = \frac{\partial A}{\partial y} \cdot \vec{\mathrm{i}} - \frac{\partial A}{\partial x} \cdot \vec{\mathrm{j}} \tag{16.216}$$

From expression (16.214), it follows that:

$$B_{\mathrm{y}} \cdot dx - B_{\mathrm{x}} \cdot dy = 0 \tag{16.217}$$

and from expressions (16.216) and (16.217), it follows that for the magnetic field line, the following holds:

$$\frac{\partial A}{\partial x} \cdot dx + \frac{\partial A}{\partial y} \cdot dy = dA = 0 \quad \Rightarrow \quad A = \text{const.} \tag{16.218}$$

This proves the statement that in 2D problems, the lines where A = const. are also magnetic field lines.

16.23. Solved Examples

Example 16.1. Using Ampère's law, derive the expression for the magnetic field intensity inside and outside an infinitely long straight cylindrical conductor carrying a stationary electric current of intensity *I*. Let the radius of the conductor be r_0, and assume the conductor is located in air.

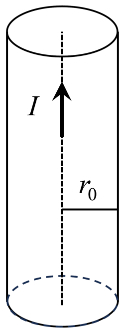


Figure 16.54. An infinitely long straight conductor

Solution:

The distribution of magnetic field strength is axisymmetric with respect to the axis of the conductor, so a cylindrical coordinate system (r, φ, z) is used.

The surface density of a stationary electric current is constant across the cross-sectional area of the conductor:

$$J = \frac{I}{S} = \frac{I}{r_0^2 \cdot \pi} = const. \tag{16.219}$$

For $r \le r_0$ Ampère's law states (Figure 16.55):

$$\oint_{C_1} \vec{H} \cdot d\vec{\ell} = H \cdot 2 \cdot \pi \cdot r = J \cdot r^2 \cdot \pi = I \cdot \frac{r^2 \cdot \pi}{r_0^2 \cdot \pi} \tag{16.220}$$

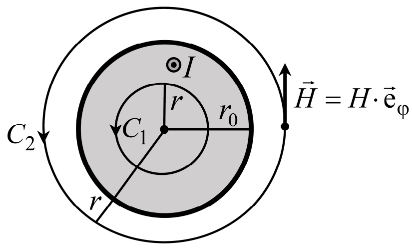


Figure 16.55. Integration curves inside and outside the conductor

From equation (16.220), it follows that the magnetic field intensity inside the conductor is described by the expression:

$$H = \frac{I \cdot r}{2 \cdot \pi \cdot r_0^2} \qquad \text{for} \quad r \le r_0 \tag{16.221}$$

For $r > r_0$ Ampère's law states (Figure 16.55):

$$\oint_{C_2} \vec{H} \cdot d\vec{\ell} = H \cdot 2 \cdot \pi \cdot r = I \tag{16.222}$$

from which it follows that the magnetic field intensity outside the conductor is described by the expression:

$$H = \frac{I}{2 \cdot \pi \cdot r} \qquad \text{for} \qquad r > r_0 \tag{16.223}$$

The magnetic field intensity is graphically shown in Figure 16.56.

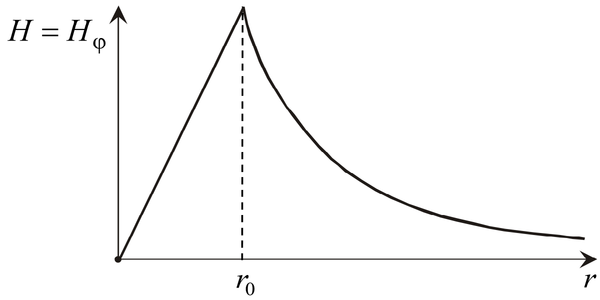


Figure 16.56. Graphical representation of the magnetic field intensity

Example 16.2. Let a stationary electric current of intensity I flow through an infinitely long straight conductor of radius R. Determine the expression for the distribution of the magnetic vector potential, and then, from the known distribution of the magnetic vector potential, determine the distribution of the magnetic field intensity. Let the conductor be in the air, let the permeability of the conductor be μ, and let the magnetic vector potential $A = 0$ along the axis of the conductor.

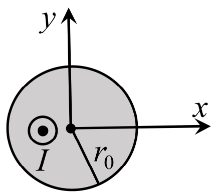


Figure 16.57. Conductor in the Cartesian coordinate system

Solution:

The magnetic vector potential satisfies Poisson's differential equation:

$$\Delta \vec{A} = -\ \mu \cdot \vec{J} \tag{16.224}$$

where in this case:

$$\vec{A} = A \cdot \vec{\mathrm{k}} \quad ; \quad \vec{J} = J \cdot \vec{\mathrm{k}} \tag{16.225}$$

The distribution of the magnetic vector potential is axisymmetric with respect to the z-axis, so the cylindrical coordinate system (r, φ, z) is used. In this case, it holds that:

$$\Delta A = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial A}{\partial r} \right) + \frac{1}{r^2} \cdot \underbrace{\frac{\partial^2 A}{\partial \varphi^2}}_{=0} + \underbrace{\frac{\partial^2 A}{\partial z^2}}_{=0} \tag{16.226}$$

Inside the conductor ($r \le r_0$) it holds that:

$$\Delta A_1 = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial A_1}{\partial r} \right) = -\mu \cdot J \tag{16.227}$$

whereas outside the conductor ($r > r_0$) it holds that:

$$\Delta A_2 = \frac{1}{r}\cdot\frac{\partial}{\partial r}\left(r\cdot\frac{\partial A_2}{\partial r}\right) = 0 \tag{16.228}$$

It follows that:

$$A = A_1 = C_1\cdot\ln r - \frac{1}{4}\cdot\mu\cdot J\cdot r^2 + C_2 \quad \text{for} \quad r \le r_0 \tag{16.229}$$

$$A = A_2 = C_3\cdot\ln r + C_4 \quad \text{for} \quad r > r_0 \tag{16.230}$$

where C_1, C_2, C_3, and C_4 are unknown constants that can be determined from the following boundary conditions:

$$A_1\big|_{r=0} = 0 \tag{16.231}$$

$$H_{1\varphi}\big|_{r=r_0} = H_{2\varphi}\big|_{r=r_0} \tag{16.232}$$

$$A_1\big|_{r=r_0} = A_2\big|_{r=r_0} \tag{16.233}$$

From the boundary condition (16.231), it follows that:

$$C_1 = C_2 = 0 \tag{16.234}$$

and, according to the expression (16.229), it follows that:

$$A = A_1 = -\frac{1}{4}\cdot\mu\cdot J\cdot r^2 \quad \text{for} \quad r \le r_0 \tag{16.235}$$

Furthermore, it holds that:

$$\vec{H} = \frac{1}{\mu}\cdot\left(\nabla\times\vec{A}\right) = \frac{1}{\mu\cdot r}\cdot\begin{vmatrix} \vec{\mathrm{e}}_\mathrm{r} & r\cdot\vec{\mathrm{e}}_\varphi & \vec{\mathrm{e}}_\mathrm{z} \\ \dfrac{\partial}{\partial r} & 0 & 0 \\ 0 & 0 & A \end{vmatrix} = -\frac{1}{\mu}\cdot\frac{\partial A}{\partial r}\cdot\vec{\mathrm{e}}_\varphi \tag{16.236}$$

from which it follows that:

$$H = H_\varphi = -\frac{1}{\mu}\cdot\frac{\partial A}{\partial r} \tag{16.237}$$

From the expressions (16.232) and (16.237), it follows that:

$$\left(-\frac{1}{\mu}\cdot\frac{\partial A_1}{\partial r}\right)_{r=r_0} = \left(-\frac{1}{\mu_0}\cdot\frac{\partial A_2}{\partial r}\right)_{r=r_0} \tag{16.238}$$

And, after differentiating the expressions (16.235) and (16.230), it is obtained that:

$$\frac{1}{4}\cdot J\cdot 2\cdot r\bigg|_{r=r_0} = -\frac{C_3}{\mu_0}\cdot\frac{1}{r}\bigg|_{r=r_0} \tag{16.239}$$

from which it follows that:

$$C_3 = -\frac{1}{2} \cdot \mu_0 \cdot J \cdot r_0^2 = -\frac{1}{2} \cdot \mu_0 \cdot J \cdot \frac{r_0^2 \cdot \pi}{\pi} = -\frac{\mu_0 \cdot I}{2 \cdot \pi} \tag{16.240}$$

$$A = A_2 = -\frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \ln r + C_4 \quad \text{for } r > r_0 \tag{16.241}$$

Furthermore, from the boundary condition (16.233) and from the expressions (16.235) and (16.241), it follows that:

$$-\frac{1}{4} \cdot \mu \cdot J \cdot r_0^2 = -\frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \ln r_0 + C_4 \tag{16.242}$$

from which it follows that:

$$C_4 = \frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \left(\ln r_0 - \frac{\mu_r}{2} \right) \tag{16.243}$$

$$A_2 = -\frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \ln r + \frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \left(\ln r_0 - \frac{\mu_r}{2} \right) \tag{16.244}$$

The final expressions are:

$$A = A_1 = -\frac{1}{4} \cdot \mu \cdot J \cdot r^2 \quad \text{for} \quad r \le r_0 \tag{16.245}$$

$$A = A_2 = -\frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \left(\ln \frac{r}{r_0} + \frac{\mu_r}{2} \right) \quad \text{for } r > r_0 \tag{16.246}$$

$$H_1 = H_{1\varphi} = -\frac{1}{\mu} \cdot \frac{\partial A_1}{\partial r} = \frac{I \cdot r}{2 \cdot \pi \cdot r_0^2} \tag{16.247}$$

$$H_2 = H_{2\varphi} = -\frac{1}{\mu_0} \cdot \frac{\partial A_2}{\partial r} = \frac{I}{2 \cdot \pi \cdot r} \tag{16.248}$$

Example 16.3. A stationary electric current of intensity I flows through an infinitely long strip of width a. Determine the distribution of the magnetic field intensity along the z-axis (in the plane $x = 0$), which is perpendicular to the strip.

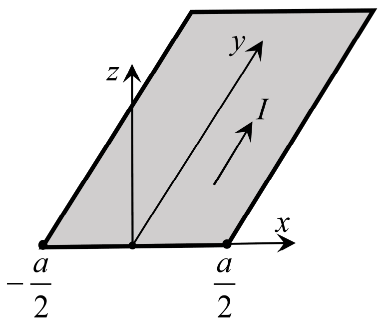


Figure 16.58. Conducting an infinitely long strip

Solution:

Due to symmetry, the resulting magnetic field intensity vector along the z-axis has only an x-component (Figure 16.59).

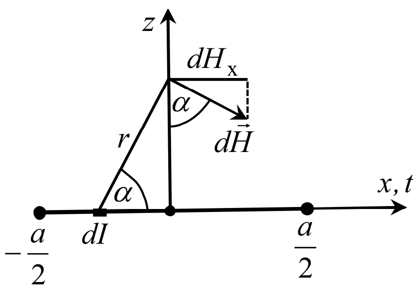


Figure 16.59. Infinitesimal vector of magnetic field intensity

According to Figure 16.59, it holds that:

$$dH_{\mathrm{x}} = dH \cdot \sin\alpha = \frac{dI}{2\cdot\pi\cdot r}\cdot\frac{z}{r} \tag{16.249}$$

where:

$$dH = \frac{dI}{2\cdot\pi\cdot r} \quad ; \quad dI = \frac{I}{a}\cdot dt \quad ; \quad r = \sqrt{z^2+t^2} \tag{16.250}$$

It follows that:

$$dH_{\mathrm{x}} = \frac{I}{2\cdot\pi\cdot a}\cdot\frac{z}{z^2+t^2}\cdot dt \tag{16.251}$$

and it is:

$$H = H_{\mathrm{x}} = \frac{I\cdot z}{2\cdot\pi\cdot a}\cdot\int_{-a/2}^{a/2}\frac{dt}{z^2+t^2} = \frac{I}{\pi\cdot a}\cdot\arctan\frac{a}{2\cdot z} \tag{16.252}$$

Example 16.4. For an infinitely long straight conductor of radius r_0, carrying a stationary electric current of intensity *I*, determine the expression for the internal inductance per unit length of the conductor: a) from the magnetic flux linkage, b) from the magnetic energy stored in the magnetostatic field. Let the material of the conductor be linear, homogeneous, and isotropic, and let its magnetic permeability be μ.

Figure 16.60. Infinitely long cylindrical conductor

Solution:

The distribution of magnetic field intensity inside the conductor is described by the expression:

$$H = \frac{I_{\mathrm{enc}}}{2\cdot\pi\cdot r} = \frac{I\cdot r}{2\cdot\pi\cdot r_0^2} \tag{16.253}$$

a) Calculation of the per-unit-length internal inductance from the magnetic flux linkage

The magnetic flux is closed around the axis of the conductor. The infinitesimal magnetic flux per unit length is given by:

$$d\Phi = \mu\cdot H\cdot dS = \mu\cdot H\cdot dr \tag{16.254}$$

whereas the infinitesimal magnetic flux linkage per unit length is described by the expression:

$$d\Psi = k_L \cdot d\Phi = \frac{I_{enc}}{I} \cdot \mu \cdot H \cdot dr = \frac{r^2}{r_0^2} \cdot \mu \cdot H \cdot dr \tag{16.255}$$

where k_L is the flux linkage factor, which is equal to the ratio of the electric current enclosed by the flux lines to the total current through which the magnetic flux linkage is expressed.
It follows that:

$$d\Psi = \mu \cdot \frac{I \cdot r^3}{2 \cdot \pi \cdot r_0^4} \cdot dr \tag{16.256}$$

and the magnetic flux linkage per unit length of the conductor is:

$$\Psi = \mu \cdot \frac{I}{2 \cdot \pi \cdot r_0^4} \cdot \int_0^{r_0} r^3 \cdot dr = \frac{\mu \cdot I}{8 \cdot \pi} = L_{int} \cdot I \tag{16.257}$$

from which it follows that the per-unit-length internal inductance of the conductor is described by the expression:

$$L_{int} = \frac{\Psi}{I} = \frac{\mu}{8 \cdot \pi} \tag{16.258}$$

b) Calculation of the per-unit-length internal inductance from the stored magnetic energy

The magnetic energy stored within one meter of length of an infinitely long conductor is described by the expression:

$$W_m = \frac{\mu}{2} \cdot \int_0^{r_o} H^2 \cdot 2 \cdot \pi \cdot r \cdot dr = \frac{\mu \cdot I^2}{4 \cdot \pi \cdot r_0^4} \cdot \int_0^{r_o} r^3 \cdot dr = \frac{\mu \cdot I^2}{16 \cdot \pi} = \frac{L_{int} \cdot I^2}{2} \tag{16.259}$$

from which it follows that the per-unit-length internal inductance of the conductor is described by the expression:

$$L_{int} = \frac{2 \cdot W_m}{I^2} = \frac{\mu}{8 \cdot \pi} \tag{16.260}$$

Note: The per-unit-length external inductance of an infinitely long straight conductor is infinite because the assumed return conductor is located at infinity.

Example 16.5. For a copper, infinitely long, straight hollow cylindrical conductor with inner radius $r_1 = 1\,\text{cm}$ and outer radius $r_2 = 5\,\text{cm}$, calculate the per-unit-length internal inductance: a) from the magnetic flux linkage, b) from the magnetic energy stored in the magnetostatic field. Assume that the conductor carries a stationary electric current.

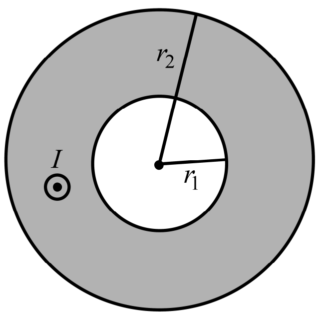


Figure 16.61. Infinitely long hollow cylindrical conductor

Solution:

The surface density of a stationary electric current in a conductor is described by the expression:

$$J = \frac{I}{(r_2^2 - r_1^2)\cdot \pi} \tag{16.261}$$

It follows that the magnetic field intensity in the conductor is described by the expression:

$$H = \frac{I_{\text{enc}}}{2\cdot \pi \cdot r} = \frac{J\cdot (r^2 - r_1^2)\cdot \pi}{2\cdot \pi \cdot r} = \frac{I}{2\cdot (r_2^2 - r_1^2)\cdot \pi}\cdot \frac{r^2 - r_1^2}{r} \tag{16.262}$$

a) Calculation of the per-unit-length internal inductance from the magnetic flux linkage

The magnetic flux is closed around the axis of the conductor. The infinitesimal magnetic flux per unit length is given by:

$$d\Phi = \mu_0 \cdot H \cdot dS = \mu_0 \cdot H \cdot dr \tag{16.263}$$

whereas the infinitesimal magnetic flux linkage per unit length is described by the expression:

$$d\Psi = k_{\text{L}} \cdot d\Phi = \frac{I_{\text{enc}}}{I}\cdot \mu_0 \cdot H \cdot dr = \frac{r^2 - r_1^2}{r_2^2 - r_1^2}\cdot \mu_0 \cdot H \cdot dr \tag{16.264}$$

where k_{L} is the flux linkage factor, which is equal to the ratio of the electric current enclosed by the flux lines to the total current through which the magnetic flux linkage is expressed.
It follows that:

$$d\Psi = \frac{\mu_0 \cdot I}{2\cdot (r_2^2 - r_1^2)^2 \cdot \pi}\cdot \left(r^3 - 2\cdot r_1^2 \cdot r + \frac{r_1^4}{r} \right) \tag{16.265}$$

and the magnetic flux linkage per unit length of the conductor is:

$$\Psi = \frac{\mu_0 \cdot I}{2\cdot (r_2^2 - r_1^2)^2 \cdot \pi}\cdot \left.\left(\frac{r^4}{4} - r_1^2 \cdot r^2 + r_1^4 \cdot \ln r \right)\right|_{r_1}^{r_2} = L_{\text{int}} \cdot I \tag{16.266}$$

from which it follows that the per-unit-length internal inductance is described by the expression:

$$L_{\text{int}} = \frac{\Psi}{I} = \frac{\mu_0}{2\cdot (r_2^2 - r_1^2)^2 \cdot \pi}\cdot \left(\frac{r_2^4 - r_1^4}{4} - r_1^2 \cdot r_2^2 + r_1^4 + r_1^4 \cdot \ln \frac{r_2}{r_1} \right) \tag{16.267}$$

By substituting the given data, it is obtained that:

$$L_{\text{int}} = \frac{\Psi}{I} = 4.639216594\times 10^{-8}\ \frac{\text{H}}{\text{m}} \tag{16.268}$$

b) Calculation of the per-unit-length internal inductance from the stored magnetic energy

The magnetic energy stored within one meter of length of an infinitely long conductor is described by the expression:

$$W_{\text{m}} = \frac{\mu_0}{2}\cdot \int_{r_1}^{r_2} H^2 \cdot 2\cdot \pi \cdot r \cdot dr = \frac{\mu_0 \cdot I^2}{4\cdot (r_2^2 - r_1^2)^2 \cdot \pi}\cdot \int_{r_1}^{r_2} \frac{\left(r^2 - r_1^2\right)^2}{r} dr = \frac{L_{\text{int}} \cdot I^2}{2} \tag{16.269}$$

from which it follows that the per-unit-length internal inductance of the conductor is described by the expression:

$$L_{\text{int}} = \frac{2 \cdot W_{\text{m}}}{I^2} = \frac{\mu_0}{2 \cdot (r_2^2 - r_1^2)^2 \cdot \pi} \cdot \left(\frac{r_2^4 - r_1^4}{4} - r_1^2 \cdot r_2^2 + r_1^4 + r_1^4 \cdot \ln \frac{r_2}{r_1} \right) \tag{16.270}$$

Example 16.6. Determine the per-unit-length inductance of a coaxial cable carrying a stationary electric current. Given: $r_1 = 1\,\text{cm}$, $r_2 = 2\,\text{cm}$, $r_3 = 3\,\text{cm}$, $\mu_{r1} = 10$, $\mu_{r3} = 5$.

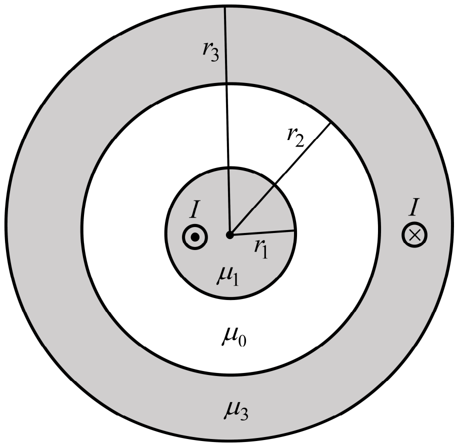


Figure 16.62. Infinitely long coaxial cable

Solution:

The per-unit-length inductance is most easily determined from the magnetic energy stored in the magnetostatic field. The magnetic field intensity is described by the expressions:

$$H = H_1 = \frac{I}{2 \cdot \pi \cdot r_1^2} \cdot r \quad \text{for } r \le r_1 \tag{16.271}$$

$$H = H_2 = \frac{I}{2 \cdot \pi \cdot r} \quad \text{for } r_1 < r \le r_2 \tag{16.272}$$

$$H = H_3 = \frac{I}{2 \cdot \pi \cdot r} \cdot \frac{r_3^2 - r^2}{r_3^2 - r_2^2} \quad \text{for } r_2 < r \le r_3 \tag{16.273}$$

The stored energy is described by the expressions:

$$W_{\text{m}} = W_{\text{m1}} + W_{\text{m2}} + W_{\text{m3}} = \frac{L \cdot I^2}{2} = \frac{L_1 + L_2 + L_3}{2} \cdot I^2 \tag{16.274}$$

$$W_{\text{mi}} = \frac{\mu_{\text{i}}}{2} \cdot \int_{V_i} H_{\text{i}}^2 \cdot 2 \cdot \pi \cdot r \cdot dr = \frac{1}{2} \cdot L_{\text{i}} \cdot I^2 \quad ; \quad \text{i} = 1, 2, 3 \tag{16.275}$$

from which it follows that:

$$L_1 = \frac{\mu_1}{8 \cdot \pi} = 0.5 \ \frac{\mu\text{H}}{\text{m}} \tag{16.276}$$

$$L_2 = \frac{\mu_0}{2 \cdot \pi} \cdot \ln \frac{r_2}{r_1} = 0.13862944 \ \frac{\mu\text{H}}{\text{m}} \tag{16.277}$$

$$L_3 = \frac{\mu_3}{2\cdot\pi}\cdot\left[\frac{r_3^4}{(r_3^2 - r_2^2)^2}\cdot\ln\frac{r_3}{r_2} - \frac{3\cdot r_3^2 - r_2^2}{4\cdot(r_3^2 - r_2^2)}\right] = 0.16370695\ \frac{\mu\text{H}}{\text{m}} \tag{16.278}$$

$$L = L_1 + L_2 + L_3 = 0.8023364\ \frac{\mu\text{H}}{\text{m}} \tag{16.279}$$

Example 16.7. Determine the per-unit-length inductance of a two-wire isolated electric line, assuming that a stationary electric current flows through the line. Let the radius of each conductor be r_0, the conductors be spaced by a distance *d*, the magnetic permeability of the conductor be μ_{cond}, and the line be located in the air. Assume that $d >> r_0$.

Figure 16.63. Two-wire isolated electric line

Solution:

Since $d >> r_0$, it can be assumed that one conductor does not affect the internal inductance of the other conductor. According to expression (16.258), the per-unit-length internal inductance of an infinitely long conductor is:

$$L_{\text{int cond}} = \frac{\mu_{\text{cond}}}{8\cdot\pi} \tag{16.280}$$

and the per-unit-length internal inductance of the electric line is:

$$L_{\text{int}} = 2\cdot L_{\text{int cond}} = \frac{\mu_{\text{cond}}}{4\cdot\pi} \tag{16.281}$$

The easiest way to derive this expression is to calculate the internal inductance of the conductor from the magnetic energy stored within an isolated, infinitely long conductor.

The per-unit-length external inductance of the electric line is most easily calculated from the magnetic flux linkage, based on Figure 16.64.

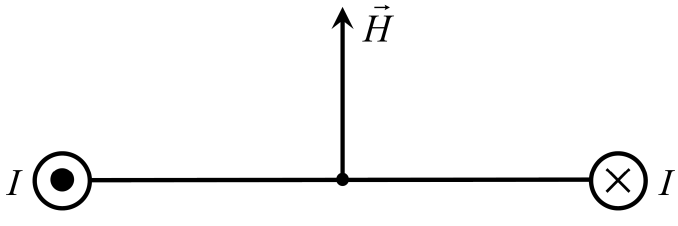


Figure 16.64. The magnetic field intensity vector between the conductors of a electric line

All the external magnetic flux of the electric line passes between the conductors. In the cylindrical coordinate system with the origin at the axis of the left conductor, the magnetic field intensity at the straight line between the conductors is described by the expression:

$$\vec{H} = \frac{I}{2\cdot\pi}\cdot\left(\frac{1}{r} + \frac{1}{d-r}\right)\cdot\vec{\text{e}}_\varphi \tag{16.282}$$

Since the conductors contribute equally to the magnetic flux passing between the conductors, to calculate the magnetic flux between the conductors, the magnetic flux can be calculated only from the electric current of the left conductor and then doubled:

$$\Psi = \Phi = \int_S \vec{B} \cdot d\vec{S} = 2 \cdot \frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \int_{r_0}^{d-r_0} \frac{dr}{r} = \frac{\mu_0 \cdot I}{\pi} \cdot \ln \frac{d - r_0}{r_0} = L_{\text{ext}} \cdot I \tag{16.283}$$

from which it follows that the per-unit-length external inductance of the two-wire electric line for $d >> r_0$ is:

$$L_{\text{ext}} = \frac{\Psi}{I} = \frac{\mu_0}{\pi} \cdot \ln \frac{d - r_0}{r_0} \approx \frac{\mu_0}{\pi} \cdot \ln \frac{d}{r_0} \tag{16.284}$$

The total per-unit-length inductance of the two-wire electric line is described by the expression:

$$L = L_{\text{int}} + L_{\text{ext}} = \frac{\mu_{\text{cond}}}{4 \cdot \pi} + \frac{\mu_0}{\pi} \cdot \ln \frac{d}{r_0} = \frac{\mu_0}{\pi} \cdot \left(\frac{\mu_r}{4} + \ln \frac{d}{r_0} \right) \tag{16.285}$$

Example 16.8. Determine the per-unit-length mutual inductance of two isolated, infinitely long, mutually parallel two-wire electric lines. Let the radius of the conductors be negligible compared to the distance between them. Let r_a, r_b, r_c, and r_d be given. Electric lines are in the air and carry stationary electric currents.

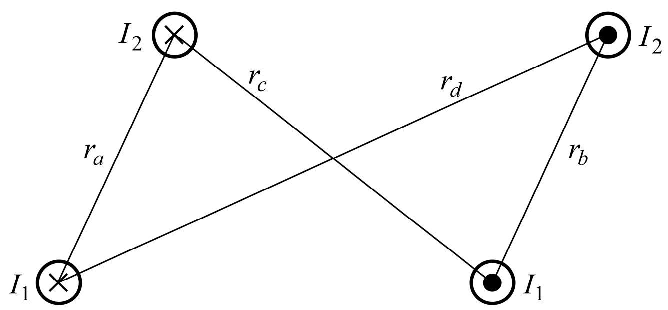


Figure 16.65. Two isolated two-wire electric lines

Solution:

The magnetic flux linkage that links line 2 and is produced by the stationary electric current in line 1 (Figure 16.66) is described by the expression:

$$\Psi_{2,1} = \Phi_{2,1} = \int_{r_a}^{r_d} \frac{\mu_0 \cdot I_1}{2 \cdot \pi \cdot r} \cdot dr - \int_{r_c}^{r_b} \frac{\mu_0 \cdot I_1}{2 \cdot \pi \cdot r} \cdot dr \tag{16.286}$$

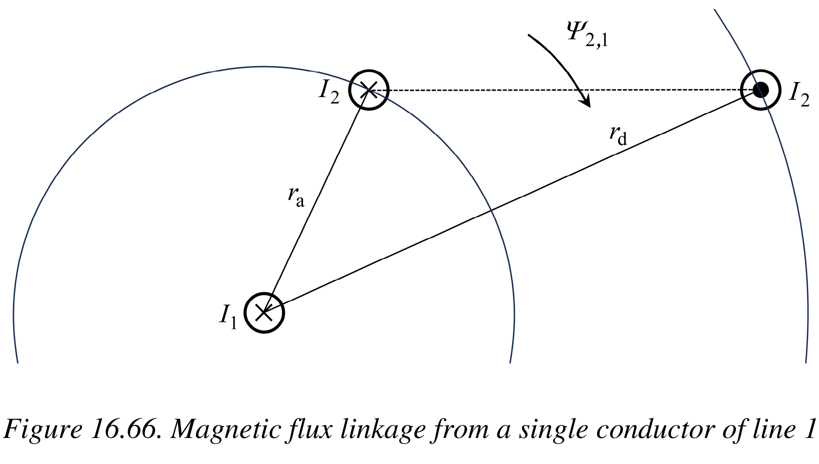


Figure 16.66. Magnetic flux linkage from a single conductor of line 1

From expression (16.286), it follows that:

$$\Psi_{2,1} = \frac{\mu_0 \cdot I_1}{2 \cdot \pi} \cdot \ln \frac{r_c \cdot r_d}{r_a \cdot r_b} = M_{2,1} \cdot I_1 \tag{16.287}$$

Therefore, the per-unit-length mutual inductance of two infinitely long, mutually parallel electric lines is:

$$M = M_{1,2} = M_{2,1} = \frac{\Psi_{2,1}}{I_1} = \frac{\mu_0}{2 \cdot \pi} \cdot \ln \frac{r_c \cdot r_d}{r_a \cdot r_b} \tag{16.288}$$

Example 16.9. A two-wire electric line in the air is located above a permeable half-space, whose relative magnetic permeability is $\mu_r = 5$. Determine the per-unit-length external inductance of the line using the method of images. Assume that a stationary electric current flows through the line. Let the following be given: $h_1 = 5$ m, $h_2 = 5.5$ m, $r_0 = 2$ cm, $d = 2$ m. Assume that $d >> r_0$.

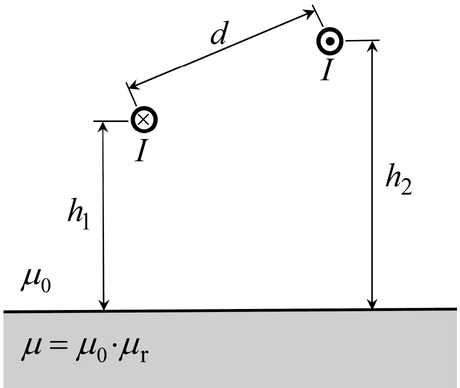


Figure 16.67. Two-wire electric line above a permeable half-space

Solution:

The effect of medium heterogeneity can be taken into account by satisfying the boundary condition at the boundary of two LIH (linear, isotropic, homogeneous) media using the method of images. According to the method of images, two conductors in the air are replaced by four conductors in an unbounded LIH medium with magnetic permeability equal to that of air (Figure 16.68).
According to expression (16.64), the reflection factor is:

$$k_R = \frac{\mu_r - 1}{\mu_r + 1} = \frac{2}{3} \tag{16.289}$$

while the parameter D (Figure 16.68) is described by the expression:

$$D = \sqrt{(h_1 + h_2)^2 + d^2 - (h_1 - h_2)^2} = \sqrt{d^2 + 4 \cdot h_1 \cdot h_2} = 10.67707825 \text{ m} \tag{16.290}$$

The external magnetic flux linkage per unit length of the line, assuming that $d >> r_0$, is described by the expression:

$$\Psi = \Phi = \int_S \vec{B} \cdot d\vec{S} = 2 \cdot \frac{\mu_0 \cdot I}{2 \cdot \pi} \cdot \int_{r_0}^{d} \frac{dr}{r} + \frac{\mu_0 \cdot k_R \cdot I}{2 \cdot \pi} \cdot \left(\int_{2 \cdot h_1}^{D} \frac{dr}{r} - \int_{D}^{2 \cdot h_2} \frac{dr}{r} \right) = L_{ext} \cdot I \tag{16.291}$$

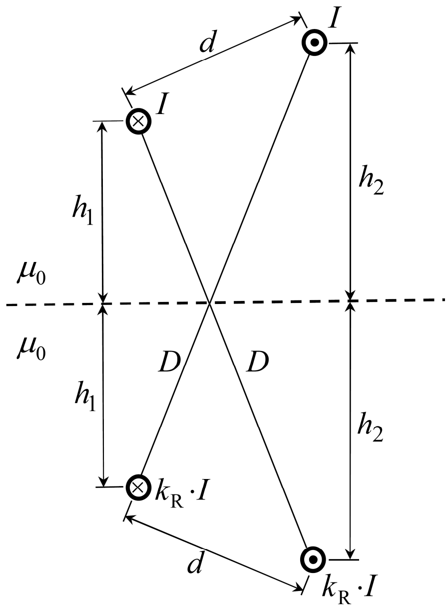


Figure 16.68. Real and image two-wire electric line in an unbounded LIH air

From expression (16.291), it follows that:

$$L_{\text{ext}} = \frac{\mu_0}{\pi} \cdot \ln\frac{d}{r_0} + \frac{\mu_0}{2 \cdot \pi} \cdot k_R \cdot \ln\frac{D^2}{4 \cdot h_1 \cdot h_2} = \frac{\mu_0}{\pi} \cdot \left(\ln\frac{d}{r_0} + k_R \cdot \ln\frac{D}{2 \cdot \sqrt{h_1 \cdot h_2}} \right) \tag{16.292}$$

By substituting the given data, it is obtained that:

$$L_{\text{ext}} = 1.846830485 \ \frac{\mu\text{H}}{\text{m}} \tag{16.293}$$

Example 16.10. Calculate the per-unit-length leakage inductance of an electric machine slot, assuming that the magnetic permeability of the iron is infinite. Let a stationary electric current flow through the copper conductor. The calculation should also include the leakage inductance of the air gap above the copper conductor, which has a height of a. Assume that the part of the magnetic field line in the copper conductor and the air gap is straight and parallel to the x-axis, and that the magnetic field intensity along this part of each field line in the copper conductor and air gap is constant. Determine the solution: a) from the magnetic flux linkage; b) from the magnetic energy stored in the magnetostatic field. Let the following be given: a = 1 cm, b = 3 cm, h = 6 cm.

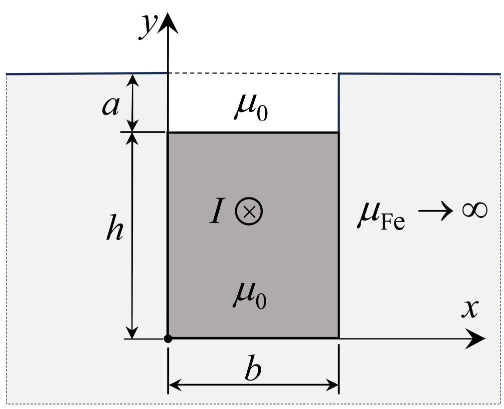


Figure 16.69. Electric machine slot

Solution:

Under the given assumptions, it holds that:

$$H = H_x = H_1 = \frac{I_{enc}}{b} = I \cdot \frac{y}{h \cdot b} \quad \text{for} \quad 0 \le y \le h \tag{16.294}$$

$$H = H_x = H_2 = \frac{I_{enc}}{b} = \frac{I}{b} \quad \text{for} \quad h < y \le h + a \tag{16.295}$$

a) Calculation of leakage inductance from the magnetic flux linkage

The magnetic field lines close as shown in Figure 16.70.

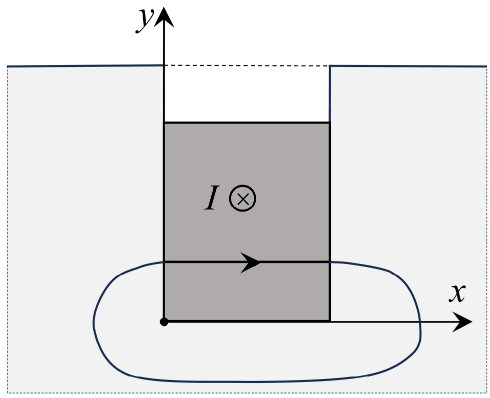


Figure 16.70. Graphical representation of the magnetic field line

The infinitesimal magnetic flux linkage in the copper conductor is described by the expression:

$$d\Psi_1 = \frac{I_{enc}}{I} \cdot d\Phi_1 = \frac{y}{h} \cdot d\Phi_1 = \frac{y}{h} \cdot \mu_0 \cdot H_1 \cdot dy = \mu_0 \cdot I \cdot \frac{y^2}{h^2 \cdot b} \cdot dy \tag{16.296}$$

from which it follows that the magnetic flux linkage per unit length of the copper conductor is:

$$\Psi_1 = \frac{\mu_0 \cdot I}{h^2 \cdot b} \cdot \int_0^h y^2 \cdot dy = \frac{\mu_0 \cdot h}{3 \cdot b} \cdot I = L_1 \cdot I \tag{16.297}$$

The infinitesimal magnetic flux linkage in the air gap above the copper conductor is described by the expression:

$$d\Psi_2 = \frac{I_{enc}}{I} \cdot d\Phi_2 = d\Phi_2 = \mu_0 \cdot H_2 \cdot dy = \mu_0 \cdot \frac{I}{b} \cdot dy \tag{16.298}$$

from which it follows that the magnetic flux linkage per unit length of the air gap is:

$$\Psi_2 = \frac{\mu_0 \cdot I}{b} \cdot \int_h^{h+a} dy = \frac{\mu_0 \cdot a}{b} \cdot I = L_2 \cdot I \tag{16.299}$$

The total per-unit-length leakage inductance of the slot is described by the expression:

$$L = \frac{\Psi}{I} = \frac{\Psi_1 + \Psi_2}{I} = L_1 + L_2 = \frac{\mu_0}{b} \cdot \left(\frac{h}{3} + a \right) \tag{16.230}$$

By substituting the given data, it is obtained that:

$$L = 1.256637061\ \frac{\mu\mathrm{H}}{\mathrm{m}} \tag{16.301}$$

b) Calculation of leakage inductance from the stored magnetic energy

The magnetic energy stored per unit length of the copper conductor is described by the expression:

$$W_{\mathrm{m1}} = \frac{\mu_0}{2} \cdot \int_0^h H_1^2 \cdot b \cdot dy = \frac{\mu_0 \cdot I^2}{2 \cdot h^2 \cdot b} \cdot \int_0^h y^2 \cdot dy = \frac{I^2}{2} \cdot \frac{\mu_0 \cdot h}{3 \cdot b} = \frac{I^2}{2} \cdot L_1 \tag{16.302}$$

whereas the magnetic energy stored per unit length of the air gap above the copper conductor is described by the expression:

$$W_{\mathrm{m2}} = \frac{\mu_0}{2} \cdot \int_h^{h+a} H_2^2 \cdot b \cdot dy = \frac{\mu_0 \cdot I^2}{2 \cdot b} \cdot \int_h^{h+a} dy = \frac{I^2}{2} \cdot \frac{\mu_0 \cdot a}{b} = \frac{I^2}{2} \cdot L_2 \tag{16.303}$$

The total per-unit-length leakage inductance of the slot is described by the expression:

$$L = \frac{2 \cdot W_{\mathrm{m}}}{I^2} = \frac{2 \cdot \left(W_{\mathrm{m1}} + W_{\mathrm{m2}}\right)}{I^2} = L_1 + L_2 = \frac{\mu_0}{b} \cdot \left(\frac{h}{3} + a\right) \tag{16.304}$$

Example 16.11. The electric line is formed by two mutually parallel copper busbars carrying a stationary electric current of intensity I. Using Ampère's law, determine the distribution of the magnetic field intensity, the magnetic flux linkage, and the per-unit-length inductance of the line. Neglect edge effects and assume that the length of the line $\ell >> h$ and that $h >> a, b$.

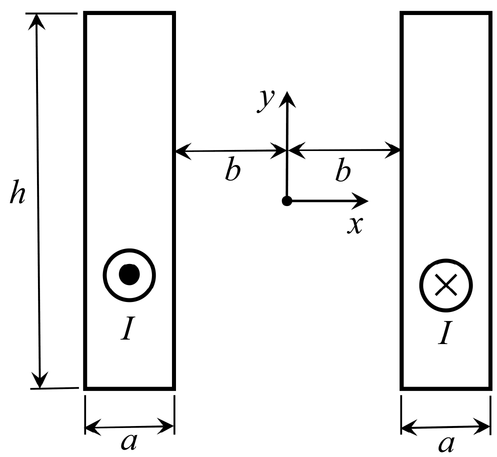


Figure 16.71. Two-wire electric line

Solution:

The 2D region can be divided into five distinct subregions (Figure 16.72). Under the given assumptions, the 3D problem is numerically reduced to a 1D problem. According to Figure 16.72, it holds that:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \oint_C \left(\underbrace{H_{\mathrm{x}}}_{0} \cdot dx + H_{\mathrm{y}} \cdot dy + \underbrace{H_{\mathrm{z}}}_{0} \cdot dz \right) = \oint_C H_{\mathrm{y}} \cdot dy = I_{\mathrm{enc}} \tag{16.305}$$

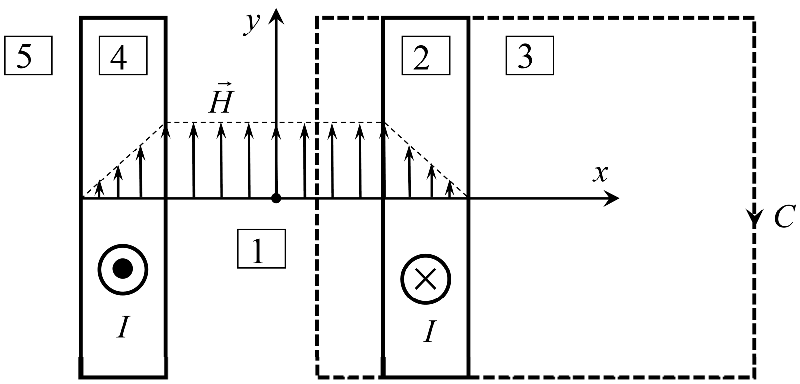


Figure 16.72. Division of the 2D region into five subregions

From expression (16.305), it follows that:

$$H_1 = H_{1y} = \frac{I_{enc}}{h} = \frac{I}{h} \tag{16.306}$$

$$H_2 = H_{2y} = \frac{I_{enc}}{h} = \frac{I}{h} \cdot \frac{a+b-x}{a} \tag{16.307}$$

$$H_3 = H_{3y} = \frac{I_{enc}}{h} = 0 \tag{16.308}$$

$$H_4 = H_{4y} = \frac{I_{enc}}{h} = \frac{I}{h} \cdot \frac{a+b+x}{a} \tag{16.309}$$

$$H_5 = H_{5y} = \frac{I_{enc}}{h} = 0 \tag{16.310}$$

It is important to note that the distribution of the magnetic field intensity is symmetric with respect to the y-axis, meaning it is an even function dependent only on the variable x.
The magnetic flux linkage in the direction of the y-axis in subregion 1 is:

$$\Psi_1 = \Phi_1 = \mu_0 \cdot H_1 \cdot 2 \cdot b \cdot 1 = 2 \cdot \mu_0 \cdot \frac{I \cdot b}{h} \tag{16.311}$$

The infinitesimal magnetic flux linkage in the direction of the y-axis in subregion 2 is:

$$d\Psi_2 = \frac{I_{enc}}{I} \cdot d\Phi_2 = \frac{a+b-x}{a} \cdot \mu_0 \cdot H_2 \cdot dx \tag{16.312}$$

from which it follows that:

$$\Psi_2 = \frac{\mu_0 \cdot I}{a^2 \cdot h} \cdot \int_b^{a+b} (a+b-x)^2 \cdot dx = -\frac{\mu_0 \cdot I}{a^2 \cdot h} \cdot \frac{(a+b-x)^3}{3} \Bigg|_b^{a+b} \tag{16.313}$$

and it is:

$$\Psi_2 = \frac{\mu_0 \cdot I}{3 \cdot a^2 \cdot h} \cdot a^3 = \frac{\mu_0 \cdot I \cdot a}{3 \cdot h} \tag{16.314}$$

The total magnetic flux linkage per unit length is:

$$\Psi = \Psi_1 + 2 \cdot \Psi_2 = \frac{\mu_0 \cdot I}{h} \cdot \left(2 \cdot b + \frac{2 \cdot a}{3} \right) \tag{16.315}$$

It follows that the per-unit-length inductance of the electric line is described by the expression:

$$L = \frac{\Psi}{I} = \frac{2 \cdot \mu_0}{h} \cdot \left(b + \frac{a}{3} \right) \tag{16.316}$$

Example 16.12. Using Neumann's formula, determine the self-inductance of a rectangular thin-wire loop located in the air. Let the conductors be made of copper with a circular cross-sectional radius r_0. Assume that a stationary electric current flows through the loop. Also, assume that $r_0 << a, b$.

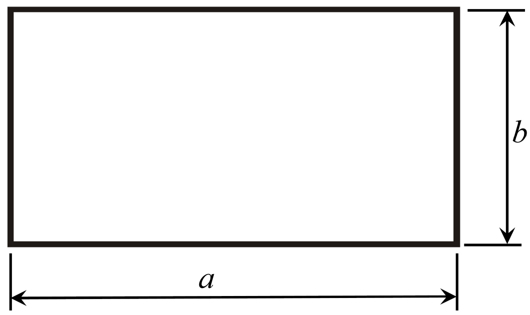


Figure 16.73. Thin-wire rectangular loop

Solution:

The thin-wire approximation of the conductor is used, meaning that the stationary electric current is assumed to be concentrated along the axis of the conductor. The self-inductance of the thin-wire loop, according to expression (16.192), is described by Neumann's formula:

$$L = L_{\text{int}} + L_{\text{ext}} \quad ; \quad L_{\text{ext}} = \frac{\mu_0}{4 \cdot \pi} \cdot \oint_C \oint_{C'} \frac{d\vec{\ell} \cdot d\vec{\ell}\,'}{r} \tag{16.317}$$

where L_{ext} is the external inductance of the thin-wire loop, whereas L_{int} is the internal inductance of the thin-wire loop, which, in the case of a copper thin-wire conductor with a circular cross-section can, according to expression (16.193), be approximated by the following expression:

$$L_{\text{int}} = \frac{\mu_{\text{cond}}}{8 \cdot \pi} \cdot \ell_{\text{loop}} = \frac{\mu_0}{4 \cdot \pi} \cdot (a + b) \tag{16.318}$$

In the general case, the integration paths for calculating the self-inductance of the thin-wire loop are given in Figure 16.74.

In the first step, the thin-wire loop is divided into four oriented thin-wire conductor segments (Figure 16.74).

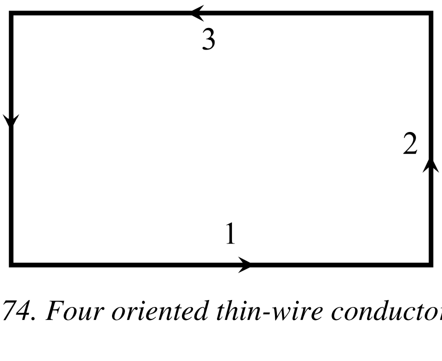


Figure 16.74. Four oriented thin-wire conductor segments

In this case, the general expression (16.317) takes the form:

$$L = L_{\text{int}} + \frac{\mu_0}{4\cdot\pi}\cdot\sum_{\text{i}=1}^{4}\sum_{\text{k}=1}^{4}\int_{C_{1\text{i}}}\int_{C_{2\text{k}}}\frac{d\vec{\ell}_{1\text{i}}\cdot d\vec{\ell}_{2\text{k}}}{r} = L_{\text{int}} + \sum_{\text{i}=1}^{4}\sum_{\text{k}=1}^{4} L_{\text{i,k}} \tag{16.319}$$

where, therefore, the self and mutual external inductances of the conductor segments are described by the expression:

$$L_{\text{i,k}} = L_{\text{k,i}} = \frac{\mu_0}{4\cdot\pi}\cdot\int_{C_{1\text{i}}}\int_{C_{2\text{k}}}\frac{d\vec{\ell}_{1\text{i}}\cdot d\vec{\ell}_{2\text{k}}}{r} \tag{16.320}$$

The mutual inductances of the mutually perpendicular conductor segments are zero, and furthermore, it holds that:

$$L_{1,1} = L_{3,3} \quad ; \quad L_{2,2} = L_{4,4} \tag{16.321}$$

and it is:

$$L = \frac{\mu_0}{4\cdot\pi}\cdot(a+b) + 2\cdot\left(L_{1,1} + L_{2,2} + L_{1,3} + L_{2,4}\right) \tag{16.322}$$

According to subchapter 16.21, the self external inductances of the segments are described by the expressions:

$$L_{1,1} = F(a, r_0) \quad ; \quad L_{2,2} = F(b, r_0) \tag{16.323}$$

whereas the mutual inductances of the segments are described by the expressions:

$$L_{1,3} = -F(a,b) \quad ; \quad L_{2,4} = -F(b,a) \tag{16.324}$$

where, according to expression (16.204), the auxiliary function $F(\ell, v)$ is described by:

$$F(\ell, v) = \frac{\mu_0}{2\cdot\pi}\cdot\left[\ell\cdot\ln\frac{\sqrt{\ell^2+v^2}+\ell}{v} - \frac{\ell^2}{\sqrt{\ell^2+v^2}+v}\right] \tag{16.325}$$

From the previous expressions, it follows that:

$$L_{1,1} = \frac{\mu_0}{2\cdot\pi}\cdot\left[a\cdot\ln\frac{\sqrt{a^2+r_0^2}+a}{r_0} - \frac{a^2}{\sqrt{a^2+r_0^2}+r_0}\right] \tag{16.326}$$

$$L_{2,2} = \frac{\mu_0}{2\cdot\pi}\cdot\left[b\cdot\ln\frac{\sqrt{b^2+r_0^2}+b}{r_0} - \frac{b^2}{\sqrt{b^2+r_0^2}+r_0}\right] \tag{16.327}$$

$$L_{1,3} = -\frac{\mu_0}{2\cdot\pi}\cdot\left[a\cdot\ln\frac{\sqrt{a^2+b^2}+a}{b} - \frac{a^2}{\sqrt{a^2+b^2}+b}\right] \tag{16.328}$$

$$L_{2,4} = -\frac{\mu_0}{2\cdot\pi}\cdot\left[b\cdot\ln\frac{\sqrt{b^2+a^2}+b}{a} - \frac{b^2}{\sqrt{b^2+a^2}+a}\right] \tag{16.329}$$

Example 16.13. For a straight segment of a copper thin-wire conductor of length $\ell = 10$ m and radius $r_0 = 5$ mm, calculate the self-inductance using Neumann's formula. Let the conductor be in the air, and assume that a stationary electric current flows through the wire. Assume that $r_0 << \ell$.

Solution:

$$L_{\text{int}} = \frac{\mu_0}{8 \cdot \pi} \cdot \ell = 0.5 \ \mu\text{H} \tag{16.330}$$

$$L_{\text{ext}} = F(\ell, r_0) = \frac{\mu_0}{2 \cdot \pi} \cdot \left[\ell \cdot \ln \frac{\sqrt{\ell^2 + r_0^2} + \ell}{r_0} - \frac{\ell^2}{\sqrt{\ell^2 + r_0^2} + r_0} \right] \tag{16.331}$$

$$L = L_{\text{int}} + L_{\text{ext}} = 15.0890991552 \ \mu\text{H} \tag{16.332}$$

Example 16.14. Determine the distribution of magnetic field intensity along the axis of a straight thin solenoid of length ℓ and radius a placed in the air. Assume the solenoid is tightly and uniformly wound on a cylinder made of a material with magnetic permeability μ_0 and that it has a total of N turns. A stationary electric current of intensity I flows through the solenoid.

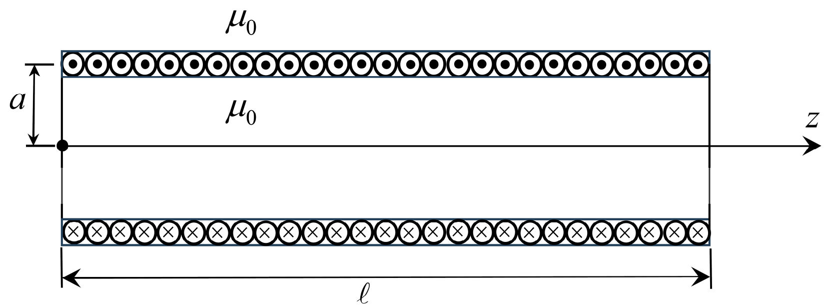


Figure 16.75. Tightly and uniformly wound straight solenoid

Solution:

To derive the expression for the distribution of the magnetic field intensity along the axis of a straight thin solenoid, the surface electric current* with linear density K can be used (Figure 16.76). It follows that:

$$K = \frac{N \cdot I}{\ell} \tag{16.333}$$

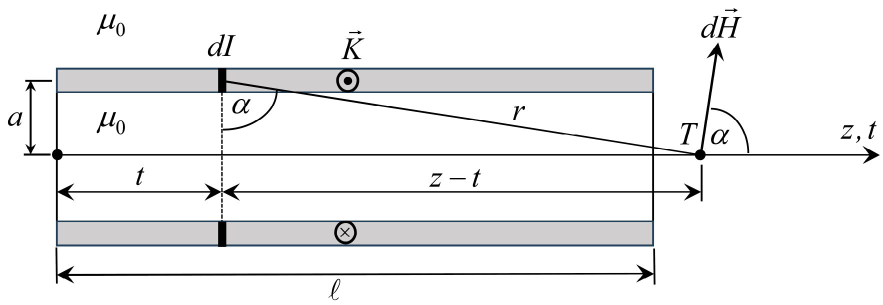


Figure 16.76. Straight solenoid and its corresponding surface electric current

* The surface electric current flowing around the cylinder in a circular (azimuthal) direction**.** Its linear density is measured in a direction perpendicular to the current flow.

According to the Biot-Savart law, for a thin-wire circular loop carrying an infinitesimal stationary electric current of intensity $dI = K \cdot dt$ (Figure 16.76), it holds that:

$$d\vec{H} = \frac{dI}{4 \cdot \pi} \cdot \frac{d\vec{\ell} \times \vec{r}}{r^3} = \frac{K \cdot dt}{4 \cdot \pi} \cdot \frac{a \cdot d\varphi \cdot \vec{e}_\varphi \times \vec{r}}{r^3} \tag{16.334}$$

Due to symmetry, along the z-axis, the magnetic field intensity has only a z-component. Thus, according to expression (16.334) and Figure 16.76, it holds that:

$$dH_z = dH \cdot \cos\alpha = \frac{a}{r} \cdot dH = \frac{K \cdot du}{4 \cdot \pi} \cdot \frac{a^2 \cdot d\varphi}{r^3} \tag{16.335}$$

from which it follows that:

$$H = H_z = \frac{K \cdot a^2}{4 \cdot \pi} \cdot \int_0^{2\cdot\pi} d\varphi \cdot \int_0^{\ell} \frac{dt}{r^3} = \frac{K \cdot a^2}{2} \cdot \int_0^{\ell} \frac{dt}{r^3} \tag{16.336}$$

where:

$$r = \sqrt{(z-t)^2 + a^2} = \sqrt{(t-z)^2 + a^2} \tag{16.337}$$

and it is:

$$H = H_z = \frac{K \cdot a^2}{2} \cdot \int_0^{\ell} \frac{dt}{\left(\sqrt{(t-z)^2 + a^2}\right)^3} \tag{16.338}$$

It follows that:

$$H = H_z = \frac{K \cdot a^2}{2} \cdot \left. \frac{t-z}{a^2 \cdot \sqrt{(t-z)^2 + a^2}} \right|_0^{\ell} \tag{16.339}$$

and it is:

$$H = H_z = \frac{N \cdot I}{2 \cdot \ell} \cdot \left[\frac{\ell - z}{\sqrt{(\ell - z)^2 + a^2}} + \frac{z}{\sqrt{z^2 + a^2}} \right] \tag{16.340}$$

If the field point is inside the solenoid, then expression (16.340) can be written in the following form:

$$H = \frac{N \cdot I}{2 \cdot \ell} \cdot (\cos\beta_1 + \cos\beta_2) \tag{16.341}$$

where:

$$\cos\beta_1 = \frac{\ell - z}{\sqrt{(\ell - z)^2 + a^2}} \tag{16.342}$$

$$\cos\beta_2 = \frac{z}{\sqrt{z^2 + a^2}} \tag{16.343}$$

The angles β_1 and β_2 are shown in Figure 16.77.

For a point T deep inside a very long solenoid ($\ell >> a$), it holds that:

$$\beta_1 \to 0 \quad ; \quad \beta_2 \to 0 \quad ; \quad H \approx \frac{N \cdot I}{\ell} \tag{16.344}$$

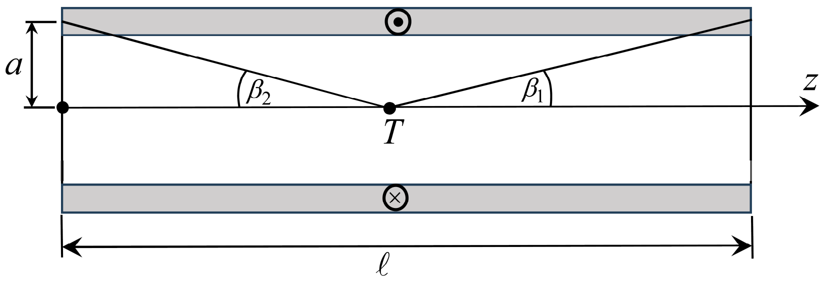


Figure 16.77. Field point inside a straight solenoid

Example 16.15. Determine the external self-inductance of a straight thin solenoid of length ℓ and radius a, placed in the air and shown in Figure 16.75. Assume the solenoid is tightly and uniformly wound on a cylinder made of a material with magnetic permeability μ_0 and that it has a total of N turns. A stationary electric current of intensity I flows through the solenoid. Assume that the magnetic field inside the solenoid is homogeneous and that the intensity of the homogeneous magnetic field is equal to the average value of the magnetic field intensity along the axis of the solenoid.

Solution:

According to the previous example, the magnetic field intensity along the z-axis is described by expression (16.339), which reads:

$$H = H(z) = \frac{N \cdot I}{2 \cdot \ell} \cdot \left(\frac{\ell - z}{\sqrt{(\ell - z)^2 + a^2}} + \frac{z}{\sqrt{z^2 + a^2}} \right) \tag{16.345}$$

and the average value of the magnetic field intensity along the z-axis, inside the solenoid, is given by the expression:

$$H_{\text{av}} = \frac{1}{\ell} \cdot \int_0^\ell H \cdot dz = \frac{N \cdot I}{2 \cdot \ell^2} \cdot \int_0^\ell \left(\frac{z}{\sqrt{z^2 + a^2}} - \frac{z - \ell}{\sqrt{(z - \ell)^2 + a^2}} \right) \cdot dz \tag{16.346}$$

which implies that:

$$H_{\text{av}} = \frac{N \cdot I}{\ell^2} \cdot \left(\sqrt{\ell^2 + a^2} - a \right) = \frac{N \cdot I}{\sqrt{\ell^2 + a^2} + a} \tag{16.347}$$

The approximation of the magnetic flux linkage of the solenoid is described by the following expression:

$$\Psi = N \cdot \Phi = N \cdot \mu_0 \cdot H_{\text{av}} \cdot a^2 \cdot \pi \tag{16.348}$$

and it is:

$$\Psi = \mu_0 \cdot \frac{N^2 \cdot I \cdot a^2 \cdot \pi}{\sqrt{a^2 + \ell^2} + a} = L \cdot I \tag{16.349}$$

It follows that the external self-inductance of the solenoid is approximated by the expression:

$$L = \frac{\Psi}{I} = \mu_0 \cdot \frac{N^2 \cdot a^2 \cdot \pi}{\sqrt{a^2 + \ell^2} + a} \tag{16.350}$$

For a very long solenoid (x > a), the following holds:

$$L \approx \frac{N^2 \cdot \mu_0 \cdot a^2 \cdot \pi}{\ell} = \frac{N^2 \cdot \mu_0 \cdot S}{\ell} = \frac{N^2}{R_\mathrm{m}} \tag{16.351}$$

where S is the cross-sectional area of the solenoid, and R_m is the magnetic reluctance of the very long solenoid.

Example 16.16. Let a stationary electric current of intensity I flow through the electromagnet coil. Determine the expression for the force by which the ends of the electromagnet attract each other: a) from the magnetic energy stored in the magnetostatic field, b) using stress (force at the boundary between two media). Assume that the total average length of the magnetic flux line ℓ is constant. Neglect the fringing of magnetic field lines in the air gap and the leakage magnetic flux. Let the following be given: $N, I, \ell_\mathrm{Fe}, \delta, \mu_\mathrm{r}, S$.

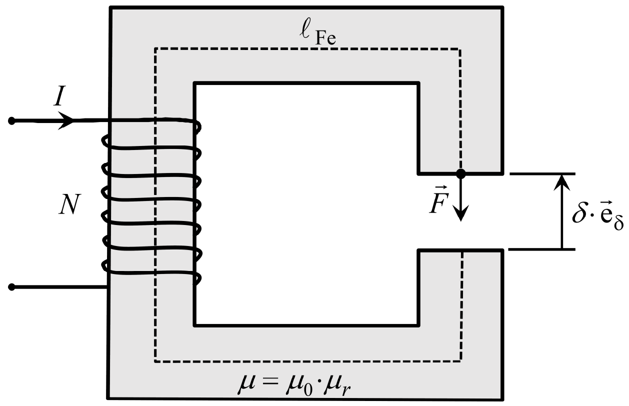


Figure 16.78. Attractive force between the poles of the electromagnet

Solution:

Since the fringing of magnetic field lines in the air gap and the leakage magnetic flux are neglected, it holds that:

$$B_\mathrm{Fe} = B_0 = B \quad ; \quad S_\mathrm{Fe} = S_0 = S \tag{16.352}$$

where:

B_Fe - the magnetic flux density in the iron,

B_0 - the magnetic flux density in the air gap,

S_Fe - the surface area through which the magnetic flux passes in the iron,

S_0 - the surface area through which the magnetic flux passes in the air gap.

For the magnetic circuit shown in Figure 16.78, under the given assumptions, the following expression holds:

$$N \cdot I = H_\mathrm{Fe} \cdot \ell_\mathrm{Fe} + H_0 \cdot \delta = \frac{B}{\mu_0} \cdot \left(\frac{\ell_\mathrm{Fe}}{\mu_\mathrm{r}} + \delta \right) \tag{16.353}$$

where:

ℓ_Fe - the average length of the magnetic field line in the iron,

δ - the average length of the magnetic field line in the air gap; width of the air gap.

From expression (16.353), it follows that the magnetic flux density in the iron as well as in the air gap is described by the following expression:

$$B=\frac{\mu_0\cdot N\cdot I}{\dfrac{\ell_{\mathrm{Fe}}}{\mu_{\mathrm{r}}}+\delta}=\frac{\mu_0\cdot\mu_{\mathrm{r}}\cdot N\cdot I}{\ell_{\mathrm{Fe}}+\mu_{\mathrm{r}}\cdot\delta} \tag{16.354}$$

a) Calculation of force from the stored magnetic energy

The magnetic energy stored in the magnetostatic field is described by the expression:

$$W_{\mathrm{m}}=\frac{B\cdot H_{Fe}\cdot S\cdot\ell_{Fe}}{2}+\frac{B\cdot H_0\cdot S\cdot\delta}{2}=\frac{B^2\cdot S}{2\cdot\mu_0\cdot\mu_{\mathrm{r}}}\cdot\left(\ell_{\mathrm{Fe}}+\mu_{\mathrm{r}}\cdot\delta\right) \tag{16.355}$$

If expression (16.354) is substituted into expression (16.355), the following expression for the stored magnetic energy is obtained:

$$W_{\mathrm{m}}=\frac{\mu_0\cdot\mu_{\mathrm{r}}\cdot N^2\cdot I^2\cdot S}{2\cdot\left(\ell_{\mathrm{Fe}}+\mu_{\mathrm{r}}\cdot\delta\right)} \tag{16.356}$$

The expression for the magnetic force in a non-isolated system (Figure 16.78) is given by:

$$\vec{F}=\frac{\partial W_{\mathrm{m}}}{\partial\delta}\cdot\vec{\mathrm{e}}_{\delta} \tag{16.357}$$

Before differentiating, the following substitution should be introduced:

$$\ell_{\mathrm{Fe}}=\ell-\delta \tag{16.358}$$

where ℓ is the total average length of the magnetic field line, which is, according to the given assumption, a constant parameter.

If substitution (16.358) is inserted into expression (16.356), an alternative expression for the stored magnetic energy is obtained:

$$W_{\mathrm{m}}=\frac{\mu_0\cdot\mu_{\mathrm{r}}\cdot N^2\cdot I^2\cdot S}{2\cdot\left(\ell_{\mathrm{Fe}}+\mu_{\mathrm{r}}\cdot\delta\right)}=\frac{\mu_0\cdot\mu_{\mathrm{r}}\cdot N^2\cdot I^2\cdot S}{2\cdot\left(\ell-\delta+\mu_{\mathrm{r}}\cdot\delta\right)} \tag{16.359}$$

which is suitable for differentiating the stored magnetic energy with respect to the air gap width δ.
It follows that:

$$\vec{F}=\frac{\partial W_{\mathrm{m}}}{\partial\delta}\cdot\vec{\mathrm{e}}_{\delta}=\frac{\mu_0\cdot\mu_{\mathrm{r}}\cdot N^2\cdot I^2\cdot S}{2\cdot\left(\ell_{\mathrm{Fe}}+\mu_{\mathrm{r}}\cdot\delta\right)^2}\cdot\left(\mu_{\mathrm{r}}-1\right)\cdot\left(-\vec{\mathrm{e}}_{\delta}\right) \tag{16.360}$$

or, written differently:

$$\vec{F}=\frac{B^2\cdot S}{2\cdot\mu_0}\cdot\frac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}}\cdot\left(-\vec{\mathrm{e}}_{\delta}\right)=\frac{1}{2}\cdot B\cdot H_0\cdot S\cdot\frac{\mu_{\mathrm{r}}-1}{\mu_{\mathrm{r}}}\cdot\left(-\vec{\mathrm{e}}_{\delta}\right) \tag{16.361}$$

From expressions (16.360) and (16.361), it follows that the ends of the electromagnet attract each other, i.e., there is a tendency to reduce the air gap.

b) Calculation of the force at the boundary between two media

The pressure at the boundary between two media is described by expression (16.180), which in this particular case is given by:

$$t_{\mathrm{n}}^{\mathrm{m}}=\frac{\left|\mu_0\cdot\mu_{\mathrm{r}}-\mu_0\right|}{2}\cdot\left(\frac{B_{\mathrm{n}}^2}{\mu_0\cdot\mu_{\mathrm{r}}\cdot\mu_0}+H_{\mathrm{t}}^2\right) \tag{16.362}$$

where, in this case, it holds that:

$$B_{\mathrm{n}} = \mathrm{B} \quad ; \quad H_{\mathrm{t}} = 0 \tag{16.363}$$

and the pressure at the boundary between the iron and air is:

$$t_{\mathrm{n}}^{\mathrm{m}} = \frac{B^2}{2 \cdot \mu_0} \cdot \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \right) \tag{16.364}$$

The magnetic force at the boundary between two media is described by the expression:

$$\vec{F} = t_{\mathrm{n}}^{\mathrm{m}} \cdot S \cdot \left(-\vec{\mathrm{e}}_{\delta} \right) = \frac{B^2 \cdot S}{2 \cdot \mu_0} \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \left(-\vec{\mathrm{e}}_{\delta} \right) = \frac{1}{2} \cdot B \cdot H_0 \cdot S \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \left(-\vec{\mathrm{e}}_{\delta} \right) \tag{16.365}$$

If the magnetic permeability of the iron tends to infinity, then the magnetic force is described by the expression:

$$\vec{F} = \frac{1}{2} \cdot B \cdot H_0 \cdot S \cdot \left(-\vec{\mathrm{e}}_{\delta} \right) \tag{16.366}$$

whereas the magnetic field intensity in the iron and the magnetic energy stored in the iron are equal to zero.

Example 16.17. A straight solenoid of length $\ell = 0.25$ m, with N =500 turns and carrying a stationary electric current of intensity $I = 1$ A, is partially inserted with an iron core. Assuming negligible edge effects (i.e., the magnetic flux density is homogeneous inside the solenoid and zero outside), determine the force on the core at $z = 0.05$ m in two ways: a) from the magnetic energy stored in the magnetostatic field, b) using stress (force at the boundary between two media). Assume that both the iron core and the solenoid have circular cross-sections and that the radius of the iron core is $a = 0.02$ m. The relative magnetic permeability of the iron is $\mu_{\mathrm{r}} = 600$.

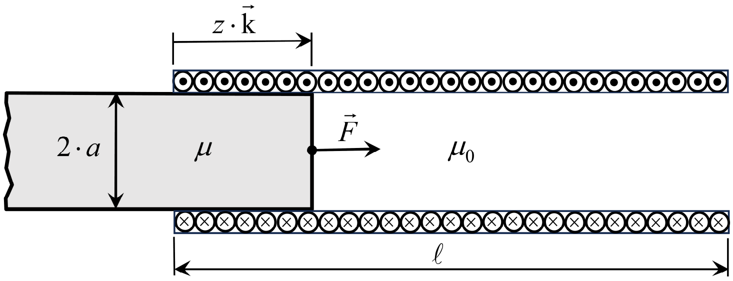


Figure 16.79. Straight solenoid with a partially inserted ferromagnetic core

Solution:

Under the given assumptions, it holds that:

$$B_{\mathrm{Fe}} = B_0 = B \quad ; \quad S_{\mathrm{Fe}} = S_0 = S \tag{16.367}$$

$$N \cdot I = H_{\mathrm{Fe}} \cdot z + H_0 \cdot (\ell - z) = \frac{B}{\mu_0} \cdot \left(\frac{z}{\mu_{\mathrm{r}}} + \ell - z \right) \tag{16.368}$$

from which it follows that the magnetic flux density is described by the following expression:

$$B = \frac{\mu_0 \cdot \mu_{\mathrm{r}} \cdot N \cdot I}{z + \mu_{\mathrm{r}} \cdot (\ell - z)} = \frac{\mu_0 \cdot \mu_{\mathrm{r}} \cdot N \cdot I}{\mu_{\mathrm{r}} \cdot \ell - z \cdot (\mu_{\mathrm{r}} - 1)} \tag{16.369}$$

a) Calculation of force from the stored magnetic energy

The magnetic energy stored in the magnetostatic field is described by the expression:

$$W_{\mathrm{m}} = \frac{B \cdot H_{Fe} \cdot S \cdot z}{2} + \frac{B \cdot H_0 \cdot S \cdot (\ell - z)}{2} = \frac{B^2 \cdot S}{2 \cdot \mu_0 \cdot \mu_{\mathrm{r}}} \cdot \left[\mu_{\mathrm{r}} \cdot \ell - z \cdot (\mu_{\mathrm{r}} - 1)\right] \tag{16.370}$$

where $S = a^2 \cdot \pi$ is the cross-sectional area of the iron core.

If expression (16.369) is substituted into expression (16.370), the following expression for the stored magnetic energy is obtained:

$$W_{\mathrm{m}} = \frac{1}{2} \cdot \frac{\mu_0 \cdot \mu_{\mathrm{r}} \cdot N^2 \cdot I^2 \cdot S}{\mu_{\mathrm{r}} \cdot \ell - z \cdot (\mu_{\mathrm{r}} - 1)} \tag{16.371}$$

The expression for the magnetic force in a non-isolated system (Figure 16.79) is given by:

$$\vec{F} = \frac{\partial W_{\mathrm{m}}}{\partial z} \cdot \vec{\mathrm{k}} = \frac{1}{2} \cdot \frac{\mu_0 \cdot \mu_{\mathrm{r}} \cdot N^2 \cdot I^2 \cdot S}{\left[\mu_{\mathrm{r}} \cdot \ell - z \cdot (\mu_{\mathrm{r}} - 1)\right]^2} \cdot (\mu_{\mathrm{r}} - 1) \cdot \vec{\mathrm{k}} \tag{16.372}$$

or, written differently:

$$\vec{F} = \frac{B^2 \cdot S}{2 \cdot \mu_0} \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \vec{\mathrm{k}} = \frac{1}{2} \cdot B \cdot H_0 \cdot S \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \vec{\mathrm{k}} \tag{16.373}$$

By substituting the given data, the magnetic force at $z = 0.05$ m is obtained as:

$$\left.\vec{F}\right|_{z=0.05\ \mathrm{m}} = 4.922474613 \cdot \vec{\mathrm{k}}\ \ \mathrm{mN} \tag{16.374}$$

b) Calculation of the force at the boundary between two media

The pressure at the boundary between two media is described by expression (16.180), which in this particular case is given by:

$$t_{\mathrm{n}}^{\mathrm{m}} = \frac{\left|\mu_0 \cdot \mu_{\mathrm{r}} - \mu_0\right|}{2} \cdot \left(\frac{B_{\mathrm{n}}^2}{\mu_0 \cdot \mu_{\mathrm{r}} \cdot \mu_0} + H_{\mathrm{t}}^2\right) \tag{16.375}$$

where, in this case, it holds that:

$$B_{\mathrm{n}} = \mathrm{B} \quad ; \quad H_{\mathrm{t}} = 0 \tag{16.376}$$

and the pressure at the boundary between the iron and air is:

$$t_{\mathrm{n}}^{\mathrm{m}} = \frac{B^2}{2 \cdot \mu_0} \cdot \left(\frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}}\right) \tag{16.377}$$

The magnetic force at the boundary between two media is described by the expression:

$$\vec{F} = t_{\mathrm{n}}^{\mathrm{m}} \cdot S \cdot \vec{\mathrm{k}} = \frac{B^2 \cdot S}{2 \cdot \mu_0} \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \vec{\mathrm{k}} = \frac{1}{2} \cdot B \cdot H_0 \cdot S \cdot \frac{\mu_{\mathrm{r}} - 1}{\mu_{\mathrm{r}}} \cdot \vec{\mathrm{k}} \tag{16.378}$$

If the magnetic permeability of the iron tends to infinity, then the magnetic force is described by the expression:

$$\vec{F} = \frac{1}{2} \cdot B \cdot H_0 \cdot S \cdot \vec{\mathrm{k}} \tag{16.379}$$

whereas the magnetic field intensity in the iron and the magnetic energy stored in the iron are zero.

Example 16.18. An infinitely long straight conductor, located in the air, is parallel to the boundary plane that divides the entire space into two half-spaces with different magnetic permeabilities. Let a stationary electric current of intensity I flow through the conductor. Reflect the conductor across the boundary plane and then determine the expression for the force acting per unit length of the conductor. Let the following be given: $I = 1\,\text{A}$, $a = 1\,\text{m}$, $\mu_r = 500$.

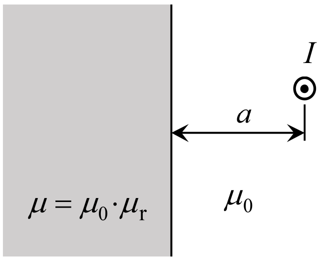


Figure 16.80. Infinitely long straight conductor parallel to the boundary plane between air and a ferromagnetic material

Solution:

According to the method of images, if the solution is sought in the air, the infinitely long thin-wire conductor in a heterogeneous medium is replaced by two infinitely long thin-wire conductors in an unbounded LIH medium with magnetic permeability equal to that of air (Figure 16.81).

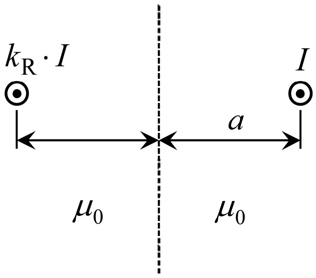


Figure 16.81. Real and image conductor

According to expression (16.64), the reflection factor is described by the expression:

$$k_R = \frac{\mu_0 \cdot \mu_r - \mu_0}{\mu_0 \cdot \mu_r + \mu_0} = \frac{\mu_r - 1}{\mu_r + 1} \tag{16.380}$$

The magnetic force per unit length between two straight, infinitely long, mutually parallel thin-wire conductors located in an unbounded LIH medium is described by expression (16.166), which in this case is given by:

$$F = \mu_0 \cdot \frac{|k_R| \cdot I \cdot I}{4 \cdot \pi \cdot a} = \mu_0 \cdot \frac{k_R \cdot I^2}{4 \cdot \pi \cdot a} = 99.6007984\ \frac{\text{nN}}{\text{m}} \tag{16.381}$$

An attractive force acts between the ferromagnetic material and the infinitely long conductor, as the conductor tends to move from air into the more permeable medium. This force is equal in magnitude to the force at the boundary between the two media, but in the opposite direction.

Example 16.19. Let three-quarters of the entire space be filled with a medium of relative magnetic permeability $\mu_r = 9$. Let the remaining quarter of the space be filled with air. An infinitely long, straight, thin-wire conductor is located in the air, at a distance a = 0.1 m from the boundary half-planes. A stationary electric current of intensity I flows through the thin-wire conductor. If the conductor is subjected to a force per unit length of F = 0.0049 N/m, calculate the electric current I.

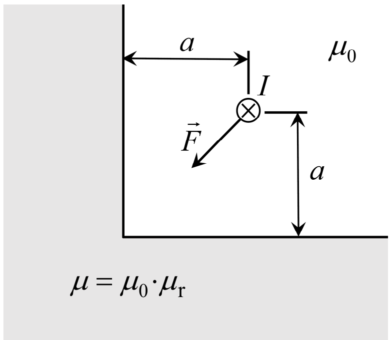


Figure 16.82. Infinitely long conductor in a heterogeneous medium

Solution:

According to the method of images, if the solution is sought in the air, the infinitely long thin-wire conductor in a heterogeneous medium is replaced by four infinitely long thin-wire conductors in an unbounded LIH medium with magnetic permeability equal to that of air (Figure 16.83).
According to expression (16.64), the reflection factor is described by the expression:

$$k_R = \frac{\mu_0 \cdot \mu_r - \mu_0}{\mu_0 \cdot \mu_r + \mu_0} = \frac{\mu_r - 1}{\mu_r + 1} = 0.8 \tag{16.382}$$

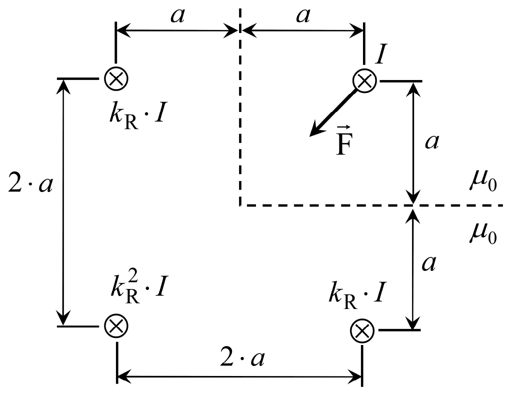


Figure 16.83. Real conductor and three image conductors in a homogeneous unbounded medium

Since the reflection coefficient is positive, all three image conductors attract the real conductor. The magnetic force per unit length between two straight, infinitely long, mutually parallel thin-wire conductors located in an unbounded LIH medium is described by expression (16.166). In this case, the magnetic force per unit length of the conductor is given by the expression:

$$F = \frac{\mu_0 \cdot I^2}{2 \cdot \pi \cdot (2 \cdot a)} \cdot \left(2 \cdot k_R \cdot \frac{\sqrt{2}}{2} + k_R^2 \cdot \frac{\sqrt{2}}{2} \right) = \frac{\sqrt{2} \cdot \mu_0 \cdot k_R \cdot (2 + k_R)}{8 \cdot \pi \cdot a} \cdot I^2 \tag{16.383}$$

from which it follows that:

$$I = \sqrt{\frac{8 \cdot \pi \cdot a \cdot F}{\sqrt{2} \cdot \mu_0 \cdot k_R \cdot (2 + k_R)}} = 55.62006983 \text{ A} \tag{16.384}$$

Example 16.20. An infinitely long straight copper solid conductor of radius $r_0 = 1\,\text{cm}$ carries a stationary electric current of intensity I = 50 A. Using the concept of stress, calculate the force per unit length by which the two longitudinal halves of the conductor attract each other. Solve the problem by integrating over the surface lying on the dividing plane between the two halves of the conductor.

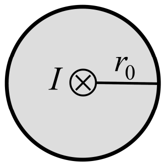


Figure 16.84. Infinitely long straight copper solid conductor

Solution:

The attractive magnetic force between the two longitudinal halves of the solid conductor, per unit length of the conductor, is described by the expression:

$$\vec{F} = \oint_S \vec{t}_m \cdot dS \tag{16.385}$$

where the closed integration surface S, enclosing one meter of one-half of the conductor, consists of a total of six parts (Figure 16.85).

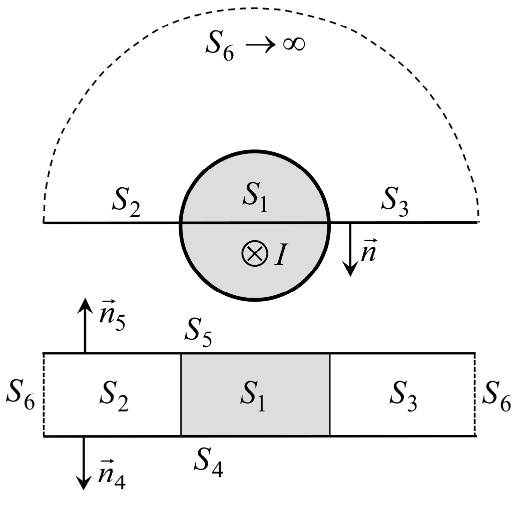


Figure 16.85. Parts of the closed integration surface S

The integration surfaces S_4 and S_5 are lateral surfaces (half-planes) that enclose a unit length of one-half of the conductor. The integrals of the magnetic stress vector over these surfaces cancel out, i.e., the forces on them are equal in magnitude and opposite in direction. The magnetic field intensity lies within the surfaces S_4 and S_5, so the force on them is opposite to the corresponding unit normal to the surface. On surface S_6, the magnetic field intensity vanishes, so the magnetic force on that surface is zero. Therefore, the integration of the magnetic stress vector over the closed surface S reduces to integration over surfaces S_1, S_2, and S_3, which lie in the same plane and therefore share a common unit normal $\vec{n}$.

On surfaces S_1, S_2, and S_3, the magnetic field intensity vector is described by the following expressions:

$$\vec{H}_1 = \pm \frac{I \cdot r}{2 \cdot \pi \cdot r_0^2} \cdot \vec{n} \quad ; \quad \vec{H}_2 = -\frac{I}{2 \cdot \pi \cdot r} \cdot \vec{n} \quad ; \quad \vec{H}_3 = \frac{I}{2 \cdot \pi \cdot r} \cdot \vec{n} \tag{16.386}$$

where r is the distance of the field point from the axis of the conductor.

In a magnetostatic field, the magnetic stress vector in an LIH medium is described by expression (16.175), which on surfaces S_1, S_2, and S_3 takes the form:

$$\vec{t}_{\mathrm{m1}} = \frac{\mu_0}{2} \cdot H_1^2 \cdot \vec{n} \quad ; \quad \vec{t}_{\mathrm{m2}} = \vec{t}_{\mathrm{m3}} = \frac{\mu_0}{2} \cdot H_2^2 \cdot \vec{n} \tag{16.387}$$

It follows that, in the cylindrical coordinate system, the magnetic force per unit length on the upper half of the conductor is:

$$\vec{F} = \vec{n} \cdot 2 \cdot \int_0^{r_0} t_{\mathrm{m1}} \cdot dr + \vec{n} \cdot 2 \cdot \int_{r_0}^{\infty} t_{\mathrm{m2}} \cdot dr \tag{16.388}$$

and, from expressions (16.386) - (16.388), it follows that:

$$\vec{F} = \vec{n} \cdot \frac{\mu_0 \cdot I^2}{4 \cdot \pi^2} \cdot \left(\frac{1}{r_0^4} \cdot \int_0^{r_0} r^2 \cdot dr + \int_{r_0}^{\infty} \frac{dr}{r^2} \right) = \frac{\mu_0 \cdot I^2}{3 \cdot \pi^2 \cdot r_0} \cdot \vec{n} \tag{16.389}$$

If the given data are substituted, it follows that:

$$\vec{F} = 1.061032954 \times 10^{-2} \cdot \vec{n} \ \frac{\mathrm{N}}{\mathrm{m}} \tag{16.390}$$

Example 16.21. An infinitely long straight copper solid conductor of radius $r_0 = 1\,\mathrm{cm}$ carries a stationary electric current of intensity $I = 50$ A. Using the concept of stress, calculate the force per unit length by which the two longitudinal halves of the conductor attract each other. Solve the problem by integrating over a surface that partially coincides with the mantle (lateral surface) of the conductor. The infinitely long, straight copper conductor is shown in Figure 16.84.

Solution:

The attractive magnetic force between the two longitudinal halves of the solid conductor, per unit length of the conductor, is described by the expression:

$$\vec{F} = \oint_S \vec{t}_{\mathrm{m}} \cdot dS \tag{16.391}$$

where the closed integration surface S, enclosing one meter of one-half of the conductor, consists of a total of four parts (Figure 16.86).

The integration surfaces S_3 and S_4 are lateral surfaces (semicircles) that enclose a unit length of one-half of the conductor. The integrals of the magnetic stress vector over these surfaces cancel out, i.e., the forces on them are equal in magnitude and opposite in direction. The magnetic field intensity lies within the surfaces S_3 and S_4, so the force on them is opposite to the corresponding unit normal to the surface. Therefore, the integration of the magnetic stress vector over the closed surface S reduces to integration over surfaces S_1 and S_2.

On surfaces S_1 and S_2, the magnetic field intensity vector is described by the following expressions:

$$\vec{H}_1 = -\frac{I \cdot r}{2 \cdot \pi \cdot r_0^2} \cdot \vec{e}_\varphi \quad ; \quad \vec{H}_2 = -\frac{I}{2 \cdot \pi \cdot r_0} \cdot \vec{e}_\varphi \tag{16.392}$$

where r is the distance of the field point from the axis of the conductor.

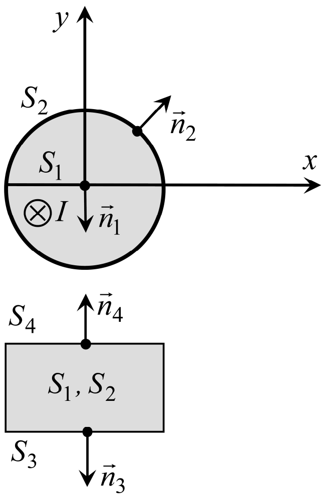


Figure 16.86. Parts of the closed integration surface S

In a magnetostatic field, the magnetic stress vector in an LIH medium is described by expression (16.175), which on surfaces S_1 and S_2 takes the form:

$$\vec{t}_{\mathrm{m1}} = \frac{\mu_0}{2} \cdot H_1^2 \cdot \vec{n}_1 = \frac{\mu_0}{2} \cdot H_1^2 \cdot \left(-\vec{\mathrm{j}}\right) \tag{16.393}$$

$$\vec{t}_{\mathrm{m2}} = -\frac{\mu_0}{2} \cdot H_2^2 \cdot \vec{n}_2 = -\frac{\mu_0}{2} \cdot H_2^2 \cdot \vec{\mathrm{e}}_{\mathrm{r}} = \frac{\mu_0}{2} \cdot H_2^2 \cdot \left(-\vec{\mathrm{e}}_{\mathrm{r}}\right) \tag{16.394}$$

It follows that, in the cylindrical coordinate system, the magnetic force per unit length on the upper half of the solid conductor is:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = \left(-\vec{\mathrm{j}}\right) \cdot 2 \cdot \int_0^{r_0} t_{\mathrm{m1}} \cdot dr - t_{\mathrm{m2}} \cdot r_0 \cdot \int_0^{\pi} \vec{e}_{\mathrm{r}} \cdot d\varphi \tag{16.395}$$

From expressions (16.391) - (16.393), it follows that:

$$\vec{F}_1 = \left(-\vec{\mathrm{j}}\right) \cdot \frac{\mu_0 \cdot I^2}{4 \cdot \pi^2 \cdot r_0^4} \cdot \int_0^{r_0} r^2 \cdot dr = \frac{\mu_0 \cdot I^2}{12 \cdot \pi^2 \cdot r_0} \cdot \left(-\vec{\mathrm{j}}\right) \tag{16.396}$$

Since the magnetic force $\vec{F}_2$ acts in the direction of the y-axis, it is sufficient to integrate only the y-component of the magnetic stress vector over surface S_2, so:

$$\vec{F}_2 = -t_{\mathrm{m2}} \cdot r_0 \cdot \int_0^{\pi} \vec{e}_{\mathrm{r}} \cdot d\varphi = -\vec{\mathrm{j}} \cdot t_{\mathrm{m2}} \cdot r_0 \cdot \int_0^{\pi} \left(\vec{e}_{\mathrm{r}} \cdot \vec{\mathrm{j}}\right) \cdot d\varphi \tag{16.397}$$

from which it follows that:

$$\vec{F}_2 = -\vec{\mathrm{j}} \cdot t_{\mathrm{m2}} \cdot r_0 \cdot 2 \cdot \int_0^{\pi/2} \sin\varphi \cdot d\varphi = 2 \cdot t_{\mathrm{m2}} \cdot r_0 \cdot \left(-\vec{\mathrm{j}}\right) = \mu_0 \cdot r_0 \cdot H_2^2 \cdot \left(-\vec{\mathrm{j}}\right) \tag{16.398}$$

From expressions (16.392) and (16.398), it follows that:

$$\vec{F}_2 = \frac{\mu_0 \cdot I^2}{4 \cdot \pi^2 \cdot r_0} \cdot \left(-\vec{\mathrm{j}}\right) \tag{16.399}$$

and the total magnetic force per unit length on the upper half of the solid conductor is:

$$\vec{F} = \vec{F}_1 + \vec{F}_2 = \frac{\mu_0 \cdot I^2}{3 \cdot \pi^2 \cdot r_0} \cdot \left(-\vec{\mathrm{j}}\right) = \frac{\mu_0 \cdot I^2}{3 \cdot \pi^2 \cdot r_0} \cdot \vec{n}_1 \tag{16.400}$$

If the given data are substituted, it follows that:

$$\vec{F} = 1.061032954 \times 10^{-2} \cdot \vec{n}_1 \ \ \frac{\mathrm{N}}{\mathrm{m}} \tag{16.401}$$

17. TIME-VARYING ELECTROMAGNETIC FIELD

The time-varying electromagnetic field is described by Maxwell's differential equations, from which Maxwell's integral equations can be easily derived. Maxwell's differential and integral equations in the time domain are thoroughly explained in Chapter 6 of this textbook, whereas Chapter 7 covers Maxwell's differential and integral equations in the phasor domain. For the sake of completeness, they are briefly restated in this chapter.

17.1. Maxwell's Differential Equations in a Slowly Moving Medium

In the time domain, Maxwell's differential equations in a slowly moving conducting medium (slowly moving system) are described by expressions (6.31) - (6.34), which are as follows:

$$\nabla\times\vec{H} = \vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \vec{J}_{\text{c}} + \frac{d\vec{D}}{dt} \tag{17.1}$$

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \tag{17.2}$$

$$\nabla\cdot\left(\vec{J}_{\text{c}} + \frac{d\vec{D}}{dt}\right) = -\nabla\cdot\vec{J}_{\text{s}} = g_{\text{st}} \tag{17.3}$$

$$\nabla\cdot\vec{B} = 0 \tag{17.4}$$

where:

$\vec{H}$ - the magnetic field intensity vector,

$\vec{E}$ - the electric field intensity vector,

$\vec{D}$ - the electric displacement vector,

$\vec{B}$ - the magnetic flux density vector,

$\vec{J}_{\text{tot}}$ - the vector of the surface density of the total electric current,

$\vec{J}_{\text{s}}$ - the vector of the surface density of the source (impressed) electric current,

$\vec{J}_{\text{c}}$ - the vector of the surface density of the conduction electric current,

g_{st} - the volume density of the source leakage electric current.

In the time domain, Maxwell's differential equations in a slowly moving perfect dielectric are described by expressions (6.1) - (6.4), which are as follows:

$$\nabla\times\vec{H} = \vec{J}_{\text{tot}} = \vec{J}_{\text{s}} + \frac{d\vec{D}}{dt} \tag{17.5}$$

$$\nabla\times\vec{E} = -\frac{d\vec{B}}{dt} \tag{17.6}$$

$$\nabla\cdot\vec{D} = \rho_{\text{s}} \tag{17.7}$$

$$\nabla\cdot\vec{B} = 0 \tag{17.8}$$

where ρ_{s} volume density of source (impressed) electric charge.

In the time domain, Maxwell's differential equations in a slowly moving perfect dielectric are a special case of Maxwell's differential equations in a slowly moving conducting medium. In a perfect dielectric, the following holds:

$$\vec{J}_{\mathrm{c}} = 0 \tag{17.9}$$

and according to expression (6.21), in a perfect dielectric, the following holds:

$$g_{\mathrm{st}} = \frac{d\rho_{\mathrm{s}}}{dt} \tag{17.10}$$

where, in this special case, g_{st} is the volume density of the displacement current.

If the total time derivatives are expressed using partial time derivatives, in the time domain, Maxwell's differential equations in a slowly moving conducting medium can be described by expressions (6.38) - (6.41), which are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{D}\right) \tag{17.11}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla \times \left(\vec{v} \times \vec{B}\right) \tag{17.12}$$

$$\nabla \cdot \left(\vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} \right) = g_{\mathrm{st}} \tag{17.13}$$

$$\nabla \cdot \vec{B} = 0 \tag{17.14}$$

where $\vec{v}$ is a vector of constant relative velocity with respect to the observer.

If the total time derivatives are expressed using partial time derivatives, in the time domain, Maxwell's differential equations in a slowly moving perfect dielectric can be described by expressions (6.9) - (6.12), which are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \rho_{\mathrm{s}} \cdot \vec{v} + \frac{\partial \vec{D}}{\partial t} - \nabla \times \left(\vec{v} \times \vec{D}\right) \tag{17.15}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} + \nabla \times \left(\vec{v} \times \vec{B}\right) \tag{17.16}$$

$$\nabla \cdot \vec{D} = \rho_{\mathrm{s}} \tag{17.17}$$

$$\nabla \cdot \vec{B} = 0 \tag{17.18}$$

In the phasor domain, from expressions (17.11) - (17.14), Maxwell's differential equations in a slowly moving conducting medium can be easily derived, and they are given as follows:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + \underline{\vec{J}}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}}\right) + j \cdot \omega \cdot \underline{\vec{D}} - \nabla \times \left(\vec{v} \times \underline{\vec{D}}\right) \tag{17.19}$$

$$\nabla \times \underline{\vec{E}} = -\, j \cdot \omega \cdot \underline{\vec{B}} + \nabla \times \left(\vec{v} \times \underline{\vec{B}}\right) \tag{17.20}$$

$$\nabla \cdot \left(\underline{\vec{J}}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}}\right) + j \cdot \omega \cdot \underline{\vec{D}}\right) = -\nabla \cdot \underline{\vec{J}}_{\mathrm{s}} = \underline{g}_{\mathrm{st}} \tag{17.21}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{17.22}$$

where:

$\underline{\vec{H}}$ - the phasor of the magnetic field intensity vector,

$\underline{\vec{E}}$ - the phasor of the electric field intensity vector,

$\underline{\vec{D}}$ - the phasor of the electric displacement vector,

$\underline{\vec{B}}$ - the phasor of the magnetic flux density vector,

$\underline{\vec{J}}_{\mathrm{s}}$ - the phasor of the vector of the surface density of source (impressed) electric current,

$\underline{\vec{J}}_{\mathrm{c}}$ - the phasor of the vector of the surface density of conduction electric current,

$\overline{g}_{\mathrm{st}}$ - the phasor representing the volume density of the source leakage electric current flowing into the surrounding conducting medium,

ω - the angular frequency,

$\vec{v}$ - the vector of the constant velocity relative to the observer,

j - the imaginary unit.

In the phasor domain, from expressions (17.15) - (17.18), Maxwell's differential equations in a slowly moving perfect dielectric can be easily derived, and they are given as follows:

$$\nabla\times\underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + \overline{\rho}_{\mathrm{s}}\cdot\vec{v} + j\cdot\omega\cdot\underline{\vec{D}} - \nabla\times\left(\vec{v}\times\underline{\vec{D}}\right) \tag{17.23}$$

$$\nabla\times\underline{\vec{E}} = -\ j\cdot\omega\cdot\underline{\vec{B}} + \nabla\times\left(\vec{v}\times\underline{\vec{B}}\right) \tag{17.24}$$

$$\nabla\cdot\underline{\vec{D}} = \overline{\rho}_{\mathrm{s}} \tag{17.25}$$

$$\nabla\cdot\underline{\vec{B}} = 0 \tag{17.26}$$

where $\overline{\rho}_{\mathrm{s}}$ is the phasor of the volume density of source (impressed) electric charge.

17.2. Maxwell's Differential Equations in a Stationary Medium

Maxwell's differential equations in a stationary medium can be easily derived from Maxwell's differential equations in a slowly moving medium by replacing the total time derivatives with partial time derivatives, or by taking into account the fact that in a stationary system $\vec{v} = 0$.

From expressions (17.1) - (17.4), it is easy to derive that in the time domain, Maxwell's differential equations in a stationary conducting medium are described by the following expressions:

$$\nabla\times\vec{H} = \vec{J}_{\mathrm{tot}} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \frac{\partial\vec{D}}{\partial t} \tag{17.27}$$

$$\nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \tag{17.28}$$

$$\nabla\cdot\left(\vec{J}_{\mathrm{c}} + \frac{\partial\vec{D}}{\partial t}\right) = g_{\mathrm{st}} \tag{17.29}$$

$$\nabla\cdot\vec{B} = 0 \tag{17.30}$$

From expressions (17.5) - (17.8), it is easy to derive that in the time domain, Maxwell's differential equations in a stationary perfect dielectric are described by the following expressions:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \frac{\partial \vec{D}}{\partial t} \tag{17.31}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{17.32}$$

$$\nabla \cdot \vec{D} = \rho_{\mathrm{s}} \tag{17.33}$$

$$\nabla \cdot \vec{B} = 0 \tag{17.34}$$

In the phasor domain, from expressions (17.27) - (17.30), or alternatively from expressions (17.19) - (17.22), Maxwell's differential equations in a stationary conducting medium can be easily derived, and they are given as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \frac{\partial \vec{D}}{\partial t} \tag{17.35}$$

$$\nabla \times \underline{\vec{E}} = -\ j \cdot \omega \cdot \underline{\vec{B}} \tag{17.36}$$

$$\nabla \cdot \left(\vec{J}_{\mathrm{c}} + \frac{\partial \vec{D}}{\partial t} \right) = g_{\mathrm{st}} \tag{17.37}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{17.38}$$

In the phasor domain, from expressions (17.31) - (17.34) or alternatively from expressions (17.23) - (17.26), Maxwell's differential equations in a stationary perfect LIH dielectric can be easily derived, and they are given as follows:

$$\nabla \times \underline{\vec{H}} = \underline{\vec{J}}_{\mathrm{s}} + j \cdot \omega \cdot \underline{\vec{D}} \tag{17.39}$$

$$\nabla \times \underline{\vec{E}} = -\ j \cdot \omega \cdot \underline{\vec{B}} \tag{17.40}$$

$$\nabla \cdot \underline{\vec{D}} = \overline{\rho}_{\mathrm{s}} \tag{17.41}$$

$$\nabla \cdot \underline{\vec{B}} = 0 \tag{17.42}$$

17.3. Maxwell's Integral Equations in a Slowly Moving Medium

In the time domain, Maxwell's integral equations in a slowly moving conducting medium (slowly moving system) are described by expressions (6.90) - (6.93), which are as follows:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_{\mathrm{s}} + \vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \vec{D}\right) \cdot d\vec{\ell} \tag{17.43}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B}\right) \cdot d\vec{\ell} = -\frac{d\Phi}{dt} \tag{17.44}$$

$$\oint_{\partial V} \left(\vec{J}_{\mathrm{c}} + \vec{v} \cdot \left(\nabla \cdot \vec{D}\right) + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = \int_V g_{\mathrm{st}} \cdot dV \tag{17.45}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{17.46}$$

where e is the electromotive force (EMF) induced in the closed contour C, and Φ is the magnetic flux passing through the surface S. The surface S and the closed curve C are described in Figure 1.4, whereas the volume V and the closed surface ∂V are described in Figure 1.3.

In the time domain, Maxwell's integral equations in a slowly moving perfect dielectric (slowly moving system) are described by expressions (6.82) - (6.85), which are as follows:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_\mathrm{s} + \rho_\mathrm{s} \cdot \vec{v} + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \vec{D} \right) \cdot d\vec{\ell} \tag{17.47}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \vec{B} \right) \cdot d\vec{\ell} = -\frac{d\Phi}{dt} \tag{17.48}$$

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho_\mathrm{s} \cdot dV = q_\mathrm{s} \tag{17.49}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{17.50}$$

where q_s is the electric charge of independent sources within the volume V (Figure 1.3).

In the phasor domain, from expressions (17.43) - (17.46), Maxwell's integral equations in a slowly moving conducting medium can be easily derived, and they are given as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_\mathrm{s} + \underline{\vec{J}}_\mathrm{c} + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}} \right) + j \cdot \omega \cdot \underline{\vec{D}} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \underline{\vec{D}} \right) \cdot d\vec{\ell} \tag{17.51}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \underline{\vec{B}} \right) \cdot d\vec{\ell} \tag{17.52}$$

$$\oint_{\partial V} \left(\underline{\vec{J}} + \vec{v} \cdot \left(\nabla \cdot \underline{\vec{D}} \right) + j \cdot \omega \cdot \underline{\vec{D}} \right) \cdot d\vec{S} = \int_V \underline{g}_\mathrm{st} \cdot dV \tag{17.53}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{17.54}$$

In the phasor domain, from expressions (17.47) - (17.50), Maxwell's integral equations in a moving perfect dielectric can be easily derived, and they are given as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \left(\underline{\vec{J}}_\mathrm{s} + \overline{\rho}_\mathrm{s} \cdot \vec{v} + j \cdot \omega \cdot \underline{\vec{D}} \right) \cdot d\vec{S} - \oint_C \left(\vec{v} \times \underline{\vec{D}} \right) \cdot d\vec{\ell} \tag{17.55}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} + \oint_C \left(\vec{v} \times \underline{\vec{B}} \right) \cdot d\vec{\ell} \tag{17.56}$$

$$\oint_{\partial V} \underline{\vec{D}} \cdot d\vec{S} = \int_V \overline{\rho}_\mathrm{s} \cdot dV = \overline{Q}_\mathrm{s} \tag{17.57}$$

$$\oint_{\partial V} \underline{\vec{B}} \cdot d\vec{S} = 0 \tag{17.58}$$

where $\overline{Q}_\mathrm{s}$ is the phasor of the source electric charge inside volume V (Figure 1.3).

17.4. Maxwell's Integral Equations in a Stationary Medium

Maxwell's integral equations in a stationary medium can be easily derived from Maxwell's integral equations in a slowly moving medium by replacing the total time derivatives with partial time derivatives, or by taking into account the fact that in a stationary system $\vec{v} = 0$.

From expressions (17.43) - (17.46), it is easy to derive that in the time domain, Maxwell's integral equations in a stationary conducting medium are described by the following expressions:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \left(\vec{J}_s + \vec{J}_c + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = i_s + i_c + i_{disp} \tag{17.59}$$

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{17.60}$$

$$\oint_{\partial V} \left(\vec{J}_c + \frac{\partial \vec{D}}{\partial t} \right) \cdot d\vec{S} = \int_V g_{st} \cdot dV = i_{st} \tag{17.61}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{17.62}$$

where:

i_s - the source electric current flowing through the surface S (Figure 1.4), which is described by expression (6.76),

i_c - the conduction electric current flowing through the surface S (Figure 1.4),

i_{disp} - the displacement electric current flowing through the surface S (Figure 1.4), which is described by expression (6.77),

i_{st} - the source leakage electric current flowing into the surrounding conducting medium within the considered volume V.

From expressions (17.47) - (17.50), it is easy to derive that in the time domain, Maxwell's integral equations in a stationary perfect dielectric are described by the following expression:

$$\oint_C \vec{H} \cdot d\vec{\ell} = \int_S \vec{J}_s \cdot d\vec{S} + \int_S \frac{\partial \vec{D}}{\partial t} \cdot d\vec{S} = i_s + i_{disp} \tag{17.63}$$

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{17.64}$$

$$\oint_{\partial V} \vec{D} \cdot d\vec{S} = \int_V \rho_s \cdot dV = q_s \tag{17.65}$$

$$\oint_{\partial V} \vec{B} \cdot d\vec{S} = 0 \tag{17.66}$$

In the phasor domain, from expressions (17.59) - (17.62) or alternatively from expressions (17.51) - (17.54), Maxwell's integral equations in a stationary conducting medium can be easily derived, and they are given as follows:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \int_S \underline{\vec{J}}_s \cdot d\vec{S} + \int_S \underline{\vec{J}}_c \cdot d\vec{S} + j \cdot \omega \cdot \int_S \underline{\vec{D}} \cdot d\vec{S} = \bar{I}_s + \bar{I}_c + \bar{I}_{disp} \tag{17.67}$$

$$\oint_C \underline{\vec{E}} \cdot d\vec{\ell} = -j \cdot \omega \cdot \int_S \underline{\vec{B}} \cdot d\vec{S} = -j \cdot \omega \cdot \overline{\Phi} \tag{17.68}$$

$$\oint_{\partial V}\left(\underline{\vec{J}}_{\mathrm{c}}+j\cdot\omega\cdot\underline{\vec{D}}\right)\cdot d\vec{S}=\int_{V}\overline{g}_{\mathrm{st}}\cdot dV=\overline{I}_{\mathrm{st}} \tag{17.69}$$

$$\oint_{\partial V}\underline{\vec{B}}\cdot d\vec{S}=0 \tag{17.70}$$

where:

$\overline{I}_{\mathrm{s}}$ - the phasor of the source electric current flowing through the surface S (Figure 1.4),

$\overline{I}_{\mathrm{c}}$ - the phasor of the conduction electric current flowing through the surface S (Figure 1.4),

$\overline{I}_{\mathrm{disp}}$ - the phasor of the displacement electric current flowing through the surface S (Figure 1.4),

$\overline{I}_{\mathrm{st}}$ - the phasor of the source leakage electric current flowing into the surrounding conducting medium within the considered volume V.

In the phasor domain, from expressions (17.63) - (17.66) or alternatively from expressions (17.55) - (17.58), Maxwell's integral equations in a stationary perfect dielectric can be easily derived, and they are given as follows:

$$\oint_{C}\underline{\vec{H}}\cdot d\vec{\ell}=\int_{S}\underline{\vec{J}}_{\mathrm{s}}\cdot d\vec{S}+j\cdot\omega\cdot\int_{S}\underline{\vec{D}}\cdot d\vec{S}=\overline{I}_{\mathrm{s}}+\overline{I}_{\mathrm{disp}} \tag{17.71}$$

$$\oint_{C}\underline{\vec{E}}\cdot d\vec{\ell}=-j\cdot\omega\cdot\int_{S}\underline{\vec{B}}\cdot d\vec{S}=-j\cdot\omega\cdot\overline{\Phi} \tag{17.72}$$

$$\oint_{\partial V}\underline{\vec{D}}\cdot d\vec{S}=\int_{V}\overline{\rho}_{\mathrm{s}}\cdot dV=\overline{Q}_{\mathrm{s}} \tag{17.73}$$

$$\oint_{\partial V}\underline{\vec{B}}\cdot d\vec{S}=0 \tag{17.74}$$

17.5. Components of the Electric Field Intensity and the Voltage Between Two Points

In a slowly moving coordinate system that is moving with a relative velocity $\vec{v}$ with respect to the magnetic field, the (total) electric field intensity can be described by the following expression:

$$\vec{E}=-\nabla\varphi-\frac{d\vec{A}}{dt}=-\nabla\varphi-\frac{\partial\vec{A}}{\partial t}+\vec{v}\times\vec{B} \tag{17.75}$$

whereas the magnetic flux density is described by the following expression:

$$\vec{B}=\nabla\times\vec{A} \tag{17.76}$$

where φ is the electric scalar potential, and $\vec{A}$ is the magnetic vector potential.

The total electric field intensity can be expressed as the sum of the static and the induced electric field intensities:

$$\vec{E}=\vec{E}_{\mathrm{stat}}+\vec{E}_{\mathrm{ind}} \tag{17.77}$$

Therefore, the time-varying electric field has two components:

- *Static electric field,* which is irrotational (conservative) and is generated by charges in a perfect dielectric and stationary electric currents in the conducting medium,
- *Induced electric field*, which is rotational and is caused by a time-varying magnetic field.

A static electric field is generated by charges that are either at rest or in uniform motion. An electrostatic field is produced by charges at rest, whereas a stationary current field is generated by charges in uniform motion.

Stationary electric currents also create a magnetostatic field. Time-varying electric currents (charges in non-uniform motion) generate a time-varying electric field and a time-varying magnetic field, that is, a time-varying electromagnetic field.

The static electric field intensity is described by the expression:

$$\vec{E}_{\text{stat}} = -\nabla\varphi \tag{17.78}$$

whereas in the moving coordinate system, the induced electric field intensity is described by the expression:

$$\vec{E}_{\text{ind}} = -\frac{d\vec{A}}{dt} = -\frac{\partial\vec{A}}{\partial t} + \vec{v}\times\vec{B} = \vec{E}_{\text{ind-tr}} + \vec{E}_{\text{ind-mo}} \tag{17.79}$$

where:

$$\vec{E}_{\text{ind-tr}} = -\frac{\partial\vec{A}}{\partial t} \tag{17.80}$$

is the induced electric field intensity due to transformation, that is, due to the time variation of the magnetic vector potential and thus also of the magnetic flux density, whereas:

$$\vec{E}_{\text{ind-mo}} = \vec{v}\times\vec{B} \tag{17.81}$$

is the induced electric field intensity is due to the relative motion of the magnetic field with respect to the coordinate system.

In the case of an electrostatic field and a stationary current field, the induced electric field intensity is equal to zero, so:

$$\vec{E} = \vec{E}_{\text{stat}} = -\nabla\varphi \tag{17.82}$$

from which it follows that the electrostatic field and the stationary current field are conservative fields, and the voltage between points A and B is described by the expression:

$$u_{\text{AB}} = \varphi_{\text{A}} - \varphi_{\text{B}} = \int_A^B \vec{E}\cdot d\vec{\ell} = \int_A^B \vec{E}_{\text{stat}}\cdot d\vec{\ell} \tag{17.83}$$

which means that the voltage between two points is equal to the difference between the electric scalar potentials of these points and therefore does not depend on the integration path (the integration curve).

In a time-varying electromagnetic field, the voltage between two points is equal to the line integral of the total electric field intensity:

$$u_{\text{AB}} = u_{\text{AB}}^{C_{\text{i}}} = \int_{C_{\text{i}}} \vec{E}\cdot d\vec{\ell} = \int_{C_{\text{i}}} \vec{E}_{\text{stat}}\cdot d\vec{\ell} + \int_{C_{\text{i}}} \vec{E}_{\text{ind}}\cdot d\vec{\ell} = \varphi_{\text{A}} - \varphi_{\text{B}} + e_{\text{AB}} \tag{17.84}$$

where C_{i} is the arbitrary i-th integration curve under consideration (Figure 17.1).

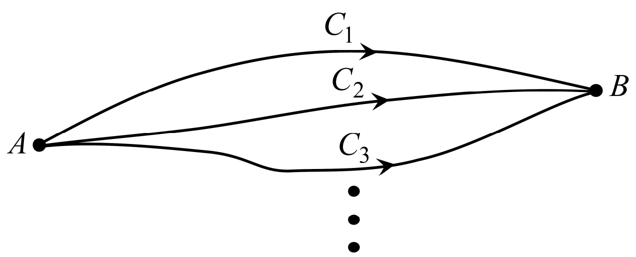


Figure 17.1. Arbitrarily chosen integration curves between points A and B

In a time-varying electromagnetic field, the voltage and the induced EMF between two points depend on the integration path, whereas the difference of the scalar electric potentials is independent of the integration path:

$$\int_{C_i} \vec{E}_{\text{stat}} \cdot d\vec{\ell} = \int_A^B \vec{E}_{\text{stat}} \cdot d\vec{\ell} = \varphi_A - \varphi_B \quad ; \quad \forall C_i \tag{17.85}$$

Therefore, the induced EMF also depends on the integration path:

$$e_{AB} = e_{AB}^{C_i} = \int_{C_i} \vec{E}_{\text{ind}} \cdot d\vec{\ell} = \int_{C_i} \vec{E}_{\text{ind-tr}} \cdot d\vec{\ell} + \int_{C_i} \vec{E}_{\text{ind-mo}} \cdot d\vec{\ell} = e_{AB-tr} + e_{AB-mo} \tag{17.86}$$

The induced EMF due to transformation is described by the expression:

$$e_{AB-tr} = \int_{C_i} \vec{E}_{\text{ind-tr}} \cdot d\vec{\ell} = -\int_{C_i} \frac{\partial \vec{A}}{\partial t} \cdot d\vec{\ell} \tag{17.87}$$

whereas the induced EMF due to motion is described by the expression:

$$e_{AB-mo} = \int_{C_i} \vec{E}_{\text{ind-mo}} \cdot d\vec{\ell} = \int_{C_i} \left(\vec{v} \times \vec{B}\right) \cdot d\vec{\ell} \tag{17.88}$$

Therefore, it holds that:

$$u_{AB}^{C_i} \neq e_{AB}^{C_i} \neq \varphi_A - \varphi_B \tag{17.89}$$

For a closed contour located in a time-varying electromagnetic field (Figure 17.2), the following holds:

$$u = u_{AA} = \oint_{C_i} \vec{E} \cdot d\vec{\ell} = \oint_{C_i} \vec{E}_{\text{stat}} \cdot d\vec{\ell} + \oint_{C_i} \vec{E}_{\text{ind}} \cdot d\vec{\ell} = \underbrace{\varphi_A - \varphi_A}_{=0} + e = \oint_{C_i} \vec{E}_{\text{ind}} \cdot d\vec{\ell} \tag{17.90}$$

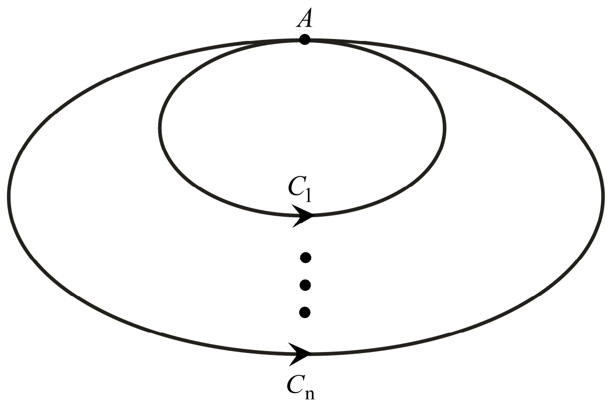


Figure 17.2. Arbitrarily chosen closed integration paths

According to expression (17.48), Faraday's law of electromagnetic induction holds for the closed contour C:

$$u = e = \oint_C \vec{E} \cdot d\vec{\ell} = \oint_C \vec{E}_{\text{ind}} \cdot d\vec{\ell} = -\frac{d\Psi}{dt} = -\frac{d\Phi}{dt} \quad ; \quad \Psi = \Phi \tag{17.91}$$

where Φ is the magnetic flux passing through the closed contour C, taking into account the right-hand rule, whereas Ψ is the magnetic flux linkage.

Furthermore, according to expression (17.48), for a closed contour in a time-varying electromagnetic field, the following expression holds:

$$e=\oint_C \vec{E}\cdot d\vec{\ell}=\oint_C \vec{E}_{\text{ind}}\cdot d\vec{\ell}=\underbrace{-\iint_S \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S}}_{e_{\text{tr}}}+\underbrace{\oint_C \left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell}}_{e_{\text{mo}}} \tag{17.92}$$

from which, in accordance with expression (17.79), it follows that:

$$e=\oint_C \vec{E}\cdot d\vec{\ell}=\oint_C \vec{E}_{\text{ind}}\cdot d\vec{\ell}=-\oint_C \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell}+\oint_C \left(\vec{v}\times\vec{B}\right)\cdot d\vec{\ell} \tag{17.93}$$

In the general case, both the EMF due to transformation and the EMF due to motion and/or deformation are induced within a closed contour located in a time-varying electromagnetic field. In the case of a closed contour, one can also speak of the induced voltage, since the voltage along the closed contour is equal to the induced EMF. This is in accordance with Kirchhoff's second law.

If Faraday's law of electromagnetic induction is neglected in time-varying electric circuits, then the voltages are equal to the potential differences. A reference node is chosen at which the potential is set to zero, and the node potentials (the voltages of the nodes relative to the reference node) do not depend on the path leading from one node to the other.

The voltage measured by a voltmeter between two points of an alternating current (AC) electrical network also depends on how the voltmeter is connected, that is; the voltage depends on the EMF induced in the loop formed by the voltmeter's leads and the electrical network.

17.6. Lorenz-Gauge Potentials in a Stationary LIH Medium

The magnetic vector potential $\vec{A}$ and the electric scalar potential φ in a stationary LIH medium are defined by the expressions:

$$\vec{B}=\nabla\times\vec{A} \tag{17.94}$$

$$\vec{E}=-\nabla\varphi-\frac{\partial \vec{A}}{\partial t} \tag{17.95}$$

The Lorenz gauge condition is most commonly used as a relationship between electromagnetic potentials and can be written as follows in a conducting LIH medium:

$$\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}+\mu\cdot\kappa\cdot\varphi=0 \tag{17.96}$$

which, in the case of a perfect LIH dielectric, takes on a new form:

$$\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}=0 \tag{17.97}$$

If the electromagnetic potentials are related by the Lorenz gauge condition, they are called Lorenz-gauge potentials.

The first and third Maxwell's differential equations in a stationary perfect LIH dielectric are described by the expressions (9.10) and (9.11), which read:

$$\nabla\times\vec{B}=\mu\cdot\vec{J}_{\text{s}}+\mu\cdot\varepsilon\cdot\frac{\partial \vec{E}}{\partial t} \tag{17.98}$$

$$\nabla\cdot\vec{E}=\frac{\rho_{\text{s}}}{\varepsilon} \tag{17.99}$$

and these expressions can also be easily obtained from expressions (17.31) and (17.33) by substituting the constitutive equations (3.11) into them.

When the expressions (17.94) and (17.95) are substituted into Maxwell's differential equations (17.98) and (17.99), the expressions (9.15) and (9.16) are obtained, which read:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\nabla\left(\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}\right)=-\mu\cdot\vec{J}_\mathrm{s} \tag{17.100}$$

$$\Delta\varphi+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)=-\frac{\rho_\mathrm{s}}{\varepsilon} \tag{17.101}$$

Based on expression (17.100), it is easy to understand why the Lorenz gauge condition (17.97) is most commonly used as a relation between electromagnetic potentials in a stationary perfect LIH dielectric. In addition to the Lorenz gauge condition, the Coulomb gauge condition is also sometimes used:

$$\nabla\cdot\vec{A}=0 \tag{17.102}$$

If the Lorenz gauge condition (17.97) is substituted into the differential equations (17.100) and (17.101), the following system of inhomogeneous undamped wave equations for the Lorenz-gauge potentials in a stationary perfect LIH dielectric is obtained:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}=-\mu\cdot\vec{J}_\mathrm{s} \tag{17.103}$$

$$\Delta\varphi-\mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2}=-\frac{\rho_\mathrm{s}}{\varepsilon} \tag{17.104}$$

where $\vec{J}_\mathrm{s}$ and ρ_s are the densities of independent sources of the electromagnetic field, which satisfy the continuity equation (6.30).

According to the expressions (6.50) and (6.52), the first and third Maxwell's differential equations in a stationary conducting LIH medium are described by the expressions:

$$\nabla\times\vec{B}=\mu\cdot\vec{J}_\mathrm{s}+\mu\cdot\kappa\cdot\vec{E}+\mu\cdot\varepsilon\cdot\frac{\partial\vec{E}}{\partial t} \tag{17.105}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\nabla\cdot\vec{E}\right)=g_\mathrm{st} \tag{17.106}$$

and these expressions can also be easily obtained from expressions (17.27) and (17.29) by substituting the constitutive equations (3.11) into them.

When the expressions (17.94) and (17.95) are substituted into Maxwell's differential equations (17.105) and (17.106), the expressions (6.27) and (6.28) are obtained, which read:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}-\nabla\left(\nabla\cdot\vec{A}+\mu\cdot\varepsilon\cdot\frac{\partial\varphi}{\partial t}+\mu\cdot\kappa\cdot\varphi\right)=-\mu\cdot\vec{J}_\mathrm{s} \tag{17.107}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\Delta\varphi+\frac{\partial}{\partial t}\left(\nabla\cdot\vec{A}\right)\right)=-g_\mathrm{st} \tag{17.108}$$

If the Lorenz gauge condition (17.96) is substituted into the differential equations (17.107) and (17.108), the following system of inhomogeneous damped differential equations for the potentials in a stationary conducting LIH medium is obtained:

$$\Delta\vec{A}-\mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t}=-\mu\cdot\vec{J}_\mathrm{s} \tag{17.109}$$

$$\left(\kappa+\varepsilon\cdot\frac{\partial}{\partial t}\right)\left(\Delta\varphi-\mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2}-\mu\cdot\kappa\cdot\frac{\partial\varphi}{\partial t}\right)=-g_\mathrm{st} \tag{17.110}$$

which, in a source-free stationary conducting LIH medium, become homogeneous damped wave equations for the Lorenz-gauge potentials:

$$\Delta\vec{A} - \mu\cdot\varepsilon\cdot\frac{\partial^2\vec{A}}{\partial t^2} - \mu\cdot\kappa\cdot\frac{\partial\vec{A}}{\partial t} = 0 \tag{17.111}$$

$$\Delta\varphi - \mu\cdot\varepsilon\cdot\frac{\partial^2\varphi}{\partial t^2} - \mu\cdot\kappa\cdot\frac{\partial\varphi}{\partial t} = 0 \tag{17.112}$$

In the phasor domain, the inhomogeneous undamped wave equations of the Lorenz-gauge potentials in a stationary perfect LIH dielectric (17.103) and (17.104) become the inhomogeneous Helmholtz differential equations (9.20) and (9.21), which read:

$$\Delta\underline{\vec{A}} - \bar{\gamma}^2\cdot\underline{\vec{A}} = -\mu\cdot\underline{\vec{J}}_{\mathrm{s}} \tag{17.113}$$

$$\Delta\bar{\varphi} - \bar{\gamma}^2\cdot\bar{\varphi} = -\frac{\bar{\rho}_{\mathrm{s}}}{\varepsilon} \tag{17.114}$$

where:

$$\bar{\gamma}^2 = -k^2 = -\omega^2\cdot\mu\cdot\varepsilon \quad ; \quad k = \omega\cdot\sqrt{\mu\cdot\varepsilon} \quad ; \quad \bar{\gamma} = j\cdot k \tag{17.115}$$

The real constant k is the *wave number*, and the complex constant $\bar{\gamma}$ is the *propagation constant* (also known as the wave constant).

In the phasor domain, the inhomogeneous damped differential equations of the Lorenz-gauge potentials in a stationary conducting LIH medium (17.109) and (17.110) become the inhomogeneous Helmholtz differential equations (9.36) and (9.37), which read:

$$\Delta\underline{\vec{A}} - \bar{\gamma}^2\cdot\underline{\vec{A}} = -\mu\cdot\underline{\vec{J}}_{s} \tag{17.116}$$

$$\Delta\bar{\varphi} - \bar{\gamma}^2\cdot\bar{\varphi} = -\frac{\bar{g}_{\mathrm{st}}}{\kappa + j\cdot\omega\cdot\varepsilon} \tag{17.117}$$

where:

$$\bar{\gamma}^2 = -\bar{k}^2 = j\cdot\omega\cdot\mu\cdot(\kappa + j\cdot\omega\cdot\varepsilon) \quad ; \quad \bar{\gamma} = j\cdot\bar{k} \tag{17.118}$$

From expressions (17.115) and (17.116), it follows that in the case of a perfect dielectric, the wave number is a real constant, whereas in the case of a conducting medium, the wave number is a complex constant.

The propagation constant of the medium can be expressed as follows:

$$\bar{\gamma} = \alpha + j\cdot\beta \tag{17.119}$$

where α is the attenuation constant, and β is the phase constant.

It can be easily shown that the constants of a conducting LIH medium are:

$$\alpha = \frac{\kappa}{N}\cdot\sqrt{\frac{\omega\cdot\mu}{2}} \quad ; \quad \beta = N\cdot\sqrt{\frac{\omega\cdot\mu}{2}} \tag{17.120}$$

where:

$$N = \sqrt{\omega\cdot\varepsilon + \sqrt{(\omega\cdot\varepsilon)^2 + \kappa^2}} \tag{17.121}$$

whereas the constants of a perfect LIH dielectric $(\kappa = 0)$ are described by the expressions:

$$\alpha = 0 \quad ; \quad \beta = k = \omega\cdot\sqrt{\mu\cdot\varepsilon} \tag{17.122}$$

Therefore, the wave number and the phase constant of a perfect LIH dielectric are identical.

It is important to emphasize once again that the equations that apply to a perfect LIH dielectric are a special case of the equations that apply to a conducting LIH medium. Accordingly, the Helmholtz differential equation (17.114), which applies to a perfect LIH dielectric, is a special case of the Helmholtz differential equation (17.117), which applies to a conducting LIH medium, with the following substitutions:

$$\kappa = 0 \quad ; \quad \overline{g}_{\mathrm{st}} = j \cdot \omega \cdot \overline{\rho}_{\mathrm{s}} \tag{17.123}$$

which follows from expression (6.30).

17.7. Darwin Approximation of the Electromagnetic Field in a Perfect Dielectric

The hybrid or Darwin approximation of the electromagnetic field in a stationary perfect LIH dielectric is obtained from the full system of Maxwell's differential equations, with partial neglect of the displacement electric currents:

$$\varepsilon \cdot \frac{\partial \vec{E}_{\mathrm{ind}}}{\partial t} = 0 \quad \Rightarrow \quad \varepsilon \cdot \frac{\partial^2 \vec{A}}{\partial t^2} = 0 \tag{17.124}$$

and, with the introduced neglect, Maxwell's differential equations in a perfect LIH dielectric are:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \varepsilon \cdot \frac{\partial \vec{E}_{\mathrm{stat}}}{\partial t} \tag{17.125}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{17.126}$$

$$\nabla \cdot \vec{E} = \frac{\rho_s}{\varepsilon} \tag{17.127}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.128}$$

The undamped wave equations of the Lorenz-gauge potentials in a stationary perfect LIH dielectric (17.103) and (17.104), with the introduced neglect, are as follows:

$$\Delta \vec{A} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.129}$$

$$\Delta \varphi - \mu \cdot \varepsilon \cdot \frac{\partial^2 \varphi}{\partial t^2} = -\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.130}$$

which means that the magnetic vector potential is described by Poisson's differential equation, whereas the electric scalar potential is described by the undamped wave equation.

Since both Lorenz-gauge potentials are not described by Poisson's differential equation, the Darwin approximation of the electromagnetic field in a stationary perfect LIH dielectric is not quasistatic. It makes no sense to use it in the case of a stationary conducting LIH medium, as neither potential is described by Poisson's differential equation.

In the phasor domain, the inhomogeneous differential equations of the Lorenz-gauge potentials in a stationary perfect LIH dielectric (17.129) and (17.130) take the following form:

$$\Delta \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.131}$$

$$\Delta \overline{\varphi} - \overline{\gamma}^2 \cdot \overline{\varphi} = -\frac{\overline{\rho}_{\mathrm{s}}}{\varepsilon} \tag{17.132}$$

where the propagation constant $\overline{\gamma}$ is described by the expressions (17.119) and (17.122).

17.8. Magnetodynamic Electromagnetic Field in a Conducting Medium

The magnetodynamic electromagnetic field can be obtained from the full system of Maxwell's differential equations in a stationary conducting medium by completely neglecting the displacement electric currents:

$$\frac{\partial \vec{D}}{\partial t} = 0 \tag{17.133}$$

which is the usual approximation of the electromagnetic field in good conductors, since it is:

$$J >> \frac{\partial D}{\partial t} \tag{17.134}$$

that is, the surface density of the displacement electric current is negligible compared to the surface density of the conduction electric current.

For a sinusoidal electromagnetic field with the angular frequency ω, the condition (17.134) is as follows:

$$\kappa >> \omega \cdot \varepsilon \tag{17.135}$$

Neglecting the electric displacement currents, Maxwell's differential equations in a stationary conducting LIH medium are described by the expressions (6.68) - (6.71), which are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} \tag{17.136}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{17.137}$$

$$\nabla \cdot \vec{E} = \frac{g_{\mathrm{st}}}{\kappa} \tag{17.138}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.139}$$

and these expressions can be easily derived from the expressions (17.27) - (17.30) by using the condition (17.133) and the constitutive equations (3.11).

With the introduced neglect, i.e., for $\varepsilon = 0,$ the differential equations of the Lorenz-gauge potentials (17.109) and (17.110) take the following new form:

$$\Delta \vec{A} - \mu \cdot \kappa \cdot \frac{\partial \vec{A}}{\partial t} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.140}$$

$$\Delta \varphi - \mu \cdot \kappa \cdot \frac{\partial \varphi}{\partial t} = -\frac{g_{\mathrm{st}}}{\kappa} \tag{17.141}$$

whereas, according to expression (17.96), the Lorenz gauge condition, with the neglect of displacement electric currents, is as follows:

$$\nabla \cdot \vec{A} + \mu \cdot \kappa \cdot \varphi = 0 \tag{17.142}$$

The differential equations (17.140) and (17.141) are inhomogeneous diffusion equations. The electromagnetic field in an unbounded space described by these equations is not quasistatic. In good conductors, if the condition (17.133) is satisfied, the solutions of the diffusion equations (17.140) and (17.141) numerically approximate the solutions of the differential equations for Lorenz-gauge potentials (17.109) and (17.110) very well. Therefore, such an electromagnetic field can be appropriately called a *magnetodynamic electromagnetic field* [14]. This approximation of the electromagnetic field in the time domain is common and justified in the case of good LIH conductors, since the differential equations for Lorenz-gauge potentials (17.109) and (17.110) are considerably simplified.

In the phasor domain, the inhomogeneous diffusion differential equations of the Lorenz-gauge potentials in a stationary conducting LIH medium (17.140) and (17.141) become the inhomogeneous Helmholtz differential equations (9.39) and (9.40), which read:

$$\Delta\underline{\vec{A}} - \bar{\gamma}^2 \cdot \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_s \tag{17.143}$$

$$\Delta\bar{\varphi} - \bar{\gamma}^2 \cdot \bar{\varphi} = -\frac{\bar{g}_{\mathrm{st}}}{\kappa} \tag{17.144}$$

where:

$$\bar{\gamma}^2 = -\bar{k}^2 = j \cdot \omega \cdot \mu \cdot \kappa \quad ; \quad \bar{\gamma} = j \cdot \bar{k} \tag{17.145}$$

from which it follows that:

$$\bar{\gamma} = \alpha + j \cdot \beta = (1 + j) \cdot \alpha \quad ; \quad \alpha = \beta = \sqrt{\frac{\omega \cdot \mu \cdot \kappa}{2}} \tag{17.146}$$

which can also be obtained from the expressions (17.119) - (17.121) by substituting $\varepsilon = 0$.

If the electromagnetic field in good LIH conductors is sinusoidal, then both the dynamic sinusoidal electromagnetic field and the magnetodynamic sinusoidal electromagnetic field are described by the inhomogeneous Helmholtz differential equations of the Lorenz-gauge potentials. Therefore, in the case of a sinusoidal electromagnetic field in good LIH conductors, the neglect of displacement electric currents is not very advantageous, even if the neglect does not introduce a significant numerical error. In the time domain, however, this neglect makes a lot of sense.

17.9. Quasistatic Electromagnetic Field

Quasistatic or quasistationary electromagnetic problems are dynamic problems that can be solved like static problems without significant numerical error. In such cases, the potentials in an unbounded LIH medium are described by Poisson's differential equations:

$$\Delta \vec{A} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.147}$$

$$\Delta \varphi = -\frac{\rho_{\mathrm{s}}}{\varepsilon} \quad \text{or} \quad \Delta \varphi = -\frac{g_{\mathrm{st}}}{\kappa} \tag{17.148}$$

These Poisson's differential equations have the same formal form as in the case of static fields. However, the potentials, the surface density of source electric current, the volume density of the source electric charge and the volume density of the source leakage electric current depend on time.

The particular solutions of Poisson's differential equations for the quasistatic electromagnetic field are:

$$\vec{A} = \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\vec{J}_{\mathrm{s}} \cdot dV}{r} \tag{17.149}$$

$$\varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \int_V \frac{\rho_{\mathrm{s}} \cdot dV}{r} \quad \text{or} \quad \varphi = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \int_V \frac{g_{\mathrm{st}} \cdot dV}{r} \tag{17.150}$$

It holds that:

$$g_{\mathrm{st}} = -\nabla^{\mathrm{s}} \cdot \vec{J}_{\mathrm{s}} \tag{17.151}$$

where the expression:

$$\nabla^{\mathrm{s}} = \frac{\partial}{\partial x_{\mathrm{s}}} \cdot \vec{\mathrm{i}} + \frac{\partial}{\partial y_{\mathrm{s}}} \cdot \vec{\mathrm{j}} + \frac{\partial}{\partial z_{\mathrm{s}}} \cdot \vec{\mathrm{k}} \tag{17.152}$$

describes the Hamiltonian differential vector operator in relation to the coordinates of the source point, where the coordinates of the source point in the Cartesian coordinate system are (x_s, y_s, z_s).

It is logical that the particular solutions of Poisson's differential equations formally have the same form as in the case of static fields. This means that, with the introduced neglects, the field changes occur simultaneously at all points in space.

For a sinusoidal electromagnetic field with angular frequency ω, in a bounded LIH region, the electromagnetic field can be considered as a quasistatic (slowly varying) electromagnetic field if the following condition is satisfied:

$$\beta \cdot R_{\max} << 1 \tag{17.153}$$

where β is the phase constant of the medium, and $R_{\max}$ is the largest distance between two points in the computation region.

For a sinusoidal electromagnetic field with angular frequency ω in a perfect LIH dielectric, the quasistatic condition is as follows:

$$R_{\max} << \frac{1}{\beta} = \frac{1}{\omega \cdot \sqrt{\mu \cdot \varepsilon}} \tag{17.154}$$

where β is the phase constant of the perfect LIH dielectric, which is described by expression (17.122).

For a sinusoidal electromagnetic field with angular frequency ω in a conducting LIH medium, the quasistatic condition is as follows:

$$R_{\max} << \frac{1}{\beta} = \sqrt{\frac{2}{\omega \cdot \mu}} \cdot \frac{1}{\sqrt{\omega \cdot \varepsilon + \sqrt{(\omega \cdot \varepsilon)^2 + \kappa^2}}} \tag{17.155}$$

where β is the phase constant of the conducting LIH medium, which is described by expressions (17.120) and (17.121).

In good conductors, displacement electric currents can be neglected, i.e., it can be assumed that $\varepsilon = 0$, so for a sinusoidal electromagnetic field with angular frequency ω in a good LIH conductor, the quasistatic condition is as follows:

$$R_{\max} << \frac{1}{\beta} = \sqrt{\frac{2}{\omega \cdot \mu \cdot \kappa}} \tag{17.156}$$

where β is the phase constant of the good LIH conductor, which is described by expression (17.146).

For a sinusoidal electromagnetic field with a frequency of 50 Hz, in a bounded region, the electromagnetic field can be considered as a quasistatic electromagnetic field if the largest distance between two points in the computation region $R_{\max}$ is:

- For air: $R_{\max}$ << 954.3 km (usually, a limit of 100 km is taken),
- For seawater ($\mu_r = 1, \varepsilon_r = 80, \kappa = 5$ S/m): $R_{\max}$ << 31.83 m,
- For copper ($\mu_r = 1, \varepsilon_r = 1, \kappa = 59.6$ MS/m): $R_{\max}$ << 9.22 mm,
- For steel ($\mu_r = 100, \varepsilon_r = 1, \kappa = 6.99$ MS/m): $R_{\max}$ << 2.69 mm.

17.10. Magnetoquasistatic Electromagnetic Field in a Perfect Dielectric

The magnetoquasistatic electromagnetic field can be obtained from the full system of Maxwell's differential equations in a stationary perfect dielectric by completely neglecting the displacement electric currents:

$$\frac{\partial \vec{D}}{\partial t} = 0 \tag{17.157}$$

Neglecting the displacement electric currents, Maxwell's differential equations in a stationary perfect LIH dielectric are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} \tag{17.158}$$

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \tag{17.159}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.160}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.161}$$

and these expressions can easily be obtained from expressions (17.31) - (17.34) by substituting condition (17.157) and the constitutive equations (3.11).

The introduced neglect of displacement electric currents, which corresponds to setting $\varepsilon = 0$, requires that the potential differential equation in a perfect LIH dielectric (17.100) be replaced. As a result, the potential differential equations (17.100) and (17.101) take the following form:

$$\Delta \vec{A} - \nabla \left(\nabla \cdot \vec{A}\right) = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.162}$$

$$\Delta \varphi + \frac{\partial}{\partial t}\left(\nabla \cdot \vec{A}\right) = -\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.163}$$

which means that the Lorenz gauge condition (17.97) reduces to the Coulomb gauge condition (17.102). After substituting the Coulomb gauge condition into the potential differential equations (17.162) and (17.163), the following two Poisson's differential equations are obtained:

$$\Delta \vec{A} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.164}$$

$$\Delta \varphi = -\frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.165}$$

which, in phasor notation, are as follows:

$$\Delta \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.166}$$

$$\Delta \overline{\varphi} = -\frac{\overline{\rho}_{\mathrm{s}}}{\varepsilon} \tag{17.167}$$

Therefore, this electromagnetic field is quasistatic, and in this case, it is called a *magnetoquasistatic* electromagnetic field.

In a perfect LIH dielectric, the magnetic field is formally described by the same pair of Maxwell's differential equations as the magnetostatic field:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} \quad ; \quad \nabla \cdot \vec{H} = 0 \tag{17.168}$$

whereas the electric field, which depends on the magnetic field, is described by additional differential equations:

$$\nabla \times \vec{E} = -\mu \cdot \frac{\partial \vec{H}}{\partial t} \quad ; \quad \nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.169}$$

17.11. Electroquasistatic Electromagnetic Field in a Stationary Medium

The electroquasistatic electromagnetic field in a stationary medium can be obtained from the full system of Maxwell's differential equations by neglecting the induced component of the electric field intensity:

$$\frac{\partial \vec{A}}{\partial t} = 0 \quad \Rightarrow \quad \frac{\partial \vec{B}}{\partial t} = 0 \tag{17.170}$$

which means that the electromagnetic potentials are described by the expressions:

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla \varphi \tag{17.171}$$

This neglect results in the fact that Faraday's law of electromagnetic induction does not hold.

With the introduced neglect, Maxwell's differential equations in a stationary perfect LIH dielectric are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \tag{17.172}$$

$$\nabla \times \vec{E} = 0 \tag{17.173}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.174}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.175}$$

and these expressions can easily be obtained from expressions (17.31) - (17.34) by substituting condition (17.170) and the constitutive equations (3.11).

If expressions (17.171) are substituted into Maxwell's differential equations (17.172) and (17.174), taking into account expression (9.14), the following differential equations for the potentials are obtained:

$$\Delta \vec{A} - \nabla \left(\nabla \cdot \vec{A} + \mu \cdot \varepsilon \cdot \frac{\partial \varphi}{\partial t} \right) = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.176}$$

$$\Delta \varphi = -\frac{\rho_s}{\varepsilon} \tag{17.177}$$

and after substituting the Lorenz gauge condition (17.97) into expression (17.176), Poisson's differential equations (17.164) and (17.165) are obtained, which are described in the phasor domain by expressions (17.166) and (17.167).

Therefore, with the introduced neglect (17.170), the electromagnetic field in a perfect LIH dielectric is quasistatic, and in this case, it is called an *electroquasistatic* electromagnetic field.

In a perfect LIH dielectric, for the electroquasistatic electromagnetic field, the electric field is formally described by the same pair of Maxwell's differential equations as the electrostatic field:

$$\nabla \times \vec{E} = 0 \quad ; \quad \nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.178}$$

whereas the magnetic field, which depends on the electric field, is described by the differential equations:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \quad ; \quad \nabla \cdot \vec{H} = 0 \tag{17.179}$$

With the introduced neglect (17.170), Maxwell's differential equations in a stationary conducting LIH medium are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \tag{17.180}$$

$$\nabla \times \vec{E} = 0 \tag{17.181}$$

$$\left(\kappa + \varepsilon \cdot \frac{\partial}{\partial t} \right) \left(\nabla \cdot \vec{E} \right) = g_{\mathrm{st}} \tag{17.182}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.183}$$

and these expressions can easily be obtained from expressions (17.27) - (17.30) by substituting condition (17.168) and the constitutive equations (3.11).

If expressions (17.171) are substituted into Maxwell's differential equations (17.180) and (17.182), taking into account expression (9.14), the following differential equations for the potentials are obtained:

$$\Delta \vec{A} - \nabla \left(\nabla \cdot \vec{A} + \mu \cdot \varepsilon \cdot \frac{\partial \varphi}{\partial t} + \mu \cdot \kappa \cdot \varphi \right) = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.184}$$

$$\left(\kappa + \varepsilon \cdot \frac{\partial}{\partial t} \right) \Delta \varphi = -g_{\mathrm{st}} \tag{17.185}$$

and after substituting the Lorenz gauge condition (17.96) into expression (17.184), the following differential equations for the Lorenz-gauge potentials are obtained:

$$\Delta \vec{A} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.186}$$

$$\left(\kappa + \varepsilon \cdot \frac{\partial}{\partial t} \right) \Delta \varphi = -g_{\mathrm{st}} \tag{17.187}$$

which, in the phasor domain, are as follows:

$$\Delta \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.188}$$

$$\Delta \overline{\varphi} = -\frac{\overline{g}_{\mathrm{st}}}{\kappa + j \cdot \omega \cdot \varepsilon} \tag{17.189}$$

and these are Poisson's differential equations.

Therefore, with the introduced neglect (17.170), the electromagnetic field in a conducting LIH medium is quasistatic, and in this case, it is also called an *electroquasistatic* electromagnetic field.

In a conducting LIH medium, for the electroquasistatic electromagnetic field, the electric field is independent of the magnetic field and is described by the following pair of Maxwell's equations:

$$\nabla \times \vec{E} = 0 \quad ; \quad \left(\kappa + \varepsilon \cdot \frac{\partial}{\partial t} \right) \left(\nabla \cdot \vec{E} \right) = g_{\mathrm{st}} \tag{17.190}$$

whereas the magnetic field, which depends on the electric field, is described by Maxwell's differential equations:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \quad ; \quad \nabla \cdot \vec{H} = 0 \tag{17.191}$$

17.12. Fully Quasistatic Electromagnetic Field in a Stationary Medium

The fully quasistatic electromagnetic field in a stationary medium can be obtained from the full system of Maxwell's differential equations by simultaneously neglecting the displacement electric currents and the induced component of the electric field intensity:

$$\frac{\partial \vec{D}}{\partial t} = 0 \quad ; \quad \frac{\partial \vec{B}}{\partial t} = 0 \tag{17.192}$$

which means that the electromagnetic potentials are described by the expressions:

$$\vec{B} = \nabla \times \vec{A} \quad ; \quad \vec{E} = -\nabla \varphi \tag{17.193}$$

This means that another neglect is introduced into Maxwell's equations for the electroquasistatic field. Without further verification, it is easy to conclude that such a field in a stationary LIH medium is quasistatic.

With the introduced neglects (17.192), Maxwell's differential equations in a stationary perfect LIH dielectric are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} \tag{17.194}$$

$$\nabla \times \vec{E} = 0 \tag{17.195}$$

$$\nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.196}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.197}$$

and these expressions can easily be obtained from expressions (17.27) - (17.30) by substituting conditions (17.192) and the constitutive equations (3.11).

If expressions (17.193) are substituted into Maxwell's differential equations (17.194) and (17.196), taking into account expression (9.14), the following differential equations for the potentials are obtained:

$$\Delta \vec{A} - \nabla \left(\nabla \cdot \vec{A} \right) = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.198}$$

$$\Delta \varphi = -\frac{\rho_s}{\varepsilon} \tag{17.199}$$

and after substituting the Coulomb gauge condition (17.102) into expression (17.198), Poisson's differential equations (17.164) and (17.165) are obtained, which are described in the phasor domain by expressions (17.166) and (17.167).

Therefore, with the introduced neglects (17.192), the electromagnetic field in a perfect LIH dielectric is quasistatic, and in this case, it can be called a *fully quasistatic* electromagnetic field.

In a perfect LIH dielectric, for the fully quasistatic electromagnetic field, the electric field and the magnetic field are independent. The electric field is formally described by the same pair of Maxwell's differential equations as the electrostatic field:

$$\nabla \times \vec{E} = 0 \quad ; \quad \nabla \cdot \vec{E} = \frac{\rho_{\mathrm{s}}}{\varepsilon} \tag{17.200}$$

whereas the magnetic field is formally described by the same pair of Maxwell's differential equations as the magnetostatic field:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} \quad ; \quad \nabla \cdot \vec{H} = 0 \tag{17.201}$$

With the introduced neglects (17.192), Maxwell's differential equations in a stationary conducting LIH medium are as follows:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} \tag{17.202}$$

$$\nabla \times \vec{E} = 0 \tag{17.203}$$

$$\nabla \cdot \vec{E} = \frac{g_{\mathrm{st}}}{\kappa} \tag{17.204}$$

$$\nabla \cdot \vec{H} = 0 \tag{17.205}$$

and these expressions can easily be obtained from expressions (17.27) - (17.30) by substituting conditions (17.192) and the constitutive equations (3.11).

If expressions (17.193) are substituted into Maxwell's differential equations (17.202) and (17.204), taking into account expression (9.14), the following differential equations for the potentials are obtained:

$$\Delta \vec{A} - \nabla \left(\nabla \cdot \vec{A} + \mu \cdot \kappa \cdot \varphi \right) = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.206}$$

$$\Delta \varphi = -\frac{g_{\mathrm{st}}}{\kappa} \tag{17.207}$$

and after substituting the Lorenz gauge condition (17.142) into expression (17.206), Poisson's differential equations for the Lorenz-gauge potentials are obtained:

$$\Delta \vec{A} = -\mu \cdot \vec{J}_{\mathrm{s}} \tag{17.208}$$

$$\Delta \varphi = -\frac{g_{\mathrm{st}}}{\kappa} \tag{17.209}$$

In a conducting LIH medium, for the fully quasistatic electromagnetic field, the electric field is independent of the magnetic field and is described by the following pair of Maxwell's equations:

$$\nabla \times \vec{E} = 0 \quad ; \quad \nabla \cdot \vec{E} = \frac{g_{\mathrm{st}}}{\kappa} \tag{17.210}$$

whereas the magnetic field, which depends on the electric field, is described by the differential equations:

$$\nabla \times \vec{H} = \vec{J}_{\mathrm{s}} + \kappa \cdot \vec{E} \quad ; \quad \nabla \cdot \vec{H} = 0 \tag{17.211}$$

17.13. Comparison of Electromagnetic Field Approximations

The approximations of the electromagnetic field in a stationary unbounded LIH medium, depending on the type of neglect, are presented in Table 17.1.

Quasistatic electromagnetic fields approximate dynamic electromagnetic fields. The hybrid or Darwin approximation of the electromagnetic field is the best of the shown approximations of dynamic electromagnetic fields, but it is not a quasistatic approximation in an unbounded medium. The Darwin approximation of the electromagnetic field is meaningful only for electromagnetic fields in a stationary perfect dielectric.

The magnetoquasistatic approximation of the dynamic electromagnetic field in a stationary perfect dielectric is based on the neglect of displacement electric currents. This is a relatively good quasistatic approximation of the dynamic electromagnetic field.

If displacement electric currents are neglected in a stationary conducting medium, the electric scalar potential and the magnetic vector potential satisfy diffusion equations. This is the so-called magnetodynamic field, which many incorrectly refer to as a quasistatic field. These equations describe the so-called *eddy current problems*.

The electroquasistatic approximation of the dynamic electromagnetic field is based on the neglect of the induced electric field intensity, which means that in this case, Faraday's law of electromagnetic induction does not hold. This is a relatively rough approximation of the dynamic electromagnetic field, which is quasistatic in both a stationary conducting medium and a stationary perfect dielectric.

Table 17.1. Classification of fields depending on the type of neglect and the type of medium

Neglect	Medium type	Field approximation
$\frac{\partial \vec{E}_{\text{ind}}}{\partial t} = 0$	Perfect dielectric	Darwin approximation
$\frac{\partial \vec{D}}{\partial t} = 0$	Perfect dielectric	Magnetoquasistatic
	Conducting medium	Magnetodynamic
$\frac{\partial \vec{B}}{\partial t} = 0$	Perfect dielectric	Electroquasistatic
	Conducting medium	
$\frac{\partial \vec{D}}{\partial t} = 0$ and $\frac{\partial \vec{B}}{\partial t} = 0$	Perfect dielectric	Fully quasistatic
	Conducting medium	

The fully quasistatic approximation of the dynamic electromagnetic field is based on the neglect of displacement electric currents and the induced electric field intensity, which means that in this case, all time derivatives are neglected in Maxwell's differential equations. This is a very rough approximation of the dynamic electromagnetic field, which is quasistatic in both a stationary conducting medium and a stationary perfect dielectric.

17.14. Particular Solutions of the Helmholtz Differential Equations

In the phasor domain, the Lorenz-gauge potentials in an unbounded LIH medium are solutions to Helmholtz differential equations. If the quasistatic approximation of the dynamic electromagnetic field is used, the inhomogeneous Helmholtz differential equations reduce to Poisson's differential equations for the electromagnetic potentials.

In a conducting LIH medium, the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials are described by the expressions (17.116) and (17.117), which are as follows:

$$\Delta \underline{\vec{A}} \; - \; \overline{\gamma}^2 \cdot \underline{\vec{A}} \; = \; -\mu \cdot \underline{\vec{J}}_s \tag{17.212}$$

$$\Delta \overline{\varphi} \; - \; \overline{\gamma}^2 \cdot \overline{\varphi} \; = \; -\frac{\overline{g}_{\text{st}}}{\kappa + j \cdot \omega \cdot \varepsilon} \tag{17.213}$$

and their particular solutions, which apply to an unbounded medium, are (Figure 17.3):

$$\underline{\vec{A}} \; = \; \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\underline{\vec{J}}_{\text{s}} \cdot e^{-\overline{\gamma} \cdot r} \cdot dV}{r} \; = \; \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\underline{\vec{J}}_{\text{s}} \cdot e^{-\alpha \cdot r} \cdot e^{-j \cdot \beta \cdot r} \cdot dV}{r} \tag{17.214}$$

$$\begin{aligned} \overline{\varphi} \; &= \; \frac{1}{4 \cdot \pi \cdot (\kappa + j \cdot \omega \cdot \varepsilon)} \cdot \int_V \frac{\overline{g}_{\text{st}} \cdot e^{-\overline{\gamma} \cdot r} \cdot dV}{r} \\ &= \; \frac{1}{4 \cdot \pi \cdot (\kappa + j \cdot \omega \cdot \varepsilon)} \cdot \int_V \frac{\overline{g}_{\text{st}} \cdot e^{-\alpha \cdot r} \cdot e^{-j \cdot \beta \cdot r} \cdot dV}{r} \end{aligned} \tag{17.215}$$

where the propagation constant $\overline{\gamma}$, attenuation constant α, and phase constant β are described by the expressions (17.119) - (17.121).

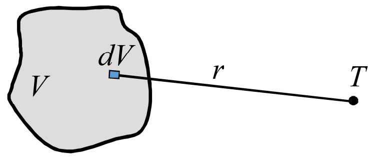


Figure 17.3. The volume V containing the field sources and the field point

In good LIH conductors, with the neglect of displacement electric currents, the dynamic electromagnetic field can be approximated by the magnetodynamic electromagnetic field. In this case, the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials are described by expressions (17.143) and (17.144), which are as follows:

$$\Delta\underline{\vec{A}} - \overline{\gamma}^2 \cdot \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_s \tag{17.216}$$

$$\Delta\overline{\varphi} - \overline{\gamma}^2 \cdot \overline{\varphi} = -\frac{\overline{g}_{\mathrm{st}}}{\kappa} \tag{17.217}$$

and their particular solutions are also described by the expressions (17.214) and (17.215), whereas the propagation constant $\overline{\gamma}$, attenuation constant α, and phase constant β are described by the expressions (17.145) and (17.146).

If the electroquasistatic approximation of the electromagnetic field is used in a conducting LIH medium, then the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials (17.212) and (17.213) reduce to Poisson's differential equations (17.188) and (17.189), which are as follows:

$$\Delta\underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.218}$$

$$\Delta\overline{\varphi} = -\frac{\overline{g}_{\mathrm{st}}}{\kappa + j \cdot \omega \cdot \varepsilon} \tag{17.219}$$

and their particular solutions are:

$$\underline{\vec{A}} = \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\underline{\vec{J}}_{\mathrm{s}} \cdot dV}{r} \tag{17.220}$$

$$\overline{\varphi} = \frac{1}{4 \cdot \pi \cdot (\kappa + j \cdot \omega \cdot \varepsilon)} \cdot \int_V \frac{\overline{g}_{\mathrm{st}} \cdot dV}{r} \tag{17.221}$$

which can be obtained from the particular solutions (17.214) and (17.215) for $\overline{\gamma} = 0$.

If the fully quasistatic approximation of the electromagnetic field is used in a conducting LIH medium, then the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials (17.212) and (17.213) reduce to Poisson's differential equations:

$$\Delta\underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.222}$$

$$\Delta\overline{\varphi} = -\frac{\overline{g}_{\mathrm{st}}}{\kappa} \tag{17.223}$$

and their particular solutions are:

$$\underline{\vec{A}} = \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\underline{\vec{J}}_{\mathrm{s}} \cdot dV}{r} \tag{17.224}$$

$$\overline{\varphi} = \frac{1}{4 \cdot \pi \cdot \kappa} \cdot \int_V \frac{\overline{g}_{\mathrm{st}} \cdot dV}{r} \tag{17.225}$$

which can be obtained from the particular solutions (17.214) and (17.215) by substituting $\overline{\gamma} = 0$ and $\varepsilon = 0$.

In a perfect LIH dielectric, the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials are described by the expressions (17.113) and (17.114), which are as follows:

$$\Delta\underline{\vec{A}} - \overline{\gamma}^2 \cdot \underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.226}$$

$$\Delta\overline{\varphi} - \overline{\gamma}^2 \cdot \overline{\varphi} = -\frac{\overline{\rho}_{\mathrm{s}}}{\varepsilon} \tag{17.227}$$

and their particular solutions are (Figure 17.3):

$$\underline{\vec{A}} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_{\mathrm{s}} \cdot e^{-\overline{\gamma}\cdot r} \cdot dV}{r} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_{\mathrm{s}} \cdot e^{-j\cdot\beta\cdot r} \cdot dV}{r} \tag{17.228}$$

$$\overline{\varphi} = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\int_V \frac{\overline{\rho}_{\mathrm{s}} \cdot e^{-\overline{\gamma}\cdot r} \cdot dV}{r} = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\int_V \frac{\overline{\rho}_{\mathrm{s}} \cdot e^{-j\cdot\beta\cdot r} \cdot dV}{r} \tag{17.229}$$

where the propagation constant $\overline{\gamma}$ and the phase constant β are described by the expressions (17.119) and (17.122).

If the electroquasistatic, magnetoquasistatic, or fully quasistatic approximation of the electromagnetic field is used in a perfect LIH medium, then the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials (17.226) and (17.227) reduce to Poisson's differential equations:

$$\Delta\underline{\vec{A}} = -\mu \cdot \underline{\vec{J}}_{\mathrm{s}} \tag{17.230}$$

$$\Delta\overline{\varphi} = -\frac{\overline{\rho}_{\mathrm{s}}}{\varepsilon} \tag{17.231}$$

and their particular solutions are:

$$\underline{\vec{A}} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_{\mathrm{s}} \cdot dV}{r} \tag{17.232}$$

$$\overline{\varphi} = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\int_V \frac{\overline{\rho}_{\mathrm{s}} \cdot dV}{r} \tag{17.233}$$

which can be obtained from the particular solutions (17.228) and (17.229) for $\overline{\gamma} = j \cdot \beta = 0$.

17.15. Phase Velocity of the Electromagnetic Wave

The propagation velocity of a sinusoidal electromagnetic wave, or the phase velocity of the electromagnetic wave, is described by the expression:

$$v = \frac{\omega}{\beta} \tag{17.234}$$

where ω is the angular frequency of the sinusoidal electromagnetic field.

It follows from expressions (17.120), (17.121), and (17.234) that the phase velocity of a sinusoidal electromagnetic wave in a conducting LIH medium is described by the expression:

$$v = \sqrt{\frac{2\cdot\omega}{\mu}}\cdot\frac{1}{\sqrt{\omega\cdot\varepsilon + \sqrt{(\omega\cdot\varepsilon)^2 + \kappa^2}}} \tag{17.235}$$

and in good conductors, with the neglect of displacement electric currents, or for $\varepsilon = 0$, the phase velocity of the electromagnetic wave is approximated by the expression:

$$v = \sqrt{\frac{2 \cdot \omega}{\mu \cdot \kappa}} \tag{17.236}$$

which can easily be obtained from expressions (17.146) and (17.234).

From expressions (17.235) and (17.236), it follows that the phase velocity of the electromagnetic wave in a conducting LIH medium depends on the properties of the medium and the frequency of the sinusoidal electromagnetic wave.

From expressions (17.122) and (17.234), it follows that the phase velocity of the sinusoidal electromagnetic wave in a perfect LIH dielectric is described by the expression:

$$v = \frac{1}{\sqrt{\mu \cdot \varepsilon}} \tag{17.237}$$

and depends only on the properties of the medium.

Therefore, unlike in a conducting LIH medium, in a perfect LIH dielectric, sinusoidal electromagnetic waves of all frequencies have the same velocity. For a vacuum (air), the phase velocity of the electromagnetic wave is:

$$v = \frac{1}{\sqrt{\mu_0 \cdot \varepsilon_0}} = c = 2.99792458 \times 10^8 \text{ m/s} \approx 3 \times 10^8 \text{ m/s} \tag{17.238}$$

where c is the speed of light in a vacuum.

17.16. Fourier Transform and Inverse Fourier Transform

Electromagnetic problems described in the time domain by linear partial differential equations can be solved in the time domain, but more complex problems are solved by integral transforms. The most commonly used transforms are the Laplace transform and the Fourier transform, which transform partial differential equations from the time domain into systems of linear algebraic equations. The Fourier transform, which maps functions from the time domain to the frequency domain (also known as the spectral domain), is defined by the following expression:

$$\overline{G}(\omega) = \int_{-\infty}^{+\infty} g(t) \cdot e^{-j \cdot \omega \cdot t} \cdot dt \tag{17.239}$$

where:

g(t) - the given real function in the time domain,

$\overline{G}(\omega)$ - the complex function in the frequency domain,

$t \geq 0$ - the time,

$\omega = 2 \cdot \pi \cdot f$ - the angular frequency.

In the special case when $t \geq 0,$ the Fourier transform is given by the following expression:

$$\overline{G}(\omega) = \int_{0}^{+\infty} f(t) \cdot e^{-j \cdot \omega \cdot t} \cdot dt \tag{17.240}$$

The inverse Fourier transform is used to map a function from the frequency domain to the time domain, and is given by the following expression:

$$g(t) = \frac{1}{2 \cdot \pi} \cdot \int_{-\infty}^{+\infty} \overline{G}(\omega) \cdot e^{j \cdot \omega \cdot t} \cdot d\omega \tag{17.241}$$

From expression (17.239), it follows that:

$$\overline{G}(-\omega) = \left(\overline{G}(\omega)\right)^* \quad ; \quad e^{-j\cdot\omega\cdot t} = \left(e^{j\cdot\omega\cdot t}\right)^* \tag{17.242}$$

and the expression for the inverse Fourier transform (17.241) can be rewritten into a new form:

$$g(t) = \frac{1}{\pi}\cdot\int_{0}^{+\infty} \mathrm{Re}\left[\overline{G}(\omega)\cdot e^{j\cdot\omega\cdot t}\right]\cdot d\omega \tag{17.243}$$

which means that the computation can be carried out under the assumption that $\omega \geq 0$.

By using the discrete fast Fourier transform (FFT) or the continuous numerical Fourier transform (CNFT), the integral described by expression (17.240) is solved for a set of so-called sample points in the frequency domain. Then, for each of these sample frequencies, the corresponding system of linear algebraic equations is solved in the frequency (phasor) domain. After that, the quantities of interest are mapped from the frequency domain to the time domain using the inverse fast Fourier transform (IFFT) or the inverse continuous numerical Fourier transform (ICNFT), and thus the integral (17.243) is solved. The integral (17.243) must be solved for each individual sample point in the time domain.

If the FFT / IFFT numerical algorithm is used, then sampling in the time domain is coupled with sampling in the frequency domain, and the sample points in each domain must be equidistant. In practical applications of the IFFT algorithm, there are two challenging circumstances. The first challenging circumstance is that the integration domain is infinite and the sample points must be equidistant, whereas the second challenging circumstance is that the integrand function is highly oscillatory, especially at high frequencies.

If the CNFT / ICNFT numerical algorithm is used, then the sampling in the time domain and in the frequency domain are completely independent and entirely arbitrary. This significantly improves the robustness and accuracy of this numerical algorithm, and it also offers other advantages over the FFT / IFFT numerical algorithm.

17.17. Retarded Potentials in an Unbounded Perfect LIH Dielectric

From expressions (17.228), (17.229), and (17.234), it follows that the particular solutions of the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials (17.113) and (17.114) in a stationary perfect LIH dielectric, in the phasor (frequency) domain, are described by the expressions:

$$\underline{\vec{A}} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_{\mathrm{s}}\cdot e^{-j\cdot\beta\cdot r}\cdot dV}{r} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_{\mathrm{s}}\cdot e^{-j\cdot\omega\cdot\frac{r}{v}}\cdot dV}{r} \tag{17.244}$$

$$\overline{\varphi} = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\int_V \frac{\overline{\rho}_{\mathrm{s}}\cdot e^{-j\cdot\beta\cdot r}\cdot dV}{r} = \frac{1}{4\cdot\pi\cdot\varepsilon}\cdot\int_V \frac{\overline{\rho}_{\mathrm{s}}\cdot e^{-j\cdot\omega\cdot\frac{r}{v}}\cdot dV}{r} \tag{17.245}$$

In a stationary perfect LIH dielectric, there is no attenuation of the electromagnetic field, and the phase constant β describes the negative phase shift in the frequency domain, which manifests as a delay (retardation) in the time domain. This is described by the inverse Fourier transform:

$$g\left(t - \frac{r}{v}\right) = \frac{1}{2\cdot\pi}\cdot\int_{-\infty}^{+\infty} \overline{G}(\omega)\cdot e^{-j\cdot\omega\cdot\frac{r}{v}}\cdot e^{j\cdot\omega\cdot t}\cdot d\omega \tag{17.246}$$

which means that the Lorenz-gauge potentials in the time domain are described by the expressions:

$$\vec{A} = \frac{\mu}{4 \cdot \pi} \cdot \int_V \frac{\vec{J}_s\left(t - \frac{r}{v}\right) \cdot dV}{r} \quad ; \quad \varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \int_V \frac{\rho_s\left(t - \frac{r}{v}\right) \cdot dV}{r} \tag{17.247}$$

which are known as the *retarded potentials*.

The retarded potentials, described by the expressions (17.247), are particular solutions to the inhomogeneous Helmholtz differential equations for the Lorenz-gauge potentials (17.113) and (17.114) in a perfect LIH dielectric, in the time domain. The retarded potentials are frequency-independent, since the wave propagation velocity v, which is described by the expression (17.237), is independent of the frequency. Therefore, expressions (17.113) and (17.114) in a perfect LIH dielectric apply to both sinusoidal and non-sinusoidal electromagnetic fields. The influence of the electromagnetic field source propagates with velocity v and is transmitted to a point at a distance r after a time interval:

$$t_0 = \frac{r}{v} \tag{17.248}$$

In a vacuum, the electromagnetic wave velocity v is equal to the speed of light c.

Let a time-varying longitudinal current flow along the axis of a thin wire located in an unbounded perfect LIH dielectric with permittivity ε (Figure 17.4).

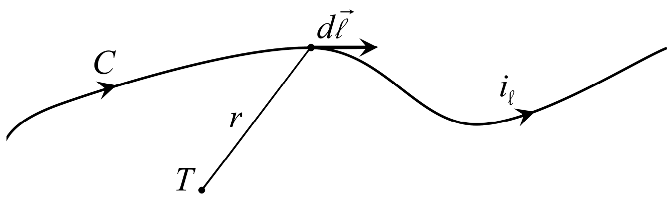


Figure 17.4. Thin-wire conductor in a perfect LIH dielectric

For a thin-wire conductor, it holds that:

$$\vec{J}_s \cdot dV \to i_\ell \cdot d\vec{\ell} \quad ; \quad \rho_s \cdot dV \to \lambda_s \cdot d\ell \tag{17.249}$$

where:

i_ℓ - the longitudinal source current,

λ_s - the linear density of source electric charge.

From expressions (17.247) and (17.249), it follows that the retarded potentials of the thin-wire conductor in a stationary perfect LIH dielectric are described by the following expressions:

$$\vec{A} = \frac{\mu}{4 \cdot \pi} \cdot \int_C \frac{i_\ell\left(t - \frac{r}{v}\right) \cdot d\vec{\ell}}{r} \quad ; \quad \varphi = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \int_C \frac{\lambda_s\left(t - \frac{r}{v}\right) \cdot d\ell}{r} \tag{17.250}$$

which, in the frequency (phasor) domain, are given by:

$$\underline{\vec{A}} = \frac{\mu}{4 \cdot \pi} \cdot \int_C \frac{\overline{I}_\ell \cdot e^{-j \cdot \omega \cdot \frac{r}{v}} \cdot d\vec{\ell}}{r} \quad ; \quad \overline{\varphi} = \frac{1}{4 \cdot \pi \cdot \varepsilon} \cdot \int_C \frac{\overline{\lambda}_s \cdot e^{-j \cdot \omega \cdot \frac{r}{v}} \cdot d\ell}{r} \tag{17.251}$$

where:

$\overline{I}_\ell$ - the phasor of the longitudinal source current,

$\overline{\lambda}_s$ - the phasor of the linear density of source electric charge.

17.18. Lorenz-Gauge Potentials of a Thin-Wire Conductor in a Conducting LIH Medium

In a stationary conducting LIH medium, the particular solutions of the inhomogeneous Helmholtz differential equations (17.116) and (17.117) are described by the expressions (17.214) and (17.215), which are as follows:

$$\underline{\vec{A}} = \frac{\mu}{4\cdot\pi}\cdot\int_V \frac{\underline{\vec{J}}_s\cdot e^{-\bar{\gamma}\cdot r}\cdot dV}{r} \tag{17.252}$$

$$\bar{\varphi} = \frac{1}{4\cdot\pi\cdot(\kappa+j\cdot\omega\cdot\varepsilon)}\cdot\int_V \frac{\bar{g}_{st}\cdot e^{-\bar{\gamma}\cdot r}\cdot dV}{r} \tag{17.253}$$

where the propagation constant $\bar{\gamma}$ of the conducting LIH medium is described by the expressions (17.119) - (17.121).

Let a sinusoidal longitudinal current flow along the axis of a thin-wire conductor located in an unbounded conducting LIH medium (Figure 17.5). The following expressions hold:

$$\underline{\vec{J}}_s\cdot dV \rightarrow \bar{I}_\ell\cdot d\vec{\ell} \tag{17.254}$$

$$\bar{g}_{st}\cdot dV \rightarrow \bar{\tau}_s\cdot d\ell \tag{17.255}$$

where:

$\bar{I}_\ell$ - the phasor of the longitudinal source current,

$\bar{\tau}_s$ - the phasor of the linear density of the source leakage (transverse) electric current.

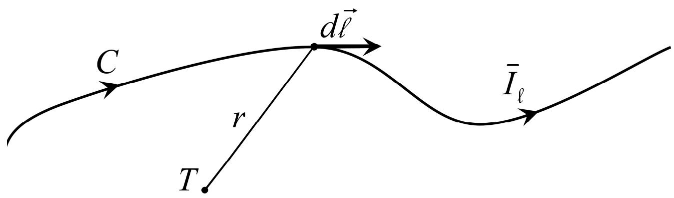


Figure 17.5. Thin-wire conductor in a conducting LIH medium

The phasor of the linear density of the leakage (transverse) electric current leaking from the axis of the thin-wire conductor into the surrounding conducting medium is given by the expression:

$$\bar{\tau}_s = -\frac{\partial\bar{I}_\ell}{\partial\ell} \tag{17.256}$$

From expressions (17.252) - (17.255), it follows that the Lorenz-gauge potentials of the thin-wire conductor in a stationary perfect LIH dielectric, in the phasor domain, are described by the following expressions:

$$\underline{\vec{A}} = \frac{\mu}{4\cdot\pi}\cdot\int_C \frac{\bar{I}_\ell\cdot e^{-\bar{\gamma}\cdot r}\cdot d\vec{\ell}}{r} \tag{17.257}$$

$$\bar{\varphi} = \frac{1}{4\cdot\pi\cdot(\kappa+j\cdot\omega\cdot\varepsilon)}\cdot\int_C \frac{\bar{\tau}_s\cdot e^{-\bar{\gamma}\cdot r}\cdot d\ell}{r} \tag{17.258}$$

In a stationary conducting LIH medium, in addition to the retardation (delay) of the electromagnetic wave, there is also attenuation of the electromagnetic wave. Furthermore, the phase velocity of the wave, described by expression (17.235), depends on the frequency of the electromagnetic field. Therefore, there is no analytical particular solution to the inhomogeneous Helmholtz differential equations for the

Lorenz-gauge potentials (17.116) and (17.117) in the time domain. In this case, the Fourier integral transform should be used to transition between the time and frequency domains.

Since the particular solutions of the Helmholtz inhomogeneous differential equations in a stationary perfect LIH dielectric, in the phasor domain, are described by the expressions (17.251), which are the special case of the particular solutions of the Helmholtz inhomogeneous differential equations in a stationary conducting LIH medium, described by expressions (17.257) and (17.258), it is important to note that the following expressions apply at the transition from a conducting LIH medium to a perfect LIH dielectric:

$$\overline{\tau}_s \to j\cdot\omega\cdot\overline{\lambda}_s \quad ; \quad \kappa \to 0 \quad ; \quad \overline{\gamma} \to j\cdot\beta = j\cdot\frac{\omega}{v} \tag{17.259}$$

17.19. Electroquasistatic and Fully Quasistatic Sinusoidal Current Field

In a poorly conducting LIH medium, such as soil, where the electrical conductivity $\kappa < 1$ S/m, the *electroquasistatic approximation* of the low-frequency current field is used. From expression (17.190), it follows that the electroquasistatic sinusoidal current field is independent of the magnetic field and is described by the differential equations:

$$\nabla\times\underline{\vec{E}} = 0 \quad ; \quad \nabla\cdot\vec{E} = \frac{\overline{g}_{\mathrm{st}}}{\kappa + j\cdot\omega\cdot\varepsilon} \tag{17.260}$$

from which it follows that:

$$\underline{\vec{E}} = -\nabla\,\overline{\varphi} \tag{17.261}$$

and that is:

$$\Delta\overline{\varphi} = -\frac{\overline{g}_{\mathrm{st}}}{\kappa + j\cdot\omega\cdot\varepsilon} \tag{17.262}$$

The particular solution of Poisson's differential equation (17.262) is:

$$\overline{\varphi} = \frac{1}{4\cdot\pi\cdot(\kappa + j\cdot\omega\cdot\varepsilon)}\cdot\int_V \frac{\overline{g}_{\mathrm{st}}\cdot dV}{r} \tag{17.263}$$

where r is the distance between the source point and the field point.

For a thin-wire conductor, expression (17.255) holds, so in this case, the particular solution of Poisson's differential equation is as follows:

$$\overline{\varphi} = \frac{1}{4\cdot\pi\cdot(\kappa + j\cdot\omega\cdot\varepsilon)}\cdot\int_C \frac{\overline{\tau}_s\cdot d\ell}{r} \tag{17.264}$$

The particular solutions of Poisson's differential equations describe the distribution of the scalar electric potential in an unbounded conducting LIH medium.

From expression (17.262), it follows that for a point source of sinusoidal electric current in a conducting LIH medium, Poisson's differential equation holds:

$$\Delta\overline{\varphi} = -\frac{\overline{I}}{\kappa + j\cdot\omega\cdot\varepsilon}\cdot\delta(\vec{r}\,) \quad ; \quad \overline{g}_{st} \to \overline{I}\cdot\delta(\vec{r}\,) \tag{17.265}$$

whose particular solution is:

$$\overline{\varphi} = \frac{\overline{I}}{4\cdot\pi\cdot(\kappa + j\cdot\omega\cdot\varepsilon)\cdot r} \tag{17.266}$$

If $\varepsilon = 0$ is substituted into the expressions listed in this chapter, then the equations of the electroquasistatic sinusoidal current field in a conducting LIH medium become the equations of the fully quasistatic sinusoidal current field in a conducting LIH medium.

17.20. Helmholtz Equations for Electromagnetic Fields in a Source-Free Stationary LIH Medium

In chapter 10 of this textbook, the wave equations for electromagnetic fields in a source-free stationary LIH medium are derived. In a source-free stationary conducting LIH medium, the wave equations are described by expressions (10.9) and (10.10), which are as follows:

$$\Delta \vec{E} - \mu \cdot \varepsilon \cdot \frac{\partial^2 \vec{E}}{\partial t^2} - \mu \cdot \kappa \cdot \frac{\partial \vec{E}}{\partial t} = 0 \qquad (17.267)$$

$$\Delta \vec{H} - \mu \cdot \varepsilon \cdot \frac{\partial^2 \vec{H}}{\partial t^2} - \mu \cdot \kappa \cdot \frac{\partial \vec{H}}{\partial t} = 0 \qquad (17.268)$$

which in a source-free stationary perfect LIH dielectric become homogeneous undamped wave equations (10.11) and (10.12), which are as follows::

$$\Delta \vec{E} - \mu \cdot \varepsilon \cdot \frac{\partial^2 \vec{E}}{\partial t^2} = 0 \qquad (17.269)$$

$$\Delta \vec{H} - \mu \cdot \varepsilon \cdot \frac{\partial^2 \vec{H}}{\partial t^2} = 0 \qquad (17.270)$$

whereas in a source-free stationary good LIH conductor, with the neglect of displacement electric currents, they become homogeneous diffusion equations (10.13), which are as follows:

$$\Delta \vec{E} - \mu \cdot \kappa \cdot \frac{\partial \vec{E}}{\partial t} = 0 \qquad (17.271)$$

$$\Delta \vec{H} - \mu \cdot \kappa \cdot \frac{\partial \vec{H}}{\partial t} = 0 \qquad (17.272)$$

Homogeneous wave equations and homogeneous diffusion equations for electromagnetic fields can be transformed from the time domain into the Helmholtz homogeneous differential equations for electromagnetic fields in the phasor domain, which are as follows:

$$\Delta \underline{\vec{E}} - \bar{\gamma}^2 \cdot \underline{\vec{E}} = 0 \quad ; \quad \Delta \underline{\vec{H}} - \bar{\gamma}^2 \cdot \underline{\vec{H}} = 0 \qquad (17.273)$$

or alternatively written as:

$$\Delta \underline{\vec{E}} = \bar{\gamma}^2 \cdot \underline{\vec{E}} \quad ; \quad \Delta \underline{\vec{H}} = \bar{\gamma}^2 \cdot \underline{\vec{H}} \qquad (17.274)$$

where the propagation constant of the medium $\bar{\gamma}$ is described by:

- Expressions (17.119) - (17.121) in a conducting LIH medium,
- Expressions (17.119) and (17.122) in a perfect LIH dielectric,
- Expression (17.146) in good LIH a conductor.

The Helmholtz homogeneous differential field equations in the phasor domain, described by the expressions (17.274), can be derived for a source-free conducting LIH medium from Maxwell's differential equations:

$$\nabla \times \underline{\vec{H}} = (\kappa + j \cdot \omega \cdot \varepsilon) \cdot \underline{\vec{E}} \qquad (17.275)$$

$$\nabla \times \underline{\vec{E}} = -j \cdot \omega \cdot \mu \cdot \underline{\vec{H}} \tag{17.276}$$

which are described in the time domain by expressions (10.1) and (10.2).
Expressions (17.275) and (17.276) should also be accompanied by the following two expressions:

$$\nabla \times \left(\nabla \times \underline{\vec{H}}\right) = \nabla \cdot \left(\nabla \cdot \underline{\vec{H}}\right) - \Delta \underline{\vec{H}} = -\Delta \underline{\vec{H}} \tag{17.277}$$

$$\nabla \times \left(\nabla \times \underline{\vec{E}}\right) = \nabla \cdot \left(\nabla \cdot \underline{\vec{E}}\right) - \Delta \underline{\vec{E}} = -\Delta \underline{\vec{E}} \tag{17.278}$$

From expressions (17.275) and (17.277), it follows that:

$$\Delta \underline{\vec{H}} = -\left(\kappa + j \cdot \omega \cdot \varepsilon\right) \cdot \left(\nabla \times \underline{\vec{E}}\right) \tag{17.279}$$

From expressions (17.276) and (17.278), it follows that:

$$\Delta \underline{\vec{E}} = -j \cdot \omega \cdot \mu \cdot \left(\nabla \times \underline{\vec{H}}\right) \tag{17.280}$$

From expressions (17.276) and (17.279), it follows that:

$$\Delta \underline{\vec{H}} = -j \cdot \omega \cdot \mu \cdot \left(\kappa + j \cdot \omega \cdot \varepsilon\right) \cdot \underline{\vec{H}} = \bar{\gamma}^2 \cdot \underline{\vec{H}} \tag{17.281}$$

From expressions (17.275) and (17.280), it follows that:

$$\Delta \underline{\vec{E}} = -j \cdot \omega \cdot \mu \cdot \left(\kappa + j \cdot \omega \cdot \varepsilon\right) \cdot \underline{\vec{E}} = \bar{\gamma}^2 \cdot \underline{\vec{E}} \tag{17.282}$$

In a source-free perfect LIH dielectric, Maxwell's differential equation (17.275) takes on a new form:

$$\nabla \times \underline{\vec{H}} = j \cdot \omega \cdot \varepsilon \cdot \underline{\vec{E}} \tag{17.283}$$

In a source-free good LIH conductor, with the neglect of displacement electric currents, Maxwell's differential equation (17.275) takes on a new form:

$$\nabla \times \underline{\vec{H}} = \kappa \cdot \underline{\vec{E}} \tag{17.284}$$

Maxwell's differential equation (17.276) also holds in a source-free perfect LIH dielectric and in a source-free good LIH conductor.

17.21. Skin Depth of the Electromagnetic Wave

The skin depth of the electromagnetic wave d is described by the expression:

$$d = \frac{1}{\alpha} \tag{17.285}$$

where α is the attenuation constant of the medium.

From expressions (17.120), (17.121), and (17.285), it follows that the skin depth of the electromagnetic wave in a conducting LIH medium is:

$$d = \frac{1}{\kappa} \cdot \sqrt{\frac{2}{\omega \cdot \mu}} \cdot \sqrt{\omega \cdot \varepsilon + \sqrt{\left(\omega \cdot \varepsilon\right)^2 + \kappa^2}} \tag{17.286}$$

whereas in a good conductor, with the neglect of displacement electric currents, the skin depth of the electromagnetic wave is described by the expression:

$$d=\sqrt{\frac{2}{\omega\cdot\mu\cdot\kappa}} \tag{17.287}$$

In a perfect LIH dielectric, there is no attenuation of the electromagnetic wave, and the attenuation constant of the medium is zero, so the skin depth of the electromagnetic wave in such a medium is infinite.

If a sinusoidal electromagnetic wave in a conducting medium travels a distance equal to the skin depth, its amplitude (and thus its RMS value) decreases to 36.78794% of its initial value, because:

$$e^{-\alpha\cdot d}=e^{-1}\approx 0.3678794 \tag{17.288}$$

17.22 Conductor Impedance and Conductor Internal Impedance

Let a sinusoidal electric current flow through a solid conductor made of an LIH material (Figure 17.6).

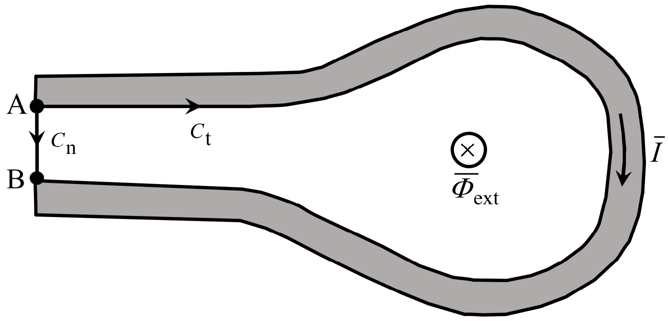


Figure 17.6. Conductor made of an LIH material carrying a sinusoidal electric current

Let the curve C_n be such that the magnetic vector potential is perpendicular to it. Then the electric voltage along this curve is called the transverse voltage or simply the electric voltage. In a time-varying electromagnetic field, the voltage is not uniquely defined. Therefore, the term "voltage" refers to the transverse voltage, which is a special case of the voltage corresponding to the difference between scalar electric potentials.

The following holds:

$$\int_{C_n}\underline{\vec{E}}\cdot d\vec{\ell} = \overline{\varphi}_A-\overline{\varphi}_B-\int_{C_n} j\cdot\omega\cdot\underbrace{\underline{\vec{A}}\cdot d\vec{\ell}}_{=0} = \overline{\varphi}_A-\overline{\varphi}_B=\overline{U} \tag{17.289}$$

where $\overline{U}$ denotes the phasor of the (transverse) voltage of the conductor.

Furthermore, we have:

$$\overline{U} = \overline{\varphi}_A-\overline{\varphi}_B = \overline{Z}\cdot\overline{I} = \left(R+j\cdot\omega\cdot L\right)\cdot\overline{I} \tag{17.290}$$

where:

$\overline{Z}$ - the conductor impedance,

R - the conductor resistance,

ω - the angular frequency of the electric current,

L - the conductor total inductance,

$\overline{I}$ - the phasor of the electric current flowing through the conductor.

According to Faraday's law of electromagnetic induction, it holds that:

$$\int_{C_t}\underline{\vec{E}}\cdot d\vec{\ell} - \int_{C_n}\underline{\vec{E}}\cdot d\vec{\ell} = -j\cdot\omega\cdot\overline{\Phi}_{ext} = -j\cdot\omega\cdot L_{ext}\cdot\overline{I} \tag{17.291}$$

where:

$\overline{\Phi}_{\text{ext}}$ - the phasor of the external magnetic flux (Figure 17.6),

L_{ext} - the conductor external inductance.

It follows from the expressions (17.288) and (17.290) that:

$$\int_{C_t} \underline{\vec{E}} \cdot d\vec{\ell} = \overline{U} - j \cdot \omega \cdot L_{\text{ext}} \cdot \overline{I} = \left(\overline{Z} - j \cdot \omega \cdot L_{\text{ext}}\right) \cdot \overline{I} = \overline{Z}_{\text{int}} \cdot \overline{I} \tag{17.292}$$

where $\overline{Z}_{\text{int}}$ is the conductor internal impedance.

Thus, the internal impedance of the solid conductor is defined by the expression:

$$\overline{Z}_{\text{int}} = \overline{Z} - j \cdot \omega \cdot L_{\text{ext}} = \frac{1}{\overline{I}} \cdot \int_{C_t} \underline{\vec{E}} \cdot d\vec{\ell} \tag{17.293}$$

In the case of an infinitely long straight conductor with a circular cross-section, the per-unit-length internal impedance of the conductor is defined by the following expression:

$$\overline{Z}_{\text{int}}^{1} = \frac{\overline{E}_{\text{t}}}{\overline{I}} \tag{17.294}$$

where $\overline{E}_{\text{t}}$ denotes the phasor of the tangential electric field intensity at the conductor's surface.

In the general case, according to expressions (12.29) and (12.41), the internal impedance of the conductor can be calculated using the formula derived from Poynting's theorem:

$$\overline{S}_{\text{ap,in}} = \overline{Z}_{\text{int}} \cdot I^2 = -\,\overline{S}_{\text{ap,out}} = -\oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot d\vec{S} \tag{17.295}$$

where:

$\overline{S}_{\text{ap,in}}$ - the complex apparent electromagnetic power entering the volume V,

$\overline{S}_{\text{ap, out}}$ - the complex apparent electromagnetic power leaving the volume V,

S - the outer surface of the conductor or the outer surface of a unit conductor segment when calculating the per-unit-length internal impedance.

From expression (17.295), it follows that the internal impedance of the conductor is given by:

$$\overline{Z}_{\text{int}} = -\frac{1}{I^2} \cdot \oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot d\vec{S} = -\frac{1}{I^2} \cdot \oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot \vec{n} \cdot dS \tag{17.296}$$

It follows that, in the case of an infinitely long straight cylindrical conductor of radius r_c, the per-unit-length internal impedance of the conductor is given by:

$$\overline{Z}_{\text{int}}^{1} = -\frac{1}{I^2} \cdot \oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot d\vec{S} = \frac{\overline{E}_{\text{t}} \cdot \overline{H}_{\text{t}}^*}{I^2} \cdot 2 \cdot \pi \cdot r_{\text{c}} \tag{17.297}$$

where:

$$\overline{H}_{\text{t}}^* = \frac{\overline{I}^*}{2 \cdot \pi \cdot r_{\text{c}}} \tag{17.298}$$

and it is:

$$\overline{Z}_{\text{int}}^{1} = \frac{\overline{E}_{\text{t}} \cdot \overline{I}^*}{I^2} = \frac{\overline{E}_{\text{t}} \cdot \overline{I}^*}{\overline{I} \cdot \overline{I}^*} = \frac{\overline{E}_{\text{t}}}{\overline{I}} = \frac{1}{2 \cdot r_{\text{c}} \cdot \pi} \cdot \frac{\overline{E}_{\text{t}}}{\overline{H}_{\text{t}}} \tag{17.299}$$

17.23. Internal Impedance of a Multilayer Cylindrical Conductor

Let an infinitely long, straight, multilayer cylindrical conductor, consisting of *m* conducting LIH layers, carry a sinusoidal current. The multilayer cylindrical conductor may be hollow (Figure 17.7-a) or solid (Figure 17.7-b). Let the field within the conductor be magnetodynamic, which means that displacement electric currents are neglected.

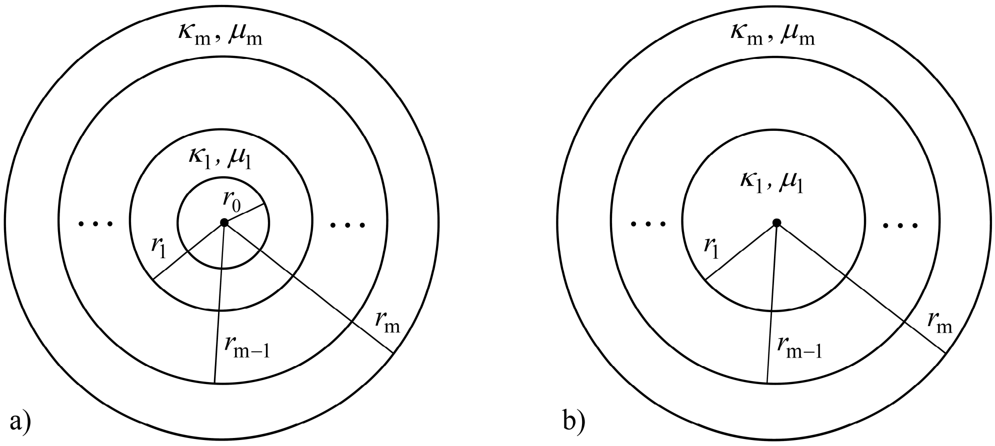


Figure 17.7. Hollow and solid m-layer cylindrical conductor

Let an infinitely long, m-layered solid or hollow cylindrical conductor be aligned along the *z*-axis of the cylindrical coordinate system (*r*, *φ*, *z*), with the conductor's axis coinciding with the *z*-axis. Let a sinusoidal electric current flow in the direction of the *z*-axis.

For the cylindrical conductor under consideration, the axisymmetric electric and magnetic fields in the i-th layer of the m-layered conductor each have only one component, which means that:

$$\underline{\vec{E}}_{\mathrm{i}} = \overline{E}_{\mathrm{i}} \cdot \vec{e}_{\mathrm{z}} \quad ; \quad \underline{\vec{H}}_{\mathrm{i}} = \overline{H}_{\mathrm{i}} \cdot \vec{e}_{\varphi} \tag{17.300}$$

Neglecting displacement electric currents, the following equations apply for the i-th layer:

$$\Delta \underline{\vec{E}}_{\mathrm{i}} = \overline{\gamma}_{\mathrm{i}}^{2} \cdot \underline{\vec{E}}_{\mathrm{i}} \tag{17.301}$$

$$\Delta \underline{\vec{H}}_{\mathrm{i}} = \overline{\gamma}_{\mathrm{i}}^{2} \cdot \underline{\vec{H}}_{\mathrm{i}} \tag{17.302}$$

$$\nabla \times \underline{\vec{H}}_{\mathrm{i}} = \kappa_{\mathrm{i}} \cdot \underline{\vec{E}}_{\mathrm{i}} \tag{17.303}$$

$$\overline{\gamma}_{\mathrm{i}}^{2} = j \cdot \omega \cdot \mu_{\mathrm{i}} \cdot \kappa_{\mathrm{i}} \tag{17.304}$$

$$\overline{\gamma}_{\mathrm{i}} = \sqrt{\omega \cdot \mu_{\mathrm{i}} \cdot \kappa_{\mathrm{i}}} \cdot e^{j \cdot 45^{o}} = (1 + j) \cdot \alpha_{\mathrm{i}} \quad ; \quad \alpha_{\mathrm{i}} = \sqrt{\frac{\omega \cdot \mu_{\mathrm{i}} \cdot \kappa_{\mathrm{i}}}{2}} \tag{17.305}$$

For the Laplacian of the phasor of magnetic field intensity vector, the following holds:

$$\Delta \underline{\vec{H}}_{\mathrm{i}} = \nabla \underbrace{\left(\nabla \cdot \underline{\vec{H}}_{\mathrm{i}}\right)}_{=0} - \nabla \times \left(\nabla \times \underline{\vec{H}}_{\mathrm{i}}\right) = -\nabla \times \left(\nabla \times \underline{\vec{H}}_{\mathrm{i}}\right) \tag{17.306}$$

In the cylindrical coordinate system (*r*, *φ*, *z*), it holds that:

$$\nabla\times\underline{\vec{H}}_{\mathrm{i}}=\frac{1}{r}\cdot\begin{vmatrix}\vec{e}_{\mathrm{r}} & r\cdot\vec{e}_{\varphi} & \vec{e}_{\mathrm{z}}\\ \dfrac{\partial}{\partial r} & 0 & 0\\ 0 & r\cdot\overline{H}_{\mathrm{i}} & 0\end{vmatrix}=\frac{1}{r}\cdot\frac{\partial\left(r\cdot\overline{H}_{\mathrm{i}}\right)}{\partial r}\cdot\vec{e}_{\mathrm{z}} \tag{17.307}$$

and it is:

$$\nabla\times\left(\nabla\times\underline{\vec{H}}_{\mathrm{i}}\right)=\frac{1}{r}\cdot\begin{vmatrix}\vec{e}_{\mathrm{r}} & r\cdot\vec{e}_{\varphi} & \vec{e}_{\mathrm{z}}\\ \dfrac{\partial}{\partial r} & 0 & 0\\ 0 & 0 & \dfrac{1}{r}\cdot\dfrac{\partial\left(r\cdot\overline{H}_{\mathrm{i}}\right)}{\partial r}\end{vmatrix}=-\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial\left(r\cdot\overline{H}_{\mathrm{i}}\right)}{\partial r}\right]\cdot\vec{e}_{\varphi} \tag{17.308}$$

From the expressions (17.302), (17.306), and (17.308), it follows that:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\frac{\partial\left(r\cdot\overline{H}_{\mathrm{i}}\right)}{\partial r}\right]=\overline{\gamma}_{\mathrm{i}}^{2}\cdot\underline{\vec{H}}_{\mathrm{i}}=-\overline{k}_{\mathrm{i}}^{2}\cdot\underline{\vec{H}}_{\mathrm{i}} \tag{17.309}$$

where $\overline{k}_{\mathrm{i}}$ is the complex wave number of the i-th layer, for which the following expression holds:

$$\overline{k}_{\mathrm{i}}=-j\cdot\overline{\gamma}_{\mathrm{i}}=\sqrt{\omega\cdot\mu_{\mathrm{i}}\cdot\kappa_{\mathrm{i}}}\cdot e^{-j\cdot45^{\circ}}=(1-j)\cdot\alpha_{\mathrm{i}}\quad;\quad\alpha_{\mathrm{i}}=\sqrt{\frac{\omega\cdot\mu_{\mathrm{i}}\cdot\kappa_{\mathrm{i}}}{2}} \tag{17.310}$$

From the expression (17.309), it follows that:

$$\frac{\partial}{\partial r}\left[\frac{1}{r}\cdot\overline{H}_{\mathrm{i}}+\frac{\partial\overline{H}_{\mathrm{i}}}{\partial r}\right]=\overline{\gamma}_{\mathrm{i}}^{2}\cdot\overline{H}_{i}=-\overline{k}_{\mathrm{i}}^{2}\cdot\overline{H}_{\mathrm{i}} \tag{17.311}$$

from which the two following forms of this differential equation can be easily derived:

$$r^{2}\cdot\frac{\partial^{2}\overline{H}_{\mathrm{i}}}{\partial r^{2}}+r\cdot\frac{\partial\overline{H}_{\mathrm{i}}}{\partial r}+\left(\overline{k}_{\mathrm{i}}^{2}\cdot r^{2}-1\right)\cdot\overline{H}_{\mathrm{i}}=0 \tag{17.312}$$

$$r^{2}\cdot\frac{\partial^{2}\overline{H}_{\mathrm{i}}}{\partial r^{2}}+r\cdot\frac{\partial\overline{H}_{\mathrm{i}}}{\partial r}-\left(\overline{\gamma}_{\mathrm{i}}^{2}\cdot r^{2}+1\right)\cdot\overline{H}_{\mathrm{i}}=0 \tag{17.313}$$

The differential equation described by the expression (17.312) is a special case of the Bessel differential equation for $n^{2}=1,$ whereas the differential equation described by the expression (17.313) is a special case of the modified Bessel differential equation for $n^{2}=1.$

A particular solution of the Bessel differential equation (17.312) is:

$$\overline{H}_{\mathrm{i}}=\overline{C}_{\mathrm{i}}\cdot\overline{J}_{1}\left(\overline{k}_{\mathrm{i}}\cdot r\right)+\overline{D}_{\mathrm{i}}\cdot\overline{N}_{1}\left(\overline{k}_{\mathrm{i}}\cdot r\right) \tag{17.314}$$

where:

$\overline{J}_{1}\left(\overline{k}_{\mathrm{i}}\cdot r\right)$ - the complex Bessel function of the first kind of the first order,

$\overline{N}_{1}\left(\overline{k}_{\mathrm{i}}\cdot r\right)$ - the complex Bessel function of the second kind of the first order; the complex Neumann function of the first order,

$\overline{C}_{\mathrm{i}}$, $\overline{D}_{\mathrm{i}}$ - the unknown complex coefficients.

From the expressions (17.303) and (17.307), it follows that:

$$\vec{\underline{E}}_{\mathrm{i}} = \frac{1}{\kappa_{\mathrm{i}}} \cdot \left(\nabla \times \vec{\underline{H}}_{\mathrm{i}} \right) = \frac{1}{\kappa_{\mathrm{i}}} \cdot \frac{1}{r} \cdot \frac{\partial \left(r \cdot \overline{H}_{\mathrm{i}} \right)}{\partial r} \cdot \vec{e}_{\mathrm{z}} \quad ; \quad \overline{E}_{\mathrm{i}} = \frac{1}{\kappa_{\mathrm{i}}} \cdot \frac{1}{r} \cdot \frac{\partial \left(r \cdot \overline{H}_{\mathrm{i}} \right)}{\partial r} \tag{17.315}$$

and it holds that:

$$\frac{d}{d\,r}\left[r \cdot \overline{J}_1\left(\overline{k}_{\mathrm{i}} \cdot r \right) \right] = \overline{k}_{\mathrm{i}} \cdot r \cdot \overline{J}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right) \;\; ; \;\; \frac{d}{d\,r}\left[r \cdot \overline{N}_1\left(\overline{k}_{\mathrm{i}} \cdot r \right) \right] = \overline{k}_{\mathrm{i}} \cdot r \cdot \overline{N}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right) \tag{17.316}$$

where:

$\overline{J}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right)$ - the complex Bessel function of the first kind of zero order,

$\overline{N}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right)$ - the complex Bessel function of the second kind of zero order; the complex Neumann function of zero order.

From the expressions (17.314) - (17.316), it follows that:

$$\overline{E}_{\mathrm{i}} = \frac{\overline{k}_{\mathrm{i}}}{\kappa_{\mathrm{i}}} \cdot \left[\overline{C}_{\mathrm{i}} \cdot \overline{J}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right) + \overline{D}_{\mathrm{i}} \cdot \overline{N}_0\left(\overline{k}_{\mathrm{i}} \cdot r \right) \right] \tag{17.317}$$

The numerical results show that, especially for large arguments, instead of the particular solution (17.314) of the Bessel differential equation (17.312), it is much better to use the particular solution of the modified Bessel differential equation (17.313), which is:

$$\overline{H}_{\mathrm{i}} = \overline{C}_{\mathrm{i}} \cdot \overline{\mathcal{I}}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) + \overline{D}_{\mathrm{i}} \cdot \overline{K}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \tag{17.318}$$

where:

$\overline{\mathcal{I}}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right)$ - the complex modified Bessel function of the first kind of order one,

$\overline{K}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right)$ - the complex modified Bessel function of the second kind of order one,

$\overline{C}_{\mathrm{i}}$, $\overline{D}_{\mathrm{i}}$ - the unknown complex coefficients.

The following expressions relate the complex Bessel functions and the complex modified Bessel functions of order n:

$$\overline{J}_{\mathrm{n}}\left(\overline{k}_{\mathrm{i}} \cdot r \right) = (-j)^{\mathrm{n}} \cdot \overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{\gamma}_i \cdot r \right) \quad ; \quad \overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) = j^{\mathrm{n}} \cdot \overline{J}_{\mathrm{n}}\left(\overline{k}_{\mathrm{i}} \cdot r \right) \tag{17.319}$$

$$\overline{K}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) = (-j)^{\mathrm{n}+1} \cdot \frac{\pi}{2} \cdot \left[\overline{J}_{\mathrm{n}}\left(\overline{k}_{\mathrm{i}} \cdot r \right) - j \cdot \overline{N}_{\mathrm{n}}\left(\overline{k}_{\mathrm{i}} \cdot r \right) \right] \tag{17.320}$$

The following holds:

$$\frac{d}{d\,r}\left[r \cdot \overline{\mathcal{I}}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \right] = \overline{\gamma}_{\mathrm{i}} \cdot r \cdot \overline{\mathcal{I}}_0\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \;\; ; \;\; \frac{d}{d\,r}\left[r \cdot \overline{K}_1\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \right] = -\, \overline{\gamma}_{\mathrm{i}} \cdot r \cdot \overline{K}_0\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \tag{17.321}$$

From expressions (17.315), (17.318), and (17.321), it follows that:

$$\overline{E}_{\mathrm{i}} = \frac{\overline{\gamma}_{\mathrm{i}}}{\kappa_{\mathrm{i}}} \cdot \left[\overline{C}_{\mathrm{i}} \cdot \overline{\mathcal{I}}_0\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) - \overline{D}_{\mathrm{i}} \cdot \overline{K}_0\left(\overline{\gamma}_{\mathrm{i}} \cdot r \right) \right] \tag{17.322}$$

Thus, if the complex Bessel functions are used, the field distribution in the i-th layer of an m-layer cylindrical conductor is described by the expressions (17.314) and (17.317), except for the first layer of an m-layer solid cylindrical conductor, for which the following applies:

$$\overline{D}_1 = 0 \;\; ; \;\; \overline{H}_1 = \overline{C}_1 \cdot \overline{J}_1\left(\overline{k}_1 \cdot r \right) \quad ; \quad \overline{E}_1 = \frac{\overline{k}_1}{\kappa_1} \cdot \overline{C}_1 \cdot \overline{J}_0\left(\overline{k}_1 \cdot r \right) \tag{17.323}$$

because the Neumann functions at $r = 0$ are infinite.

If the complex modified Bessel functions are used, the field distribution in the i-th layer of an m-layer cylindrical conductor is described by expressions (17.318) and (17.322), except for the first layer of an m-layer solid cylindrical conductor, for which the following applies:

$$\overline{D}_1 = 0 \;\; ; \quad \overline{H}_1 = \overline{C}_1 \cdot \bar{\mathcal{I}}_1(\bar{\gamma}_1 \cdot r) \quad ; \quad \overline{E}_1 = \frac{\bar{\gamma}_1}{\kappa_1} \cdot \overline{C}_1 \cdot \bar{\mathcal{I}}_0(\bar{\gamma}_1 \cdot r) \tag{17.324}$$

because the modified Bessel functions of the second kind at $r = 0$ are infinite.

The unknown coefficients are determined from a system of equations obtained by substituting the expressions (17.314) and (17.317) into the boundary conditions if complex Bessel functions are used, or by substituting the expressions (17.318) and (17.322) into the boundary conditions if complex modified Bessel functions are used.

For a solid m-layer cylindrical conductor, the boundary conditions are:

$$\left.\overline{H}_{\mathrm{i}}\right|_{r=r_{\mathrm{i}}} = \left.\overline{H}_{\mathrm{i}+1}\right|_{r=r_{\mathrm{i}}} \qquad \text{for} \quad \mathrm{i} = 1, 2, \ldots, \mathrm{m}-1 \tag{17.325}$$

$$\left.\overline{E}_{\mathrm{i}}\right|_{r=r_{i}} = \left.\overline{E}_{\mathrm{i}+1}\right|_{r=r_{\mathrm{i}}} \qquad \text{for} \quad \mathrm{i} = 1, 2, \ldots, \mathrm{m}-1 \tag{17.326}$$

$$\left.\overline{H}_{\mathrm{m}}\right|_{r=r_{\mathrm{m}}} = \frac{\bar{I}}{2 \cdot r_{\mathrm{m}} \cdot \pi} \tag{17.327}$$

For a solid m-layer cylindrical conductor, a system of 2·m – 1 equations is obtained, and the unknowns are:

- $\overline{C}_1$,
- $\overline{C}_{\mathrm{i}}, \overline{D}_{\mathrm{i}}$; i $= 2, 3, \ldots, \mathrm{m}$.

For a hollow m-layer cylindrical conductor, the boundary conditions (17.325) - (17.327) apply, as well as an additional boundary condition:

$$\left.\overline{H}_1\right|_{r=r_0} = 0 \tag{17.328}$$

For a hollow m-layer cylindrical conductor, a system of 2·m equations is obtained, and the unknowns are: $\overline{C}_{\mathrm{i}}, \overline{D}_{\mathrm{i}}$; i $= 1, 2, \ldots, \mathrm{m}$.

From expression (17.299), it follows that the per-unit-length internal impedance of an m-layer cylindrical conductor, whether hollow or solid, is described by the expression:

$$\overline{Z}_{\mathrm{int}}^{1} = \frac{1}{2 \cdot r_{\mathrm{m}} \cdot \pi} \cdot \frac{\left.\overline{E}_{\mathrm{m}}\right|_{r=r_{\mathrm{m}}}}{\left.\overline{H}_{\mathrm{m}}\right|_{r=r_{\mathrm{m}}}} = \frac{\left.\overline{E}_{\mathrm{m}}\right|_{r=r_{\mathrm{m}}}}{\bar{I}} \tag{17.329}$$

where r_{m} is the outer radius of the m-layer cylindrical conductor.

If complex Bessel functions are used, according to expressions (17.314), (17.317), and (17.329), the per-unit-length internal impedance of an m-layer cylindrical conductor is described by the expression:

$$\overline{Z}_{\mathrm{int}}^{1} = \frac{\bar{k}_{\mathrm{m}}}{2 \cdot r_{\mathrm{m}} \cdot \pi \cdot \kappa_{\mathrm{m}}} \cdot \frac{\overline{C}_{\mathrm{m}} \cdot \bar{J}_0\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right) + \overline{D}_{\mathrm{m}} \cdot \overline{N}_0\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right)}{\overline{C}_{\mathrm{m}} \cdot \bar{J}_1\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right) + \overline{D}_{\mathrm{m}} \cdot \overline{N}_1\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right)} \tag{17.330}$$

For numerical reasons, it is more convenient to use scaled values of the coefficients and scaled values of the complex Bessel functions, so that expression (17.330) takes on a new form:

$$\overline{Z}_{\mathrm{int}}^{1} = \frac{\bar{k}_{\mathrm{m}}}{2 \cdot r_{\mathrm{m}} \cdot \pi \cdot \kappa_{\mathrm{m}}} \cdot \frac{\overline{C}_{\mathrm{m}}^{\mathrm{s}} \cdot \bar{J}_0^{\mathrm{s}}\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right) + \overline{D}_{\mathrm{m}}^{\mathrm{s}} \cdot \overline{N}_0^{\mathrm{s}}\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right)}{\overline{C}_{\mathrm{m}}^{\mathrm{s}} \cdot \bar{J}_1^{\mathrm{s}}\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right) + \overline{D}_{\mathrm{m}}^{\mathrm{s}} \cdot \overline{N}_1^{\mathrm{s}}\left(\bar{k}_{\mathrm{m}} \cdot r_{\mathrm{m}}\right)} \tag{17.331}$$

where all the coefficients are scaled by the same factor, whereas the following expressions can be used to scale the complex Bessel functions:

$$\overline{J}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{-\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{J}_{\mathrm{n}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\quad;\quad\overline{J}_{\mathrm{n}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{J}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\tag{17.332}$$

$$\overline{N}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{-\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{N}_{\mathrm{n}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\quad;\quad\overline{N}_{\mathrm{n}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{N}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\tag{17.333}$$

where n = 0, 1. The parameter α_{m} is described by expression (17.310).

For clarity, it is useful to state that for a complex argument $\overline{z}=x+j\cdot y,$ the following expressions hold:

$$\overline{J}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)=\mathrm{e}^{-|y|}\cdot\overline{J}_{\mathrm{n}}\left(\overline{z}\right)\quad;\quad\overline{J}_{\mathrm{n}}\left(\overline{z}\right)=\mathrm{e}^{|y|}\cdot\overline{J}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)\tag{17.334}$$

$$\overline{N}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)=\mathrm{e}^{-|y|}\cdot\overline{N}_{\mathrm{n}}\left(\overline{z}\right)\quad;\quad\overline{N}_{\mathrm{n}}\left(\overline{z}\right)=\mathrm{e}^{|y|}\cdot\overline{N}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)\tag{17.335}$$

where n = 0, 1, 2, ...

If complex modified Bessel functions are used, according to expressions (17.318), (17.322) and (17.329), the per-unit-length internal impedance of an m-layer cylindrical conductor is described by:

$$\overline{Z}_{\mathrm{int}}^{1}=\frac{\overline{\gamma}_{\mathrm{m}}}{2\cdot r_{\mathrm{m}}\cdot\pi\cdot\kappa_{\mathrm{m}}}\cdot\frac{\overline{C}_{\mathrm{m}}\cdot\overline{\mathcal{I}}_{0}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)-\overline{D}_{\mathrm{m}}\cdot\overline{K}_{0}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)}{\overline{C}_{\mathrm{m}}\cdot\overline{\mathcal{I}}_{1}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)+\overline{D}_{\mathrm{m}}\cdot\overline{K}_{1}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)}\tag{17.336}$$

If large values of the arguments also occur, scaled values of the coefficients and scaled values of the complex modified Bessel functions should be used for numerical reasons, so that expression (17.336) takes on a new form:

$$\overline{Z}_{\mathrm{int}}^{1}=\frac{\overline{\gamma}_{\mathrm{m}}}{2\cdot r_{\mathrm{m}}\cdot\pi\cdot\kappa_{\mathrm{m}}}\cdot\frac{\overline{C}_{\mathrm{m}}^{\mathrm{s}}\cdot\overline{\mathcal{I}}_{0}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)-\overline{D}_{\mathrm{m}}^{\mathrm{s}}\cdot\overline{K}_{0}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\cdot\mathrm{e}^{-2\cdot\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}}{\overline{C}_{\mathrm{m}}^{\mathrm{s}}\cdot\overline{\mathcal{I}}_{1}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)+\overline{D}_{\mathrm{m}}^{\mathrm{s}}\cdot\overline{K}_{1}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\cdot\mathrm{e}^{-2\cdot\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}}\tag{17.337}$$

where all the coefficients are scaled by the same factor, whereas the following expressions can be used to scale the complex modified Bessel functions:

$$\overline{\mathcal{I}}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{-\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\quad;\quad\overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{\mathcal{I}}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\tag{17.338}$$

$$\overline{K}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{K}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\quad;\quad\overline{K}_{\mathrm{n}}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)=\mathrm{e}^{-\alpha_{\mathrm{m}}\cdot r_{\mathrm{m}}}\cdot\overline{K}_{\mathrm{n}}^{s}\left(\overline{\gamma}_{\mathrm{m}}\cdot r_{\mathrm{m}}\right)\tag{17.339}$$

where n = 0, 1.

For clarity, it is useful to state that for a complex argument $\overline{z}=x+j\cdot y,$ the following expressions hold:

$$\overline{\mathcal{I}}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)=\mathrm{e}^{-|x|}\cdot\overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{z}\right)\quad;\quad\overline{\mathcal{I}}_{\mathrm{n}}\left(\overline{z}\right)=\mathrm{e}^{|x|}\cdot\overline{\mathcal{I}}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)\tag{17.340}$$

$$\overline{K}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)=\mathrm{e}^{x}\cdot\overline{K}_{\mathrm{n}}\left(\overline{z}\right)\quad;\quad\overline{K}_{\mathrm{n}}\left(\overline{z}\right)=\mathrm{e}^{-x}\cdot\overline{K}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{z}\right)\tag{17.341}$$

where n = 0, 1, 2, ...

Analytical expressions for the unknown coefficients $\overline{C}_{\mathrm{m}}$ and $\overline{D}_{\mathrm{m}}$, or for the unknown coefficients $\overline{C}_{\mathrm{m}}^{\mathrm{s}}$ and $\overline{D}_{\mathrm{m}}^{\mathrm{s}}$, are meaningful to derive only for single-layer (m = 1) and two-layer (m = 2) cylindrical conductors, while for m > 2, a numerical computation based on solving a system of linear equations is recommended. In the following, analytical expressions will be derived only for the unknown coefficients of single-layer cylindrical conductors.

17.24. Internal Impedance of a Single-Layer Solid Cylindrical LIH Conductor

A single-layer solid, infinitely long, cylindrical LIH conductor is a special case of an m-layer solid cylindrical conductor. The conductor parameters are shown in Figure 17.8.

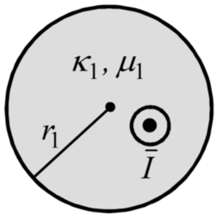


Figure 17.8. Single-layer, infinitely long, solid cylindrical LIH conductor

If complex Bessel functions are used, according to expressions (17.323), the distributions of the magnetic field intensity and the electric field intensity are described by the expressions:

$$\bar{H}_1 = \bar{C}_1 \cdot \bar{J}_1\left(\bar{k}_1 \cdot r\right) \quad ; \quad \bar{E}_1 = \frac{\bar{k}_1}{\kappa_1} \cdot \bar{C}_1 \cdot \bar{J}_0\left(\bar{k}_1 \cdot r\right) \tag{17.342}$$

and, from expression (17.329), for m = 1, it follows that the per-unit-length internal impedance of an infinitely long single-layer solid cylindrical conductor is given by the expression:

$$\bar{Z}_{\text{int}}^1 = \frac{\bar{k}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{J}_0\left(\bar{k}_1 \cdot r_1\right)}{\bar{J}_1\left(\bar{k}_1 \cdot r_1\right)} = \frac{\bar{k}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{J}_0^{\text{s}}\left(\bar{k}_1 \cdot r_1\right)}{\bar{J}_1^{\text{s}}\left(\bar{k}_1 \cdot r_1\right)} \tag{17.343}$$

which is a special case of the expressions (17.330) and (17.331), where $\bar{D}_1 = \bar{D}_1^{\text{s}} = 0.$

If complex modified Bessel functions are used, according to expressions (17.324), the distributions of the magnetic field intensity and the electric field intensity are described by the expressions:

$$\bar{H}_1 = \bar{C}_1 \cdot \bar{\mathcal{J}}_1\left(\bar{\gamma}_1 \cdot r\right) \quad ; \quad \bar{E}_1 = \frac{\bar{\gamma}_1}{\kappa_1} \cdot \bar{C}_1 \cdot \bar{\mathcal{J}}_0\left(\bar{\gamma}_1 \cdot r\right) \tag{17.344}$$

and from expression (17.329), for m = 1, it follows that the per-unit-length internal impedance of an infinitely long single-layer cylindrical conductor is given by:

$$\bar{Z}_{\text{int}}^1 = \frac{\bar{\gamma}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{\mathcal{J}}_0\left(\bar{\gamma}_1 \cdot r_1\right)}{\bar{\mathcal{J}}_1\left(\bar{\gamma}_1 \cdot r_1\right)} = \frac{\bar{\gamma}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{\mathcal{J}}_0^{\text{s}}\left(\bar{\gamma}_1 \cdot r_1\right)}{\bar{\mathcal{J}}_1^{\text{s}}\left(\bar{\gamma}_1 \cdot r_1\right)} \tag{17.345}$$

which is a special case of the expressions (17.336) and (17.337), where $\bar{D}_1 = \bar{D}_1^{\text{s}} = 0.$

In this special case, highly accurate results are obtained for both small and large arguments, regardless of whether complex Bessel functions or complex modified Bessel functions are used.
For very large arguments, the following holds:

$$\bar{Z}_{\text{un}}^1 \approx \frac{\bar{\gamma}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \tag{17.346}$$

because:

$$\lim_{\alpha_1 \to \infty} \frac{\bar{J}_0\left(\bar{k}_1 \cdot r_1\right)}{\bar{J}_1\left(\bar{k}_1 \cdot r_1\right)} = j \quad ; \quad \lim_{\alpha_1 \to \infty} \frac{\bar{\mathcal{J}}_0\left(\bar{\gamma}_1 \cdot r_1\right)}{\bar{\mathcal{J}}_1\left(\bar{\gamma}_1 \cdot r_1\right)} = 1 \quad ; \quad \bar{\gamma}_1 = j \cdot \bar{k}_1 \tag{17.347}$$

17.25. Internal Impedance of a Single-Layer Hollow Cylindrical LIH Conductor

A single-layer hollow infinitely long cylindrical conductor is a special case of an m-layer solid cylindrical conductor. The conductor parameters are shown in Figure 17.9.

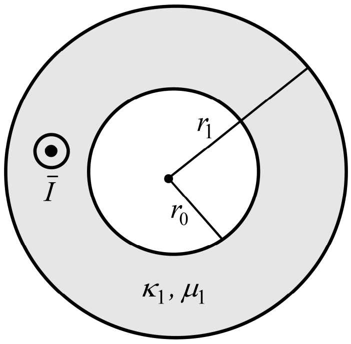


Figure 17.9. Single-layer, infinitely long, hollow cylindrical LIH conductor

For small arguments, that is, for low frequencies, the per-unit-length internal impedance of a hollow cylindrical conductor can be computed with sufficient accuracy using either scaled or unscaled complex Bessel functions. From expression (17.330), it follows that:

$$\bar{Z}_{\text{int}}^{1} = \frac{\bar{k}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{C}_1 \cdot \bar{J}_0(\bar{k}_1 \cdot r_1) + \bar{D}_1 \cdot \bar{N}_0(\bar{k}_1 \cdot r_1)}{\bar{C}_1 \cdot \bar{J}_1(\bar{k}_1 \cdot r_1) + \bar{D}_1 \cdot \bar{N}_1(\bar{k}_1 \cdot r_1)} \tag{17.348}$$

where $\bar{C}_1$ and $\bar{D}_1$ are unknown complex coefficients.

Instead of the unknown coefficients $\bar{C}_1$ and $\bar{D}_1$, their scaled values can be used, so that the expression (17.348) takes on a new form:

$$\bar{Z}_{\text{int}}^{1} = \frac{\bar{k}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{C}_1^{\text{s}} \cdot \bar{J}_0(\bar{k}_1 \cdot r_1) + \bar{D}_1^{\text{s}} \cdot \bar{N}_0(\bar{k}_1 \cdot r_1)}{\bar{C}_1^{\text{s}} \cdot \bar{J}_1(\bar{k}_1 \cdot r_1) + \bar{D}_1^{\text{s}} \cdot \bar{N}_1(\bar{k}_1 \cdot r_1)} \tag{17.349}$$

Moreover, the complex Bessel functions can also be scaled, and it follows that:

$$\bar{Z}_{\text{int}}^{1} = \frac{\bar{k}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\bar{C}_1^{\text{s}} \cdot \bar{J}_0^{\text{s}}(\bar{k}_1 \cdot r_1) + \bar{D}_1^{\text{s}} \cdot \bar{N}_0^{\text{s}}(\bar{k}_1 \cdot r_1)}{\bar{C}_1^{\text{s}} \cdot \bar{J}_1^{\text{s}}(\bar{k}_1 \cdot r_1) + \bar{D}_1^{\text{s}} \cdot \bar{N}_1^{\text{s}}(\bar{k}_1 \cdot r_1)} \tag{17.350}$$

According to expressions (17.314) and (17.317), the distributions of the magnetic field intensity and the electric field intensity are described by the following expressions:

$$\bar{H}_1 = \bar{C}_1 \cdot \bar{J}_1(\bar{k}_1 \cdot r) + \bar{D}_1 \cdot \bar{N}_1(\bar{k}_1 \cdot r) \tag{17.351}$$

$$\bar{E}_1 = \frac{\bar{k}_1}{\kappa_1} \cdot \left[\bar{C}_1 \cdot \bar{J}_0(\bar{k}_1 \cdot r) + \bar{D}_1 \cdot \bar{N}_0(\bar{k}_1 \cdot r)\right] \tag{17.352}$$

Analytical expressions for the coefficients $\bar{C}_1$ and $\bar{D}_1$ can be obtained from the boundary conditions:

$$\bar{H}_1\Big|_{r=r_0} = 0 \quad ; \quad \bar{H}_1\Big|_{r=r_1} = \frac{\bar{I}}{2 \cdot r_1 \cdot \pi} \tag{17.353}$$

From expressions (17.351) and (17.353), the following system of linear equations can be easily derived:

$$\bar{C}_1 \cdot \bar{J}_1(\bar{k}_1 \cdot r_0) + \bar{D}_1 \cdot \bar{N}_1(\bar{k}_1 \cdot r_0) = 0 \tag{17.354}$$

$$\bar{C}_1 \cdot \bar{J}_1(\bar{k}_1 \cdot r_1) + \bar{D}_1 \cdot \bar{N}_1(\bar{k}_1 \cdot r_1) = \frac{\bar{I}}{2 \cdot \pi \cdot r_1} \tag{17.355}$$

which, in matrix notation, is given by:

$$\left[\overline{B}\right]\cdot\begin{Bmatrix}\overline{C}_1\\ \overline{D}_1\end{Bmatrix}=\begin{Bmatrix}0\\ \dfrac{\overline{I}}{2\cdot\pi\cdot r_1}\end{Bmatrix}\quad ;\quad \left[\overline{B}\right]=\begin{bmatrix}\overline{J}_1\left(\overline{k}_1\cdot r_0\right) & \overline{N}_1\left(\overline{k}_1\cdot r_0\right)\\ \overline{J}_1\left(\overline{k}_1\cdot r_1\right) & \overline{N}_1\left(\overline{k}_1\cdot r_1\right)\end{bmatrix} \tag{17.356}$$

According to Cramer's rule, the solution of this system of linear equations in matrix form is given by:

$$\overline{C}_1=\frac{\begin{vmatrix}0 & \overline{N}_1\left(\overline{k}_1\cdot r_0\right)\\ \dfrac{\overline{I}}{2\cdot\pi\cdot r_1} & \overline{N}_1\left(\overline{k}_1\cdot r_1\right)\end{vmatrix}}{\det\overline{B}}\quad ;\quad \overline{D}_1=\frac{\begin{vmatrix}\overline{J}_1\left(\overline{k}_1\cdot r_0\right) & 0\\ \overline{J}_1\left(\overline{k}_1\cdot r_1\right) & \dfrac{\overline{I}}{2\cdot\pi\cdot r_1}\end{vmatrix}}{\det\overline{B}} \tag{17.357}$$

from which it follows that:

$$\overline{C}_1=-\frac{\overline{I}}{2\cdot\pi\cdot r_1}\cdot\frac{1}{\det\overline{\overline{\mathrm{B}}}}\cdot\overline{N}_1\left(\overline{k}_1\cdot r_0\right)=\frac{\overline{I}}{2\cdot\pi\cdot r_1}\cdot\frac{1}{\det\overline{\overline{\mathrm{B}}}}\cdot\overline{C}_1^{\mathrm{s}}=\overline{M}\cdot\overline{C}_1^{\mathrm{s}} \tag{17.358}$$

$$\overline{D}_1=\frac{\overline{I}}{2\cdot\pi\cdot r_1}\cdot\frac{1}{\det\overline{\overline{\mathrm{B}}}}\cdot\overline{J}_1\left(\overline{k}_1\cdot r_0\right)=\frac{\overline{I}}{2\cdot\pi\cdot r_1}\cdot\frac{1}{\det\overline{\overline{\mathrm{B}}}}\cdot\overline{D}_1^{s}=\overline{M}\cdot\overline{D}_1^{s} \tag{17.359}$$

The coefficients are scaled by subtracting the common part $\overline{M}$, described by the expression:

$$\overline{M}=\frac{\overline{I}}{2\cdot\pi\cdot r_1}\cdot\frac{1}{\det\overline{\overline{\mathrm{B}}}} \tag{17.360}$$

from which it follows that the scaled coefficients are described by the expressions:

$$\overline{C}_1^{\mathrm{s}}=-\overline{N}_1\left(\overline{k}_1\cdot r_0\right)\quad ;\quad \overline{D}_1^{\mathrm{s}}=\overline{J}_1\left(\overline{k}_1\cdot r_0\right) \tag{17.361}$$

From expressions (17.349) and (17.361), with a change of sign in both the numerator and the denominator, it follows that the per-unit-length internal impedance of the hollow cylindrical conductor is described by the expression:

$$\overline{Z}_{\mathrm{int}}^{1}=\frac{\overline{k}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{N}_1\left(\overline{k}_1\cdot r_0\right)\cdot\overline{J}_0\left(\overline{k}_1\cdot r_1\right)-\overline{J}_1\left(\overline{k}_1\cdot r_0\right)\cdot\overline{N}_0\left(\overline{k}_1\cdot r_1\right)}{\overline{N}_1\left(\overline{k}_1\cdot r_0\right)\cdot\overline{J}_1\left(\overline{k}_1\cdot r_1\right)-\overline{J}_1\left(\overline{k}_1\cdot r_0\right)\cdot\overline{N}_1\left(\overline{k}_1\cdot r_1\right)} \tag{17.362}$$

According to expressions (17.332) - (17.335), the complex Bessel functions of the first and second kinds are scaled as follows:

$$\overline{J}_{\mathrm{n}}\left(\overline{k}_1\cdot r\right)=\mathrm{e}^{\alpha_1\cdot r}\cdot\overline{J}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_1\cdot r\right)\quad ;\quad \overline{N}_{\mathrm{n}}\left(\overline{k}_1\cdot r\right)=\mathrm{e}^{\alpha_1\cdot r}\cdot\overline{N}_{\mathrm{n}}^{\mathrm{s}}\left(\overline{k}_1\cdot r\right) \tag{17.363}$$

and after their scaling, expression (17.362) takes its final form:

$$\overline{Z}_{\mathrm{int}}^{1}=\frac{\overline{k}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{N}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_0\right)\cdot\overline{J}_0^{\mathrm{s}}\left(\overline{k}_1\cdot r_1\right)-\overline{J}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_0\right)\cdot\overline{N}_0^{\mathrm{s}}\left(\overline{k}_1\cdot r_1\right)}{\overline{N}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_0\right)\cdot\overline{J}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_1\right)-\overline{J}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_0\right)\cdot\overline{N}_1^{\mathrm{s}}\left(\overline{k}_1\cdot r_1\right)} \tag{17.364}$$

However, even after scaling all complex Bessel functions, the final expression (17.364) does not provide sufficiently accurate results for large arguments. In such cases, numerical problems arise because the computation yields an indeterminate form of 0/0.

The per-unit-length internal impedance of the hollow cylindrical conductor (Figure 17.9) can be computed with sufficient accuracy using complex modified Bessel functions. From expression (17.336), it follows that:

$$\overline{Z}_{\mathrm{int}}^{1}=\frac{\overline{\gamma}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{C}_1\cdot\overline{\mathcal{I}}_0\left(\overline{\gamma}_1\cdot r_1\right)-\overline{D}_1\cdot\overline{K}_0\left(\overline{\gamma}_1\cdot r_1\right)}{\overline{C}_1\cdot\overline{\mathcal{I}}_1\left(\overline{\gamma}_1\cdot r_1\right)+\overline{D}_1\cdot\overline{K}_1\left(\overline{\gamma}_1\cdot r_1\right)} \tag{17.365}$$

where $\overline{C}_1$ and $\overline{D}_1$ are unknown complex coefficients.

Instead of the unknown coefficients $\overline{C}_1$ and $\overline{D}_1$, their scaled values can be used, so that the expression (17.365) takes on a new form:

$$\overline{Z}_{\text{int}}^1 = \frac{\overline{\gamma}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\overline{C}_1^{\text{s}} \cdot \overline{\mathcal{I}}_0(\overline{\gamma}_1 \cdot r_1) - \overline{D}_1^{\text{s}} \cdot \overline{K}_0(\overline{\gamma}_1 \cdot r_1)}{\overline{C}_1^{\text{s}} \cdot \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_1) + \overline{D}_1^{\text{s}} \cdot \overline{K}_1(\overline{\gamma}_1 \cdot r_1)} \tag{17.366}$$

Moreover, the complex modified Bessel functions can also be scaled, and it follows that:

$$\overline{Z}_{\text{int}}^1 = \frac{\overline{\gamma}_1}{2 \cdot r_1 \cdot \pi \cdot \kappa_1} \cdot \frac{\overline{C}_1^{\text{s}} \cdot \overline{\mathcal{I}}_0^{\text{s}}(\overline{\gamma}_1 \cdot r_1) - \overline{D}_1^{\text{s}} \cdot \overline{K}_0^{\text{s}}(\overline{\gamma}_1 \cdot r_1)}{\overline{C}_1^{\text{s}} \cdot \overline{\mathcal{I}}_1^{\text{s}}(\overline{\gamma}_1 \cdot r_1) + \overline{D}_1^{\text{s}} \cdot \overline{K}_1^{\text{s}}(\overline{\gamma}_1 \cdot r_1)} \tag{17.367}$$

According to expressions (17.318) and (17.322), the distributions of the magnetic field intensity and the electric field intensity are described by the following expressions:

$$\overline{H}_1 = \overline{C}_1 \cdot \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r) + \overline{D}_1 \cdot \overline{K}_1(\overline{\gamma}_1 \cdot r) \tag{17.368}$$

$$\overline{E}_1 = \frac{\overline{\gamma}_1}{\kappa_1} \cdot \left[\overline{C}_1 \cdot \overline{\mathcal{I}}_0(\overline{\gamma}_1 \cdot r) - \overline{D}_1 \cdot \overline{K}_0(\overline{\gamma}_1 \cdot r)\right] \tag{17.369}$$

Analytical expressions for the coefficients $\overline{C}_1$ and $\overline{D}_1$ can be obtained from the boundary conditions (17.353). The following system of linear equations results from the expressions (17.368) and (17.353):

$$\overline{C}_1 \cdot \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_0) + \overline{D}_1 \cdot \overline{K}_1(\overline{\gamma}_1 \cdot r_0) = 0 \tag{17.370}$$

$$\overline{C}_1 \cdot \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_1) + \overline{D}_1 \cdot \overline{K}_1(\overline{\gamma}_1 \cdot r_1) = \frac{\overline{I}}{2 \cdot \pi \cdot r_1} \tag{17.371}$$

which, in matrix notation, is given by:

$$\left[\overline{B}\right] \cdot \begin{Bmatrix} \overline{C}_1 \\ \overline{D}_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ \dfrac{\overline{I}}{2 \cdot \pi \cdot r_1} \end{Bmatrix} \quad ; \quad \left[\overline{B}\right] = \begin{bmatrix} \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_0) & \overline{K}_1(\overline{\gamma}_1 \cdot r_0) \\ \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_1) & \overline{K}_1(\overline{\gamma}_1 \cdot r_1) \end{bmatrix} \tag{17.372}$$

According to Cramer's rule, the solution of this system of linear equations in matrix form is given by:

$$\overline{C}_1 = \frac{\begin{vmatrix} 0 & \overline{K}_1(\overline{\gamma}_1 \cdot r_0) \\ \dfrac{\overline{I}}{2 \cdot \pi \cdot r_1} & \overline{K}_1(\overline{\gamma}_1 \cdot r_1) \end{vmatrix}}{\det \overline{B}} \quad ; \quad \overline{D}_1 = \frac{\begin{vmatrix} \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_0) & 0 \\ \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_1) & \dfrac{\overline{I}}{2 \cdot \pi \cdot r_1} \end{vmatrix}}{\det \overline{B}} \tag{17.373}$$

from which it follows that:

$$\overline{C}_1 = -\frac{\overline{I}}{2 \cdot \pi \cdot r_1} \cdot \frac{1}{\det \overline{\mathrm{B}}} \cdot \overline{K}_1(\overline{\gamma}_1 \cdot r_0) = \frac{\overline{I}}{2 \cdot \pi \cdot r_1} \cdot \frac{1}{\det \overline{\mathrm{B}}} \cdot \overline{C}_1^{\text{s}} = \overline{M} \cdot \overline{C}_1^{\text{s}} \tag{17.374}$$

$$\overline{D}_1 = \frac{\overline{I}}{2 \cdot \pi \cdot r_1} \cdot \frac{1}{\det \overline{\mathrm{B}}} \cdot \overline{\mathcal{I}}_1(\overline{\gamma}_1 \cdot r_0) = \frac{\overline{I}}{2 \cdot \pi \cdot r_1} \cdot \frac{1}{\det \overline{\mathrm{B}}} \cdot \overline{D}_1^{\text{s}} = \overline{M} \cdot \overline{D}_1^{\text{s}} \tag{17.375}$$

The coefficients are scaled by subtracting the common part $\overline{M}$, described by the expression (17.360), so that the scaled coefficients are given by the expressions:

$$\overline{C}_1^{s} = -\,\overline{K}_1\left(\overline{\gamma}_1\cdot r_0\right) \quad ; \quad \overline{D}_1^{s} = \overline{\mathcal{I}}_1\left(\overline{\gamma}_1\cdot r_0\right) \tag{17.376}$$

From expressions (17.366) and (17.376), with a change of sign in both the numerator and the denominator, it follows that the per-unit-length internal impedance of the hollow cylindrical conductor is described by the expression:

$$\overline{Z}_{\text{int}}^{1} = \frac{\overline{\gamma}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{K}_1\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{\mathcal{I}}_0\left(\overline{\gamma}_1\cdot r_1\right)+\overline{\mathcal{I}}_1\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{K}_0\left(\overline{\gamma}_1\cdot r_1\right)}{\overline{K}_1\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{\mathcal{I}}_1\left(\overline{\gamma}_1\cdot r_1\right)-\overline{\mathcal{I}}_1\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{K}_1\left(\overline{\gamma}_1\cdot r_1\right)} \tag{17.377}$$

According to expressions (17.338) - (17.341), the complex modified Bessel functions of the first and second kinds are scaled as follows:

$$\overline{\mathcal{I}}_{\text{n}}\left(\overline{\gamma}_1\cdot r\right)=\text{e}^{\alpha_1\cdot r}\cdot\overline{\mathcal{I}}_{\text{n}}^{s}\left(\overline{\gamma}_1\cdot r\right) \quad ; \quad \overline{K}_{\text{n}}\left(\overline{\gamma}_1\cdot r\right)=\text{e}^{-\alpha_1\cdot r}\cdot\overline{K}_{\text{n}}^{s}\left(\overline{\gamma}_1\cdot r\right) \tag{17.378}$$

and after their scaling, expression (17.377) takes its final form:

$$\overline{Z}_{\text{int}}^{1} = \frac{\overline{\gamma}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{K}_1^{s}\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{\mathcal{I}}_0^{s}\left(\overline{\gamma}_1\cdot r_1\right)+\overline{\mathcal{I}}_1^{s}\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{K}_0^{s}\left(\overline{\gamma}_1\cdot r_1\right)\cdot e^{-2\cdot\alpha_1\cdot(r_1-r_0)}}{\overline{K}_1^{s}\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{\mathcal{I}}_1^{s}\left(\overline{\gamma}_1\cdot r_1\right)-\overline{\mathcal{I}}_1^{s}\left(\overline{\gamma}_1\cdot r_0\right)\cdot\overline{K}_1^{s}\left(\overline{\gamma}_1\cdot r_1\right)\cdot e^{-2\cdot\alpha_1\cdot(r_1-r_0)}} \tag{17.379}$$

Expression (17.379) provides highly accurate results for all values of the arguments.
For large arguments, the following holds:

$$\lim_{\alpha_1\to\infty} e^{-2\cdot\alpha_1\cdot(r_1-r_0)} = 0 \tag{17.380}$$

and, thus, for large arguments:

$$\overline{Z}_{\text{int}}^{1} \approx \frac{\overline{\gamma}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1}\cdot\frac{\overline{\mathcal{I}}_0^{s}\left(\overline{\gamma}_1\cdot r_1\right)}{\overline{\mathcal{I}}_1^{s}\left(\overline{\gamma}_1\cdot r_1\right)} \tag{17.381}$$

which means that the per-unit-length internal impedance of the hollow cylindrical conductor, for large arguments, can be approximated by the expression that, according to expression (17.345), describes the per-unit-length internal impedance of a solid cylindrical conductor. This indicates a considerable redistribution of the electric current towards the surface of the conductor.
For very large arguments, the approximation (17.346) can be used, which is given by:

$$\overline{Z}_{\text{int}}^{1} \approx \frac{\overline{\gamma}_1}{2\cdot r_1\cdot\pi\cdot\kappa_1} \tag{17.382}$$

because:

$$\lim_{\alpha_1\to\infty}\frac{\overline{\mathcal{I}}_0^{s}\left(\overline{\gamma}_1\cdot r_1\right)}{\overline{\mathcal{I}}_1^{s}\left(\overline{\gamma}_1\cdot r_1\right)} = 1 \tag{17.383}$$

17.26. Solved Examples

Example 17.1. Determine the maximum value of the displacement electric current density in a copper conductor with a cross-sectional area of S = 1 mm^2 through which a sinusoidal electric current flows with an RMS value of I = 1 A and a frequency f = 50 Hz. Neglect the skin effect. The properties of the material of which the conductor is made are given as: $\kappa = 58$ MS/m, $\varepsilon = \varepsilon_0$, $\mu = \mu_0$.

Solution:

If the skin effect is neglected, then the sinusoidal electric field intensity in the copper conductor is described by the expression:

$$E = E_{max} \cdot \cos(\omega \cdot t) \tag{17.384}$$

It follows that the surface density of the conduction electric current is given by the expression:

$$J_c = \kappa \cdot E = \kappa \cdot E_{max} \cdot \cos(\omega \cdot t) = J_{c\,max} \cdot \cos(\omega \cdot t) \tag{17.385}$$

The surface density of the displacement electric current is given by the expression:

$$J_{disp} = \frac{\partial D}{\partial t} = \varepsilon_0 \cdot \frac{\partial E}{\partial t} = -\,\omega \cdot \varepsilon_0 \cdot E_{max} \cdot \sin(\omega \cdot t) \tag{17.386}$$

and, thus, the maximum value of the surface density of the displacement electric current is:

$$J_{disp\,max} = \omega \cdot \varepsilon_0 \cdot E_{max} = \frac{\omega \cdot \varepsilon_0}{\kappa} \cdot J_{c\,max} \tag{17.387}$$

It follows that the ratio of the maximum values of the surface density of the displacement electric current and the surface density of the conduction electric current is:

$$\frac{J_{disp\,max}}{J_{c\,max}} = \frac{\omega \cdot \varepsilon_0}{\kappa} = 4.7985905414 \times 10^{-17} \tag{17.388}$$

With the introduced neglect, the maximum value of the surface density of the displacement electric current is:

$$J_{disp\,max} = \frac{\omega \cdot \varepsilon_0}{\kappa} \cdot J_{c\,max} = \frac{\omega \cdot \varepsilon_0}{\kappa} \cdot \frac{\sqrt{2} \cdot I}{S} = 6.78243448 \times 10^{-11} \ \frac{\text{A}}{\text{m}^2} \tag{17.389}$$

Example 17.2. Determine the expression for the induced EMF in a rectangular loop located in a homogeneous sinusoidal magnetic field. The magnetic flux density is described by the expression: $\vec{B} = B_{max} \cdot \cos(\omega \cdot t) \cdot \vec{j}$. Assume that the loop is cut at one point so that no electric current can flow through it.

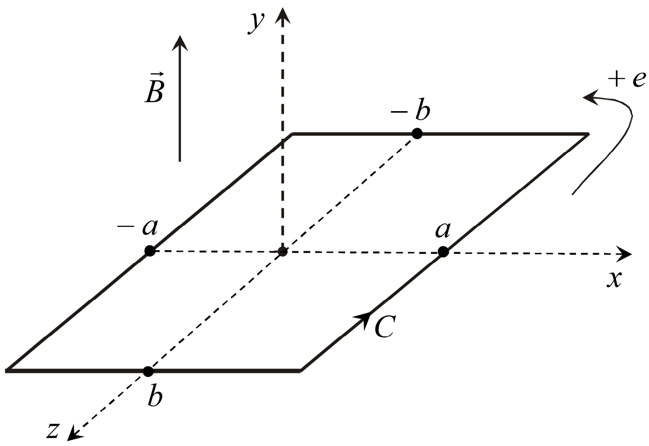


Figure 17.10. Rectangular loop in a homogeneous sinusoidal magnetic field

Solution:

The induced EMF in the rectangular loop is described by Maxwell's second integral equation (6.73), which is given by:

$$e = \oint_C \vec{E} \cdot d\vec{\ell} = -\int_S \frac{\partial \vec{B}}{\partial t} \cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{17.390}$$

where S is the surface enclosed by the loop through which the magnetic flux Φ flows.

Since, in this case, the magnetic field is homogeneous, it follows that:

$$e = -4 \cdot a \cdot b \cdot \frac{\partial B}{\partial t} = 4 \cdot a \cdot b \cdot \omega \cdot B_{\max} \cdot \sin(\omega \cdot t) \tag{17.391}$$

Example 17.3. A sinusoidal electric current $i = \sin(100 \cdot \pi t)$ A flows through an infinitely long straight thin-wire conductor. Determine the expression for the induced EMF in a square loop moving with a velocity v =1 m/s parallel to the conductor. Assume that both the loop and the conductor are in the air. Assume that the loop is cut at one point so that no electric current can flow through it. Given parameters: a = 0.2 m, b = 0.1 m.

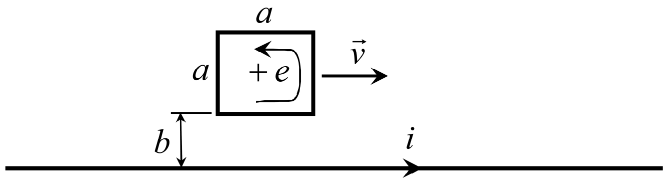


Figure 17.11. An infinitely long straight thin-wire conductor and a square loop

Solution:

The magnetic flux through a square loop, which is not affected by the given motion of the square loop, is described by the expression:

$$\Psi = \Phi = \frac{\mu_0 \cdot i}{2 \cdot \pi} \cdot \int_b^{a+b} \frac{a \cdot dr}{r} = \frac{\mu_0 \cdot a}{2 \cdot \pi} \cdot \ln \frac{b+a}{b} \cdot \sin(100 \cdot \pi \cdot t) \tag{17.392}$$

thus the induced EMF is given by the transformation:

$$e = -\frac{d\Phi}{dt} = -50 \cdot \mu_0 \cdot a \cdot \ln \frac{b+a}{b} \cdot \cos(100 \cdot \pi \cdot t) \tag{17.393}$$

By substituting the given data, it follows that:

$$e = -13.80556918 \cdot \cos(100 \cdot \pi \cdot t) \ \mu\text{V} \tag{17.394}$$

The induced EMF does not depend on the motion of the loop because the magnetic flux through the square loop does not depend on the velocity v.

Example 17.4. The square loop has the side length a = 1 m and lies in the plane (x, y). It moves uniformly along a straight line in the direction of the x-axis with a velocity of v = 2 m/s. Determine the expression for the induced EMF in the square loop. Assume that the loop is cut at one point so that no electric current can flow through it. At the initial time t = 0, let b = 0, and let the frequency of the sinusoidal magnetic flux density be f = 50 Hz. The sinusoidal magnetic flux density is described by the following expression:

$$\vec{B} = 0{,}01 \cdot \left[(x+1) \cdot \vec{\mathrm{i}} + (y+1) \cdot \vec{\mathrm{j}} + \left(x^2 - y^2\right) \cdot \vec{\mathrm{k}}\right] \cdot \sin(\omega \cdot t) \ \text{ T.}$$

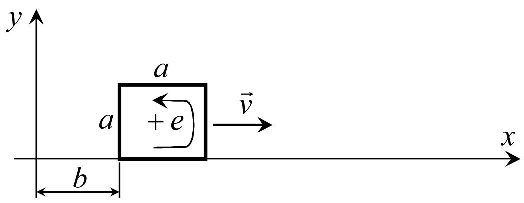


Figure 17.12. A square loop moving along the x-axis

Solution:

The magnetic flux through the square loop is generated only by the z-component of the magnetic flux density, so the magnetic flux is described by the following expression:

$$\Psi = \Phi = \int_S \vec{B} \cdot d\vec{S} = \int_S B_z \cdot dx \cdot dy \tag{17.395}$$

where:

$$B_z = 0.01 \cdot \left(\mathrm{x}^2 - \mathrm{y}^2\right) \cdot \sin\left(\omega \cdot t\right) \tag{17.396}$$

and it is:

$$\Phi = 0.01 \cdot \sin\left(\omega \cdot t\right) \cdot \int_b^{b+a} dx \int_0^a \left(x^2 - y^2\right) \cdot dy = 0.01 \cdot a^2 \cdot b \cdot \left(b + a\right) \cdot \sin\left(\omega \cdot t\right) \tag{17.397}$$

The parameter b is described by the following expression::

$$b = v \cdot t \tag{17.398}$$

By substituting the given data, it follows that:

$$\Phi = 0.02 \cdot \left(2 \cdot t^2 + t\right) \cdot \sin\left(100 \cdot \pi \cdot t\right) \ \text{Vs} \tag{17.399}$$

The induced EMF is described by the following expression:

$$e = -\frac{d\Phi}{dt} = -\,0.02 \cdot \left(4 \cdot t + 1\right) \cdot \sin\left(100 \cdot \pi \cdot t\right) - 2 \cdot \pi \cdot \left(2 \cdot t^2 + \ t\right) \cdot \cos\left(100 \cdot \pi \cdot t\right) \tag{17.400}$$

Example 17.5. Two mutually parallel, infinitely long, straight conductors, which are in the air and separated by a distance $d = 5 \cdot a$, carry a sinusoidal electric current $i = I_{\max} \cdot \cos(\omega \cdot t)$. There is a square loop with side length a in the plane in which the conductors lie. The square loop moves uniformly along a straight line with velocity v. Calculate the induced EMF in the square loop for $b = a$ and $t = 1$ s. Assume that the loop is cut at one point so that no electric current can flow through it. The given parameters are: $a = 0.2$ m, $f = 50$ Hz, $v = 0.2$ m/s, $I_{\max} = 10$ A.

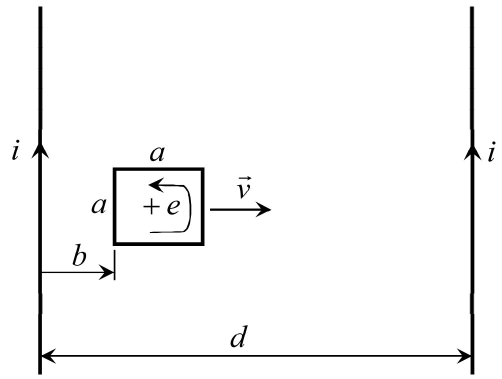


Figure 17.13. A square loop moving between two conductors

Solution:

Let the EMF induced in the square loop be described by the following expression:

$$e = +\frac{d\Psi}{dt} = \frac{d\Phi}{dt} = \frac{\partial\Phi}{\partial t} + \frac{\partial\Phi}{\partial b} \cdot \frac{\partial b}{\partial t} = \frac{\partial\Phi}{\partial t} + v \cdot \frac{\partial\Phi}{\partial b} = e_{\mathrm{tr}} + e_{\mathrm{mo}} \tag{17.401}$$

from which it follows that the magnetic flux is described by the following expression:

$$\Phi = \frac{\mu_0 \cdot i}{2\cdot\pi}\cdot\left(\int_b^{b+a}\frac{a\cdot dr}{r} - \int_{d-b-a}^{d-b}\frac{a\cdot dr}{r}\right) = \frac{\mu_0\cdot i\cdot a}{2\cdot\pi}\cdot\ln\frac{(b+a)\cdot(d-b-a)}{b\cdot(d-b)} \tag{17.402}$$

and it is:

$$\Phi = \frac{\mu_0\cdot I_{\max}\cdot a\cdot\cos(\omega\cdot t)}{2\cdot\pi}\cdot\ln\frac{(b+a)\cdot(d-b-a)}{b\cdot(d-b)} \tag{17.403}$$

It follows that:

$$e_{\mathrm{tr}} = \frac{\partial\Phi}{\partial t} = -\frac{\mu_0\cdot I_{\max}\cdot a\cdot\omega\cdot\sin(\omega\cdot t)}{2\cdot\pi}\cdot\ln\frac{(b+a)\cdot(d-b-a)}{b\cdot(d-b)} \tag{17.404}$$

$$e_{\mathrm{mo}} = v\cdot\frac{\partial\Phi}{\partial b} = v\cdot\frac{\mu_0\cdot I_{\max}\cdot a\cdot\cos(\omega\cdot t)}{2\cdot\pi}\cdot\left(\frac{1}{b+a} - \frac{1}{d-b-a} - \frac{1}{b} + \frac{1}{d-b}\right) \tag{17.405}$$

By substituting the given data, it follows that for $b = a = 0.2$ m, $t = 1$ s, the induced EMF is:

$$e_{\mathrm{tr}} = 0 \quad ; \quad e_{\mathrm{mo}} = -\,0.2333333333\ \mu\mathrm{V} \quad ; \quad e = e_{\mathrm{tr}} + e_{\mathrm{mo}} = -\,0.2333333333\ \mu\mathrm{V} \tag{17.406}$$

Example 17.6. An infinitely long straight conductor carries an electric current $i = \sin(100\cdot\pi\cdot t)$ A. Determine the expression for the induced EMF in a triangular loop moving with a velocity $v = 1$ m/s. Let the initial position be $b = b_0 = 0{,}2$ m, and $a = 0{,}15$ m, $\alpha = 30^{\circ}$. Assume that the loop is cut at one point so that no electric current can flow through it.

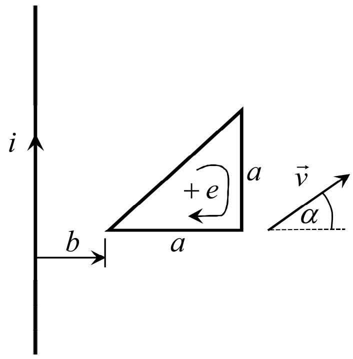


Figure 17.14. An infinitely long conductor and a triangular loop

Solution:

The magnetic flux through the triangular loop, which is affected only by the motion of the loop perpendicular to the straight conductor, is described by the following expression:

$$\Psi = \Phi = \int_S \vec{B}\cdot d\vec{S} = \frac{\mu_0\cdot i}{2\cdot\pi}\cdot\int_b^{b+a}\frac{1}{r}\cdot c\cdot dr \quad ; \quad c = r - b \tag{17.407}$$

from which it follows that:

$$\Phi = \frac{\mu_0\cdot i}{2\cdot\pi}\cdot\int_b^{b+a}\left(1 - \frac{b}{r}\right)\cdot dr = \frac{\mu_0\cdot i}{2\cdot\pi}\cdot\left(a - b\cdot\ln\frac{b+a}{b}\right) \tag{17.408}$$

By substituting the given expression for the electric current:

$$i = \sin(100\cdot\pi\cdot \mathrm{t}) \tag{17.409}$$

into equation (17.408), the magnetic flux is then described by the following expression:

$$\Phi = \frac{\mu_0}{2\cdot\pi}\cdot\left(a - b\cdot\ln\frac{b+a}{b}\right)\cdot\sin(100\cdot\pi\cdot \mathrm{t}) \tag{17.410}$$

where:

$$b = b_0 + v\cdot t\cdot\cos\alpha = 0.2 + \frac{\sqrt{3}}{2}\cdot t \tag{17.411}$$

The induced EMF, the direction of which is indicated in Figure 17.14, is described by the following expression:

$$e = -\frac{d\Phi}{dt} = -\frac{\partial\Phi}{\partial t} - \frac{\partial\Phi}{\partial b}\cdot\frac{\partial b}{\partial t} = -\frac{\partial\Phi}{\partial t} - \frac{\sqrt{3}}{2}\cdot\frac{\partial\Phi}{\partial b} = e_{\mathrm{tr}} + e_{\mathrm{mo}} \tag{17.412}$$

where:

$$e_{\mathrm{tr}} = -\frac{\partial\Phi}{\partial t} = -50\cdot\mu_0\cdot\left(a - b\cdot\ln\frac{b+a}{b}\right)\cdot\cos(100\cdot\pi\cdot \mathrm{t}) \tag{17.413}$$

$$e_{\mathrm{mo}} = -\frac{\sqrt{3}}{2}\cdot\frac{\partial\Phi}{\partial b} = -\frac{\sqrt{3}\cdot\mu_0}{4\cdot\pi}\cdot\left(\frac{a}{b+a} + \ln\frac{b}{b+a}\right)\cdot\sin(100\cdot\pi\cdot \mathrm{t}) \tag{17.414}$$

Example 17.7. In the plane $z = 0$, there is a thin-wire circular loop with a radius $a = 1$ cm, centered at the origin of the coordinate system, carrying a sinusoidal electric current $i = 100\cdot\sin(200\cdot t)$ A. Another loop is located at the intersection of a spherical surface of radius $R = 1$ m with the coordinate planes $z = 0$ and $y = 0$. Derive the expression for the induced EMF in the large loop, considering the positively selected direction of the induced EMF. Assume that the small loop behaves as a magnetic dipole. Also, assume that the large loop is cut at one point so that no electric current can flow through it.

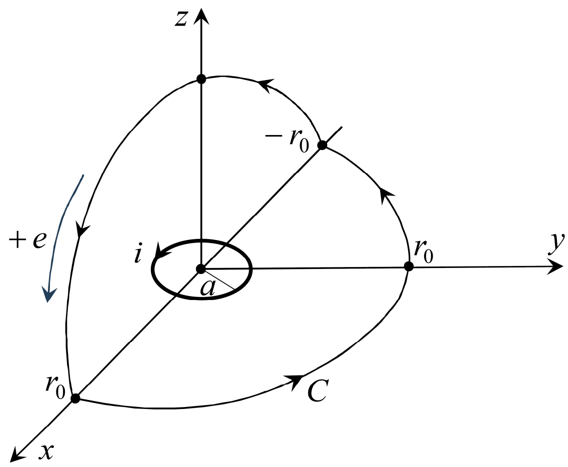


Figure 17.15. Magnetic dipole and integration curve C

Solution:

In the spherical coordinate system (r, ϑ, φ), the magnetic vector potential generated by the magnetic dipole is described by the following expression:

$$\vec{A} = \frac{\mu_0\cdot m\cdot\sin\vartheta}{4\cdot\pi\cdot r^2}\cdot\vec{\mathrm{e}}_\varphi \quad ; \quad m = i\cdot a^2\cdot\pi \tag{17.415}$$

From the expression:

$$\vec{B} = \nabla \times \vec{A} = \frac{1}{r^2 \cdot \sin\vartheta} \cdot \begin{vmatrix} \vec{e}_r & r \cdot \vec{e}_\vartheta & r \cdot \sin\vartheta \cdot \vec{e}_\varphi \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \vartheta} & 0 \\ 0 & 0 & r \cdot \sin\vartheta \cdot A \end{vmatrix} \tag{17.416}$$

it follows that the components of the magnetic flux density vector are described by the following expressions:

$$B_r = \frac{\mu \cdot m \cdot \cos\vartheta}{2 \cdot \pi \cdot r^3} \quad ; \quad B_\vartheta = \frac{\mu \cdot m \cdot \sin\vartheta}{4 \cdot \pi \cdot r^3} \quad ; \quad B_\varphi = 0 \tag{17.417}$$

The integration curve C encloses one quarter of the spherical surface, which simplifies the calculation of the induced EMF. The magnetic flux through this quarter of the spherical surface, due to the electric current of the magnetic dipole, is described by the following expression:

$$\Psi = \Phi = \int_S \vec{B} \cdot d\vec{S} = \int_0^\pi d\varphi \cdot \int_0^{\pi/2} B_r\big|_{r=r_0} \cdot r_0^2 \cdot \sin\vartheta \cdot d\vartheta \tag{17.418}$$

from which it follows that:

$$\Phi = \frac{\mu_0 \cdot m}{2 \cdot r_0} \cdot \int_0^{\pi/2} \sin\vartheta \cdot \cos\vartheta \cdot d\vartheta = \frac{\mu_0 \cdot m}{4 \cdot r_0} = \frac{\mu_0 \cdot a^2 \cdot \pi}{4 \cdot r_0} \cdot i \tag{17.419}$$

Since the electric current of the magnetic dipole is described by the following expression:

$$i = 100 \cdot \sin(200 \cdot t) \ \text{A} \tag{17.420}$$

it follows that the magnetic flux is described by the following expression:

$$\Phi = \frac{25 \cdot \mu_0 \cdot a^2 \cdot \pi}{r_0} \cdot \sin(200 \cdot t) \tag{17.421}$$

The induced EMF in the large loop is described by the following expression:

$$e = -\frac{\partial \Phi}{\partial t} = -\frac{5000 \cdot \mu_0 \cdot a^2 \cdot \pi}{r_0} \cdot \cos(200 \cdot t) \tag{17.422}$$

By substituting the given data into equation (17.422), it follows that:

$$e = -1.97392088 \cdot \cos(200 \cdot t) \ \mu\text{V} \tag{17.423}$$

Example 17.8. A solid cylindrical ferromagnetic LIH core of radius r_0 and length ℓ is uniformly wound with a solenoid having a total of N turns. Let a sinusoidal electric current with an RMS value I and frequency f flow through the solenoid. Let the characteristics of the ferromagnetic core be denoted by κ and μ. Determine the expressions for the distribution of the electric and magnetic fields inside the core, as well as the expression for the equivalent impedance of the core. Neglect displacement electric currents and the magnetic reluctance of the surrounding space.

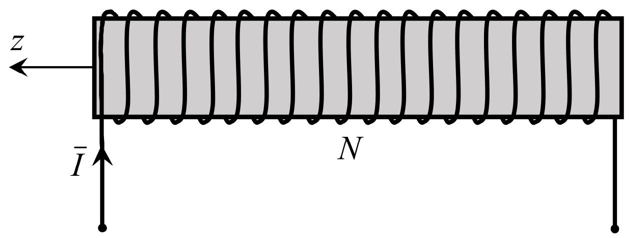


Figure 17.16. Straight solenoid wound on a solid ferromagnetic core

Solution:

Since there is axial symmetry with respect to the z-axis, the problem is solved in the cylindrical coordinate system (r, φ, z). Neglecting displacement electric currents, the following relationship holds for the sinusoidal electromagnetic field in the core, in the phasor domain:

$$\Delta \underline{\vec{H}} = \bar{\gamma}^2 \cdot \underline{\vec{H}} \quad ; \quad \Delta \underline{\vec{E}} = \bar{\gamma}^2 \cdot \underline{\vec{E}} \tag{17.424}$$

where the wave constant of the LIH ferromagnetic material (good conductor) is described by the following expression:

$$\bar{\gamma} = \alpha + j \cdot \beta = \sqrt{\omega \cdot \mu \cdot \kappa} \cdot e^{j \cdot \pi/4} = \sqrt{\frac{\omega \cdot \mu \cdot \kappa}{2}} \cdot (1 + j) \tag{17.425}$$

Since the magnetic reluctance of the surrounding medium is neglected, the electric field intensity and the magnetic field intensity each have only one component:

$$\underline{\vec{H}} = \overline{H}_z \cdot \vec{e}_z = \overline{H} \cdot \vec{e}_z \quad ; \quad \underline{\vec{E}} = \overline{E}_\varphi \cdot \vec{e}_\varphi = \overline{E} \cdot \vec{e}_\varphi \tag{17.426}$$

The magnetic field intensity has only a z-component, and the unit vector of the z-component does not change direction in space, since z is a straight coordinate axis; thus, the following expression holds:

$$\Delta \overline{H} = \frac{1}{r} \cdot \frac{\partial}{\partial r}\left(r \cdot \frac{\partial \overline{H}}{\partial r} \right) = \bar{\gamma}^2 \cdot \overline{H} \tag{17.427}$$

from which the modified Bessel differential equation is easily obtained:

$$r^2 \cdot \frac{\partial^2 \overline{H}}{\partial r^2} + r \cdot \frac{\partial \overline{H}}{\partial r} - \bar{\gamma}^2 \cdot r^2 \cdot \overline{H} = 0 \tag{17.428}$$

which is a special case ($n = 0$) of the modified Bessel differential equation:

$$r^2 \cdot \frac{\partial^2 \overline{H}}{\partial r^2} + r \cdot \frac{\partial \overline{H}}{\partial r} - \left(\bar{\gamma}^2 \cdot r^2 + n^2 \right) \cdot \overline{H} = 0 \tag{17.429}$$

The particular solution of the Bessel differential equation (17.428) is:

$$\overline{H} = \overline{C} \cdot \bar{\mathcal{I}}_0(\bar{\gamma} \cdot r) + \overline{D} \cdot \overline{K}_0(\bar{\gamma} \cdot r) \tag{17.430}$$

where:

$\bar{\mathcal{I}}_0(\bar{\gamma} \cdot r)$ - the complex modified Bessel function of the first kind of zero order,

$\overline{K}_0(\bar{\gamma} \cdot r)$ - the complex modified Bessel function of the second kind of zero order,

$\overline{C}$, $\overline{D}$ - unknown complex coefficients.

Since the function $\overline{K}_0(\overline{\gamma} \cdot r)$ takes an infinite value for $r = 0$, in the case of a solid ferromagnetic core, the complex coefficient $\overline{D} = 0,$ and in this case, the particular solution of the modified Bessel equation (17.428) is described by the following expression:

$$\overline{H} = \overline{C} \cdot \overline{\mathcal{I}}_0(\overline{\gamma} \cdot r) \tag{17.431}$$

which must satisfy the boundary condition:

$$\overline{H}\Big|_{r=r_0} = \frac{N \cdot \overline{I}}{\ell} \tag{17.432}$$

from which it follows that:

$$\overline{C} = \frac{N \cdot \overline{I}}{\ell} \cdot \frac{1}{\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r_0)} \quad ; \quad \overline{H} = \frac{N \cdot \overline{I}}{\ell} \cdot \frac{\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r)}{\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r_0)} \tag{17.433}$$

The phasor of the electric field intensity can now be determined from the following expression:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \left(\nabla \times \underline{\vec{H}}\right) = \frac{1}{\kappa} \cdot \frac{1}{r} \cdot \begin{vmatrix} \vec{e}_r & r \cdot \vec{e}_\varphi & \vec{e}_z \\ \frac{\partial}{\partial r} & 0 & 0 \\ 0 & 0 & \overline{H} \end{vmatrix} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial r} \cdot \vec{e}_\varphi \tag{17.434}$$

from which it follows that:

$$\overline{E} = \overline{E}_\varphi = -\frac{\overline{\gamma}}{\kappa} \cdot \frac{N \cdot \overline{I}}{\ell} \cdot \frac{\overline{\mathcal{I}}_1(\overline{\gamma} \cdot r)}{\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r_0)} \tag{17.435}$$

because:

$$\frac{d}{dr}\left[\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r)\right] = \overline{\gamma} \cdot \overline{\mathcal{I}}_1(\overline{\gamma} \cdot r) \tag{17.436}$$

According to equation (17.295), the complex apparent electromagnetic power entering the volume V through the closed surface S is described by the following expression:

$$\overline{S}_{\text{ap, in}} = -\overline{S}_{\text{ap, out}} = -\oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} = -\oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot d\vec{S} = \overline{Z} \cdot I^2 \tag{17.437}$$

where S is the outer surface of the ferromagnetic core, whereas $\overline{Z}$ is the equivalent impedance of the ferromagnetic core.

If the expressions (17.426) are substituted into the expression (17.437), the following expression is obtained:

$$\overline{S}_{\text{ap, in}} = -\oint_S \underline{\vec{E}} \times \underline{\vec{H}}^* \cdot \vec{n} \cdot dS = -\oint_S \overline{E} \cdot \overline{H}^* \underbrace{\left(\vec{e}_\varphi \times \vec{e}_z\right)}_{\vec{e}_r} \cdot \vec{n} \cdot dS \tag{17.438}$$

from which it follows that:

$$\overline{S}_{\text{ap, in}} = -\left(\overline{E} \cdot \overline{H}^*\right)_{r=r_0} \cdot 2 \cdot \pi \cdot r_0 \cdot \ell \tag{17.439}$$

where:

$$\overline{E}\Big|_{r=r_0} = -\frac{\overline{\gamma}}{\kappa} \cdot \frac{N \cdot \overline{I}}{\ell} \cdot \frac{\overline{\mathcal{I}}_1(\overline{\gamma} \cdot r_0)}{\overline{\mathcal{I}}_0(\overline{\gamma} \cdot r_0)} \quad ; \quad \overline{H}^*\Big|_{r=r_0} = \frac{N \cdot \overline{I}^*}{\ell} \tag{17.440}$$

It follows that:

$$\overline{S}_{\text{ap, in}} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{N^2}{\ell} \cdot \frac{\overline{\mathcal{J}}_1(\overline{\gamma} \cdot r_0)}{\overline{\mathcal{J}}_0(\overline{\gamma} \cdot r_0)} \cdot 2 \cdot \pi \cdot r_0 \cdot \overline{I} \cdot \overline{I}^* = \overline{Z} \cdot I^2 \tag{17.441}$$

from which it follows that the equivalent impedance of the ferromagnetic core is described by the following expression:

$$\overline{Z} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{N^2}{\ell} \cdot \frac{\overline{\mathcal{J}}_1(\overline{\gamma} \cdot r_0)}{\overline{\mathcal{J}}_0(\overline{\gamma} \cdot r_0)} \cdot 2 \cdot \pi \cdot r_0 \tag{17.442}$$

Note: If the ferromagnetic core is a hollow cylinder with an outer radius r_0 and an inner radius r_{in}, the distribution of the phasor of the magnetic field intensity in the ferromagnetic material would be described by the expression (17.430). To determine the expression for the phasor of the electric field intensity, the previously unused expression must also be used:

$$\frac{d}{dr}\left[\overline{K}_0(\overline{\gamma} \cdot r)\right] = -\,\overline{\gamma} \cdot \overline{K}_1(\overline{\gamma} \cdot r) \tag{17.443}$$

and it follows that:

$$\overline{E} = \overline{E}_\varphi = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial r} = -\frac{\overline{\gamma}}{\kappa} \cdot \left[\, \overline{C} \cdot \overline{\mathcal{J}}_1(\overline{\gamma} \cdot r) - \overline{D} \cdot \overline{K}_1(\overline{\gamma} \cdot r)\right] \tag{17.444}$$

If the displacement electric currents in both the ferromagnetic core and the cylindrical hole are neglected, the magnetic field in the hole is homogeneous. According to Maxwell's second integral equation, which is described by expression (7.61), the boundary condition holds:

$$\overline{E}\Big|_{r=r_{\text{in}}} \cdot 2 \cdot \pi \cdot r_{\text{in}} = -\, j \cdot \omega \cdot \mu_0 \cdot \overline{H}\Big|_{r=r_{\text{in}}} \cdot r_{\text{in}}^2 \cdot \pi \tag{17.445}$$

whereas the second boundary condition is described by the expression (17.432).
In this case, the complex apparent electromagnetic power is described by the following expression:

$$\overline{S}_{\text{ap, in}} = -\left(\overline{E} \cdot \overline{H}^*\right)_{r=r_0} \cdot 2 \cdot \pi \cdot r_0 \cdot \ell + \left(\overline{E} \cdot \overline{H}^*\right)_{r=r_{\text{in}}} \cdot 2 \cdot \pi \cdot r_{\text{in}} \cdot \ell = \overline{Z} \cdot I^2 \tag{17.446}$$

18. ELECTRIC NETWORK THEORY AND ITS LIMITATIONS

The theory of electric networks (electric circuits) is a special case of the general theory of the electromagnetic field. The theory of time-varying electric networks is a simplified representation of the phenomena that occur in the electromagnetic field. It is, in fact, an electroquasistatic approximation of the dynamic electromagnetic field.

An electric network consists of current and/or voltage sources as well as branches and nodes. Each branch of the electric network is located between two nodes.

The theory of time-varying electric networks is based on the following assumptions:

- The current along any branch of the electric network is uniform, and Kirchhoff's first law applies to every node of the network,
- Kirchhoff's second law applies to every closed loop along the branches of the network, which means that the voltage is uniquely defined and independent of the integration path.

It is important to determine when these assumptions are valid, i.e., when the application of circuit theory provides a satisfactory solution. The limitations are as follows:

- The length of each branch is limited by the condition of field quasistaticity. Otherwise, the assumption that the electric current along each branch is uniform at any given time is not valid.
- The nodes must be dimensioned in such a way that only a negligible amount of electric charge can accumulate in them. Otherwise, Kirchhoff's first law does not apply.
- It is necessary that the magnetic field generated by the electric currents in the network branches outside the network elements is negligible, so that these time-varying electric currents in the loops and branches of the electric network induce only negligible electromotive forces compared to the electric voltages of the branches. Otherwise, the electric voltages between the network nodes are not uniquely defined.

According to the expressions in subchapter 17.5, for a stationary closed loop in Figure 18.1, which is in a time-varying electromagnetic field, the electric voltages along the curves C_1 and C_2 are described by the following expressions:

$$u_{\mathrm{AB}}^{C_1} = \int\limits_{C_1} \vec{E}\cdot d\vec{\ell} = -\int\limits_{C_1} \nabla\varphi\cdot d\vec{\ell} + \int\limits_{C_1} \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell} = \varphi_{\mathrm{A}} - \varphi_{\mathrm{B}} - \int\limits_{C_1} \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell} \tag{18.1}$$

$$u_{\mathrm{AB}}^{C_2} = \int\limits_{C_2} \vec{E}\cdot d\vec{\ell} = -\int\limits_{C_2} \nabla\varphi\cdot d\vec{\ell} + \int\limits_{C_2} \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell} = \varphi_{\mathrm{B}} - \varphi_{\mathrm{A}} - \int\limits_{C_2} \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell} \tag{18.2}$$

whereas the EMF induced in the closed loop is described by the following expression:

$$e = u = u_{\mathrm{AB}}^{C_1} + u_{\mathrm{AB}}^{C_2} = \int\limits_{C_1+C_2} \vec{E}\cdot d\vec{\ell} = -\int\limits_{C_1+C_2} \frac{\partial \vec{A}}{\partial t}\cdot d\vec{\ell} = -\iint\limits_{S} \frac{\partial \vec{B}}{\partial t}\cdot d\vec{S} = -\frac{\partial \Phi}{\partial t} \tag{18.3}$$

where the EMF e induced in the closed loop is equal to the voltage u induced in the closed loop.

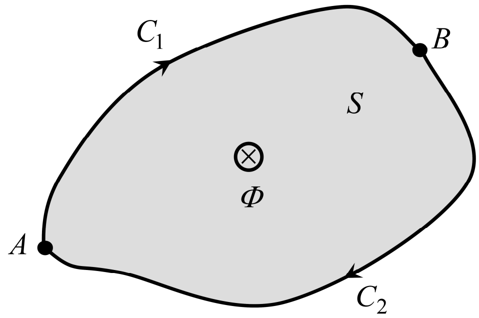


Figure 18.1. Stationary closed loop between points A and B

The following applies to the electroquasistatic approximation of the dynamic electromagnetic field:

$$\frac{\partial \vec{A}}{\partial t} = 0 \quad ; \qquad u_{\mathrm{AB}} = \int_C \vec{E} \cdot d\vec{\ell} = -\int_C \nabla \varphi \cdot d\vec{\ell} = \varphi_{\mathrm{A}} - \varphi_{\mathrm{B}} \quad \forall C \tag{18.4}$$

The basic parameters of electric networks are:

- Resistance R,
- Inductance L,
- Capacitance C.

These are the so-called *lumped parameters*, which should be distinguished from the *distributed parameters* that occur in electric transmission lines.

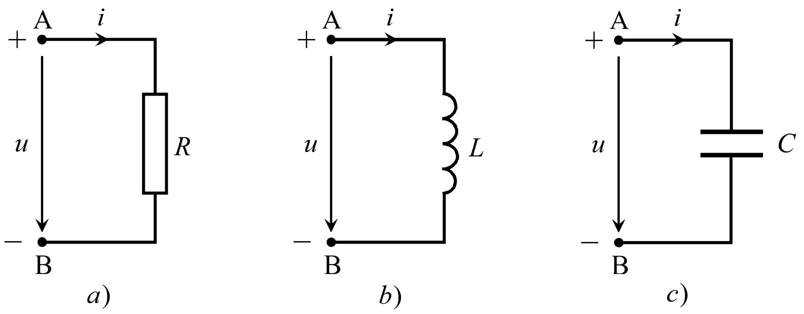


Figure 18.2. Voltages and currents of the basic parameters of the electric network

The electric current flowing through the resistance R (Figure 18.2-a) is described by the expression:

$$i = \frac{u}{R} = \frac{u_{\mathrm{AB}}}{R} = \frac{\varphi_{\mathrm{A}} - \varphi_{\mathrm{B}}}{R} \tag{18.5}$$

where:

$u = u_{\mathrm{AB}}$ - the electric voltage across the branch between nodes A and B,

$\varphi_{\mathrm{A}}, \varphi_{\mathrm{B}}$ - the electric scalar potentials of nodes A and B.

The electric voltage between nodes A and B is the so-called longitudinal voltage or branch voltage. The transverse electric voltage of a node is the voltage between the node and the reference node, whose electric scalar potential is assumed to be zero. This is also the electric scalar potential of the node.*

The longitudinal electric voltage across a perfect inductor with inductance L (Figure 18.2-b) is described by the expression:

$$u = L \cdot \frac{di}{dt} \tag{18.6}$$

whereas the displacement electric current flowing through a perfect capacitor with capacitance C (Figure 18.2-c) is described by the expression:

$$i = C \cdot \frac{du}{dt} \tag{18.7}$$

Sinusoidal electric current (AC) networks are solved in the same manner as direct electric current (DC) networks, where direct electric current refers to an electric current that remains constant over time. The only major difference is that a phasor transformation is used in sinusoidal electric networks, which

* The electric scalar potentials of the nodes (transverse node voltages) are often simply referred to as node voltages, whereas the longitudinal electric voltages of the network branches are referred to as voltage drops or branch voltages. Therefore, the branch electric voltage is equal to the difference between the transverse voltages of the nodes of that branch.

means that sinusoidal networks are solved in the phasor domain. The phasor transformation converts the integro-differential equations from the time domain into linear algebraic equations in the phasor domain.

In the phasor domain, equations (18.5) - (18.7) can be expressed by the complex form of Ohm's law, which states:

$$\bar{I} = \frac{\bar{U}}{\bar{Z}} = \frac{\bar{\varphi}_A - \bar{\varphi}_B}{\bar{Z}} \tag{18.8}$$

where:

$\bar{I}$ - the phasor of the branch current flowing between nodes A and B,

$\bar{U}$ - the phasor of the branch voltage between nodes A and B,

$\bar{Z}$ - the branch impedance,

$\bar{\varphi}_A$ - the phasor of the electric scalar potential at node A; the phasor of the node voltage at node A,

$\bar{\varphi}_B$ - the phasor of the electric scalar potential at node B; the phasor of the node voltage at node B.

From expressions (18.5) and (18.8), it follows that the impedance of the resistance R is:

$$\bar{Z} = R \tag{18.9}$$

which means that the impedance of the resistance has only a real part.

From expressions (18.6) and (18.8), it follows that the impedance of the inductance L is:

$$\bar{Z} = j \cdot \omega \cdot L = j \cdot X_L \quad ; \quad X_L = \omega \cdot L \tag{18.10}$$

where:

ω - the angular frequency of the sinusoidal electric current,

X_L - the inductive reactance.

From expressions (18.7) and (18.8), it follows that the impedance of the capacitance C is:

$$\bar{Z} = -j \cdot \frac{1}{\omega \cdot C} = -j \cdot X_C \quad ; \quad X_C = \frac{1}{\omega \cdot C} \tag{18.11}$$

where X_C is the capacitive reactance.

Kirchhoff's laws in a sinusoidal electric network can be written as follows:

$$\sum_{k=1}^{n} \bar{I}_k = 0 \quad ; \quad \sum_{j=1}^{N} \bar{E}_j = \sum_{k=1}^{M} \bar{I}_k \cdot \bar{Z}_k \tag{18.12}$$

where:

$\bar{I}_k$ - the phasor of the electric current of the k-th branch,

$\bar{E}_j$ - the phasor of the j-th EMF source in the considered closed loop,

$\bar{Z}_k$ - the impedance of the k-th branch.

If n impedances are connected in series, then the total impedance is:

$$\bar{Z} = \sum_{k=1}^{n} \bar{Z}_k \tag{18.13}$$

If n impedances are connected in parallel, then the total impedance is:

$$\frac{1}{\bar{Z}} = \sum_{k=1}^{n} \frac{1}{\bar{Z}_k} \tag{18.14}$$

and for n admittances connected in parallel, the following expression holds:

$$\bar{Y} = \sum_{k=1}^{n} \bar{Y}_k \quad ; \quad \bar{Y} = \frac{1}{\bar{Z}} \tag{18.15}$$

19. PLANE ELECTROMAGNETIC WAVE

A plane electromagnetic wave is an electromagnetic phenomenon in which the field quantities depend only on one spatial coordinate and on time.

Let the field quantities of the plane electromagnetic wave depend only on the z-coordinate of a Cartesian coordinate system. In this case, the vectors of the electric field intensity and the magnetic field intensity lie in planes perpendicular to the z-axis (z = const.), and the plane wave propagates in the direction of the z-axis. Therefore, the following holds:

$$\vec{E} = E_{\mathrm{x}}(z,t) \cdot \vec{\mathrm{i}} + E_{\mathrm{y}}(z,t) \cdot \vec{\mathrm{j}} \tag{19.1}$$

$$\vec{H} = H_{\mathrm{x}}(z,t) \cdot \vec{\mathrm{i}} + H_{\mathrm{y}}(z,t) \cdot \vec{\mathrm{j}} \tag{19.2}$$

A plane electromagnetic wave is a transverse wave because the vectors of the electromagnetic field are perpendicular to the direction of wave propagation.

In the special case where the electric field intensity has only an x-component and the magnetic field intensity has only a y-component, the following holds:

$$\vec{E} = E_{\mathrm{x}}(z,t) \cdot \vec{\mathrm{i}} = E(z,t) \cdot \vec{\mathrm{i}} \tag{19.3}$$

$$\vec{H} = H_{\mathrm{y}}(z,t) \cdot \vec{\mathrm{j}} = H(z,t) \cdot \vec{\mathrm{j}} \tag{19.4}$$

Such an electromagnetic wave is called a linearly polarized wave and propagates in the direction of the Poynting vector:

$$\vec{\Gamma} = \vec{E} \times \vec{H} = E \cdot H \cdot \vec{\mathrm{k}} \tag{19.5}$$

The vectors $\vec{E}$ and $\vec{H}$ are perpendicular to each other and to the z-axis. If these vectors are constant at every point in the plane z = const., the electromagnetic field is said to be uniform. If the field is *uniform*, the energy density at every point in the plane z = const. is also constant.

A plane electromagnetic wave can also have different types of polarization, such as circular polarization or elliptical polarization.

19.1. Linearly Polarized Plane Wave in a Conducting LIH Medium

For a linearly polarized plane electromagnetic wave, described by expressions (19.3) and (19.4), the following holds:

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ 0 & H & 0 \end{vmatrix} = -\frac{\partial H}{\partial z} \cdot \vec{\mathrm{i}} \quad ; \quad \nabla \times \vec{E} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ E & 0 & 0 \end{vmatrix} = \frac{\partial E}{\partial z} \cdot \vec{\mathrm{j}} \tag{19.6}$$

From expressions (10.1), (10.2) and (19.6), it follows that the first two of Maxwell's differential equations for a linearly polarized plane wave in a source-free stationary conducting LIH medium are:

$$-\frac{\partial H}{\partial z} = \kappa \cdot E + \varepsilon \cdot \frac{\partial E}{\partial t} \quad ; \quad \frac{\partial E}{\partial z} = -\mu \cdot \frac{\partial H}{\partial t} \tag{19.7}$$

The derivatives of Maxwell's differential equations (19.7) with respect to z and t are:

$$-\frac{\partial^2 H}{\partial z^2} = \kappa \cdot \frac{\partial E}{\partial z} + \varepsilon \cdot \frac{\partial^2 E}{\partial z \partial t} \quad ; \quad -\frac{\partial^2 H}{\partial z \partial t} = \kappa \cdot \frac{\partial E}{\partial t} + \varepsilon \cdot \frac{\partial^2 E}{\partial t^2} \tag{19.8}$$

$$\frac{\partial^2 E}{\partial z^2} = -\mu \cdot \frac{\partial^2 H}{\partial z \partial t} \quad ; \quad \frac{\partial^2 E}{\partial z \partial t} = -\mu \cdot \frac{\partial^2 H}{\partial t^2} \tag{19.9}$$

From the previous six differential equations, described by expressions (19.7) - (19.9), two differential equations can be easily derived, namely the damped wave equations of a plane electromagnetic wave in a source-free conducting LIH medium:

$$\frac{\partial^2 E}{\partial z^2} - \mu \cdot \kappa \cdot \frac{\partial E}{\partial t} - \mu \cdot \varepsilon \cdot \frac{\partial^2 E}{\partial t^2} = 0 \tag{19.10}$$

$$\frac{\partial^2 H}{\partial z^2} - \mu \cdot \kappa \cdot \frac{\partial H}{\partial t} - \mu \cdot \varepsilon \cdot \frac{\partial^2 H}{\partial t^2} = 0 \tag{19.11}$$

which are also a special case of the damped field wave equations in a source-free conducting LIH medium, described by the expressions (17.267) and (17.268), with the substitutions:

$$\Delta E = \frac{\partial^2 E}{\partial z^2} \quad ; \quad \Delta H = \frac{\partial^2 H}{\partial z^2} \tag{19.12}$$

The damped wave equations (19.10) and (19.11) can be solved in the time domain using the method of separation of variables. However, of particular interest is the solution of these differential equations for a sinusoidal electromagnetic field. In this case, the phasor representation of sinusoidal (time-harmonic) quantities is used.

By applying the phasor transformation, the wave equations of the damped plane wave from the time domain are transformed into one-dimensional homogeneous Helmholtz differential equations in the phasor domain, which are:

$$\frac{\partial^2 \overline{E}}{\partial z^2} - \overline{\gamma}^2 \cdot \overline{E} = 0 \quad ; \quad \frac{\partial^2 \overline{H}}{\partial z^2} - \overline{\gamma}^2 \cdot \overline{H} = 0 \tag{19.13}$$

which is a special case of the homogeneous Helmholtz differential equations (17.273), with the substitutions described by the expressions (19.12). The propagation constant $\overline{\gamma}$ of a conducting LIH medium is described by expressions (17.119) -(17.121).

The solutions of the 1D homogeneous Helmholtz differential equations (19.13) are:

$$\overline{E} = \overline{E}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{E}_2 \cdot e^{\overline{\gamma} \cdot z} \quad ; \quad \overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot z} \tag{19.14}$$

where $\overline{E}_1, \overline{E}_2, \overline{H}_1, \overline{H}_2$ are unknown complex constants.

From the expression (19.7), it follows that Maxwell's second differential equation for a linearly polarized sinusoidal electromagnetic wave in the phasor domain is as follows:

$$\frac{\partial \overline{E}}{\partial z} = -j \cdot \omega \cdot \mu \cdot \overline{H} \tag{19.15}$$

If the solutions of the homogeneous Helmholtz differential equations, described by expressions (19.14), are substituted into the expression (19.15), it follows that:

$$\frac{\partial \left(\overline{E}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{E}_2 \cdot e^{\overline{\gamma} \cdot z} \right)}{\partial z} = -j \cdot \omega \cdot \mu \cdot \left(\overline{H}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot z} \right) \tag{19.16}$$

from which it follows that:

$$-\overline{\gamma} \cdot \overline{E}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{\gamma} \cdot \overline{E}_2 \cdot e^{\overline{\gamma} \cdot z} = -j \cdot \omega \cdot \mu \cdot \left(\overline{H}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot z} \right) \tag{19.17}$$

and it is:

$$\left(-\overline{\gamma} \cdot \overline{E}_1 + j \cdot \omega \cdot \mu \cdot \overline{H}_1 \right) \cdot e^{-\overline{\gamma} \cdot z} + \left(\overline{\gamma} \cdot \overline{E}_2 + j \cdot \omega \cdot \mu \cdot \overline{H}_2 \right) \cdot e^{\overline{\gamma} \cdot z} = 0 \tag{19.18}$$

From expression (19.18), it follows that:

$$-\bar{\gamma}\cdot\overline{E}_1 + j\cdot\omega\cdot\mu\cdot\overline{H}_1 = 0 \quad ; \quad \bar{\gamma}\cdot\overline{E}_2 + j\cdot\omega\cdot\mu\cdot\overline{H}_2 = 0 \tag{19.19}$$

from which it follows that:

$$\frac{\overline{E}_1}{\overline{H}_1} = -\frac{\overline{E}_2}{\overline{H}_2} = \frac{j\cdot\omega\cdot\mu}{\bar{\gamma}} = \overline{Z}_{\mathrm{v}} \tag{19.20}$$

where:

$$\overline{Z}_{\mathrm{m}} = \frac{j\cdot\omega\cdot\mu}{\bar{\gamma}} = \frac{\bar{\gamma}}{\kappa + j\cdot\omega\cdot\varepsilon} \tag{19.21}$$

is the wave impedance of a conducting LIH medium. The propagation constant of a conducting LIH medium is described by the expression (17.119), which reads:

$$\bar{\gamma} = \alpha + j\cdot\beta \tag{19.22}$$

where the constants α and β are described by the expressions (17.120) and (17.121).
From expressions (19.21) and (19.22), it follows that:

$$\overline{Z}_{\mathrm{m}} = \frac{j\cdot\omega\cdot\mu}{\alpha + j\cdot\beta} = \frac{\omega\cdot\mu}{\alpha^2+\beta^2}\cdot(\beta + j\cdot\alpha) = Z_{\mathrm{m}}\cdot e^{j\cdot\varphi_{\mathrm{m}}} \tag{19.23}$$

where:

$$Z_{\mathrm{m}} = \left|\overline{Z}_{\mathrm{m}}\right| = \frac{\omega\cdot\mu}{\sqrt{\alpha^2+\beta^2}} \quad ; \quad \varphi_{\mathrm{m}} = \operatorname{arc\,tg}\frac{\alpha}{\beta} \tag{19.24}$$

The propagation velocity of the electromagnetic wave is described by the expression (17.234).
Let it be:

$$\overline{E}_1 = E_1\cdot e^{j\cdot\varphi_{1\mathrm{E}}} \quad ; \quad \overline{E}_2 = E_2\cdot e^{j\cdot\varphi_{2\mathrm{E}}} \tag{19.25}$$

$$\overline{H}_1 = H_1\cdot e^{j\cdot\varphi_{1\mathrm{H}}} \quad ; \quad \overline{H}_2 = H_2\cdot e^{j\cdot\varphi_{2\mathrm{H}}} \tag{19.26}$$

From expressions (19.14), (19.22), (19.25), and (19.26), it follows that:

$$\overline{E} = E_1\cdot e^{j\cdot\varphi_{1\mathrm{E}}}\cdot e^{-(\alpha+j\cdot\beta)\cdot z} + E_2\cdot e^{j\cdot\varphi_{2\mathrm{E}}}\cdot e^{(\alpha+j\cdot\beta)\cdot z} \tag{19.27}$$

$$\overline{H} = H_1\cdot e^{j\cdot\varphi_{1\mathrm{H}}}\cdot e^{-(\alpha+j\cdot\beta)\cdot z} + H_2\cdot e^{j\cdot\varphi_{2\mathrm{H}}}\cdot e^{(\alpha+j\cdot\beta)\cdot z} \tag{19.28}$$

and it is:

$$\overline{E} = E_1\cdot e^{-\alpha\cdot z}\cdot e^{-j\cdot(\beta\cdot z - \varphi_{1\mathrm{E}})} + E_2\cdot e^{\alpha\cdot z}\cdot e^{j\cdot(\beta\cdot z + \varphi_{2\mathrm{E}})} \tag{19.29}$$

$$\overline{H} = H_1\cdot e^{-\alpha\cdot z}\cdot e^{-j\cdot(\beta\cdot z - \varphi_{1\mathrm{H}})} + H_2\cdot e^{\alpha\cdot z}\cdot e^{j\cdot(\beta\cdot z + \varphi_{2\mathrm{H}})} \tag{19.30}$$

From expressions (19.29) and (19.30), it follows that the electric field intensity and magnetic field intensity in the time domain are described by the following expressions:

$$\begin{aligned} E(z,t) = E_{\mathrm{for}}(z,t) + E_{\mathrm{rev}}(z,t) &= \sqrt{2}\cdot E_1\cdot e^{-\alpha\cdot z}\cdot\cos(\omega\cdot t - \beta\cdot z + \varphi_{1\mathrm{E}}) \\ &+ \sqrt{2}\cdot E_2\cdot e^{\alpha\cdot z}\cdot\cos(\omega\cdot t + \beta\cdot z + \varphi_{2\mathrm{E}}) \end{aligned} \tag{19.31}$$

$$\begin{aligned} H(z,t) = H_{\mathrm{for}}(z,t) + H_{\mathrm{rev}}(z,t) &= \sqrt{2}\cdot H_1\cdot e^{-\alpha\cdot z}\cdot\cos(\omega\cdot t - \beta\cdot z + \varphi_{1\mathrm{H}}) \\ &+ \sqrt{2}\cdot H_2\cdot e^{\alpha\cdot z}\cdot\cos(\omega\cdot t + \beta\cdot z + \varphi_{2\mathrm{H}}) \end{aligned} \tag{19.32}$$

The forward electromagnetic wave propagates in the direction of the z-axis, whereas the reverse electromagnetic wave propagates in the opposite direction. Each of these waves is attenuated in the direction of its propagation.

The phase angles of the electric field intensity and the magnetic field intensity are related through the angle of the wave impedance of the medium. The following holds:

$$\frac{\overline{E}_1}{\overline{H}_1} = \frac{E_1}{H_1} \cdot e^{j\cdot(\varphi_{1\mathrm{E}} - \varphi_{1\mathrm{H}})} = Z_\mathrm{m} \cdot e^{j\cdot\varphi_\mathrm{m}} \tag{19.33}$$

$$\frac{\overline{E}_2}{\overline{H}_2} = \frac{E_2}{H_2} \cdot e^{j\cdot(\varphi_{2\mathrm{E}} - \varphi_{2\mathrm{H}})} = -Z_\mathrm{m} \cdot e^{j\cdot\varphi_\mathrm{m}} = Z_\mathrm{m} \cdot e^{j\cdot(\varphi_\mathrm{m}+\pi)} \tag{19.34}$$

from which it follows that:

$$H_1 = \frac{E_1}{Z_\mathrm{m}} \quad ; \quad \varphi_{1\mathrm{H}} = \varphi_{1\mathrm{E}} - \varphi_\mathrm{m} \tag{19.35}$$

$$H_2 = \frac{E_2}{Z_\mathrm{m}} \quad ; \quad \varphi_{2\mathrm{H}} = \varphi_{2\mathrm{E}} - \varphi_\mathrm{m} - \pi \tag{19.36}$$

From expressions (19.31), (19.32), (19.35), and (19.36), it follows that the electric field intensity and magnetic field intensity in the time domain can also be described by the following expressions:

$$\begin{aligned} E(z,t) &= \sqrt{2} \cdot E_1 \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1\mathrm{E}}) \\ &\quad + \sqrt{2} \cdot E_2 \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2\mathrm{E}}) \end{aligned} \tag{19.37}$$

$$\begin{aligned} H(z,t) &= \sqrt{2} \cdot \frac{E_1}{Z_\mathrm{m}} \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1E} - \varphi_\mathrm{m}) \\ &\quad - \sqrt{2} \cdot \frac{E_2}{Z_\mathrm{m}} \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2E} - \varphi_\mathrm{m}) \end{aligned} \tag{19.38}$$

The electromagnetic wave is attenuated in the direction of its propagation. The distance d along the z-axis for which $\alpha\cdot\Delta z = \alpha\cdot d = 1$ is the skin depth of the electromagnetic wave, which is described by the expressions (17.285) - (17.287).

19.2. Linearly Polarized Plane Wave in a Good LIH Conductor

In good conductors, neglecting displacement currents, expression (17.146) holds, according to which:

$$\alpha = \beta = \sqrt{\frac{\omega\cdot\mu\cdot\kappa}{2}} \tag{19.39}$$

and the wave impedance of a good LIH conductor is described by the expression:

$$\overline{Z}_\mathrm{m} = \frac{j\cdot\omega\cdot\mu}{\overline{\gamma}} = \frac{j\cdot\omega\cdot\mu\cdot\kappa}{\overline{\gamma}\cdot\kappa} = \frac{\overline{\gamma}}{\kappa} = \sqrt{\frac{\omega\cdot\mu}{\kappa}} \cdot e^{j\cdot\pi/4} \tag{19.40}$$

From expressions (19.37), (19.38), and (19.40), it follows that the electric field intensity and magnetic field intensity in the time domain are described by the following expressions:

$$\begin{aligned} E(z,t) &= \sqrt{2} \cdot E_1 \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1\mathrm{E}}) \\ &\quad + \sqrt{2} \cdot E_2 \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2\mathrm{E}}) \end{aligned} \tag{19.41}$$

$$H(z,t)=\sqrt{2}\cdot\frac{E_1}{Z_{\mathrm{m}}}\cdot e^{-\alpha\cdot z}\cdot\cos\left(\omega\cdot t-\beta\cdot z+\varphi_{1E}-\pi/4\right)$$
$$-\ \sqrt{2}\cdot\frac{E_2}{Z_{\mathrm{m}}}\cdot e^{\alpha\cdot z}\cdot\cos\left(\omega\cdot t+\beta\cdot z+\varphi_{2E}-\pi/4\right) \tag{19.42}$$

Here, one can also speak of the magnetodynamic approximation of the linearly polarized plane wave, or of the diffusion approximation of the linearly polarized plane wave.

19.3. Linearly Polarized Plane Wave in a Perfect LIH Dielectric

In a stationary perfect LIH dielectric, expression (17.122) holds, according to which:

$$\alpha=0 \quad ; \quad \beta=\omega\cdot\sqrt{\mu\cdot\varepsilon} \tag{19.43}$$

and the wave impedance of a perfect LIH dielectric is described by the expression:

$$\overline{Z}_{\mathrm{m}}=\frac{j\cdot\omega\cdot\mu}{\overline{\gamma}}=\frac{j\cdot\omega\cdot\mu}{j\cdot\beta}=\sqrt{\frac{\mu}{\varepsilon}}=Z_{\mathrm{m}} \quad ; \quad \varphi_{\mathrm{m}}=0 \tag{19.44}$$

It follows that the wave impedance of a perfect LIH dielectric has only a real part, and in this case, the term *wave resistance* of a perfect LIH dielectric is also used.
From expression (17.44), it follows that the wave impedance (wave resistance) of a vacuum is:

$$\overline{Z}_{\mathrm{m}}=Z_{\mathrm{m}}=Z_0=\sqrt{\frac{\mu_0}{\varepsilon_0}}=376.730313668\ \ \Omega \tag{19.45}$$

Let it be $\varphi_{1\mathrm{E}}=\varphi_{2\mathrm{E}}=0$. In this case, according to expressions (17.37) and (17.38), it is:

$$E(z,t)=\sqrt{2}\cdot E_1\cdot\cos\left(\omega\cdot t-\beta\cdot z\right)+\sqrt{2}\cdot E_2\cdot\cos\left(\omega\cdot t+\beta\cdot z\right) \tag{19.46}$$

$$H(z,t)=\sqrt{2}\cdot\frac{E_1}{Z_{\mathrm{m}}}\cdot\cos\left(\omega\cdot t-\beta\cdot z\right)-\sqrt{2}\cdot\frac{E_2}{Z_{\mathrm{m}}}\cdot\cos\left(\omega\cdot t+\beta\cdot z\right) \tag{19.47}$$

It follows that the forward components of the magnetic field intensity and the forward components of the electric field intensity are in phase.

$$E_{\mathrm{for}}(z,t)=\sqrt{2}\cdot E_1\cdot\cos\left(\omega\cdot t-\beta\cdot z\right) \tag{19.48}$$

$$H_{\mathrm{for}}(z,t)=\sqrt{2}\cdot\frac{E_1}{Z_{\mathrm{m}}}\cdot\cos\left(\omega\cdot t-\beta\cdot z\right) \tag{19.49}$$

whereas the reverse components have opposite phase:

$$E_{\mathrm{rev}}(z,t)=\sqrt{2}\cdot E_2\cdot\cos\left(\omega\cdot t+\beta\cdot z\right) \tag{19.50}$$

$$H_{\mathrm{rev}}(z,t)=-\sqrt{2}\cdot\frac{E_2}{Z_{\mathrm{m}}}\cdot\cos\left(\omega\cdot t+\beta\cdot z\right) \tag{19.51}$$

In the special case when $\varphi_{1\mathrm{E}}=\varphi_{2\mathrm{E}}=0$ and $E_1=E_2$, it is referred to as a standing wave, where the wave nodes remain stationary. From expressions (19.45) and (19.46), with the substitution $E_1=E_2$, it follows that:

$$E(z,t)=\sqrt{2}\cdot E_1\cdot\left[\cos\left(\omega\cdot t-\beta\cdot z\right)+\cos\left(\omega\cdot t+\beta\cdot z\right)\right] \tag{19.52}$$

$$H(z,t)=\sqrt{2}\cdot\frac{E_1}{Z_{\mathrm{m}}}\cdot\left[\cos\left(\omega\cdot t-\beta\cdot z\right)-\cos\left(\omega\cdot t+\beta\cdot z\right)\right] \tag{19.53}$$

from which it follows that:

$$E(z,t)=2\cdot\sqrt{2}\cdot E_1\cdot\cos(\beta\cdot z)\cdot\cos(\omega\cdot t) \tag{19.54}$$

$$H(z,t)=2\cdot\sqrt{2}\cdot\frac{E_1}{Z_\mathrm{m}}\cdot\sin(\beta\cdot z)\cdot\sin(\omega\cdot t) \tag{19.55}$$

Over the distance of one wavelength (λ), the spatial phase changes by $2\cdot\pi$, so the following holds:

$$\beta\cdot\lambda=2\cdot\pi \quad ; \quad \lambda=\frac{v}{f}=\frac{2\cdot\pi}{\beta} \tag{19.56}$$

where v is the wave velocity, and f is the wave frequency.
It follows that for f = 50 Hz in the air, the wavelength λ = 6000 km = 6 Mm.
The forward and reverse waves carry energy, which is described by the Poynting vector:

$$\underline{\vec{\varGamma}}_1=\underline{\vec{E}}_1\times\underline{\vec{H}}_1^*=\overline{E}_1\cdot\overline{H}_1^*\cdot\vec{\mathrm{k}} \tag{19.57}$$

$$\underline{\vec{\varGamma}}_2=\underline{\vec{E}}_2\times\underline{\vec{H}}_2^*=\overline{E}_2\cdot\overline{H}_2^*\cdot\left(-\vec{\mathrm{k}}\right) \tag{19.58}$$

From expression (12.30), it follows that the average value of the surface density of the forward wave power, or the surface density of the forward wave active power, is given by the following expression:

$$P_1=\varGamma_{1\,\mathrm{av}}=\mathrm{Re}\left(\overline{E}_1\cdot\overline{H}_1^*\right) \tag{19.59}$$

whereas the surface density of the reverse wave active power is:

$$P_2=\varGamma_{2\,\mathrm{av}}=\mathrm{Re}\left(\overline{E}_2\cdot\overline{H}_2^*\right) \tag{19.60}$$

In the air, the energy contained in the forward and reverse waves is transmitted at the speed of light.

19.4. Solved Examples

Example 19.1. Using the complex Poynting theorem, calculate the internal impedance, resistance, and internal inductance per unit length of a rectangular copper conductor through which a sinusoidal electric current flows, assuming that $\ell >> a >> b$. Let it be given: a = 1 cm, b = 0.05 cm, $\kappa = 56$ MS/m, $\mu=\mu_0$, $\omega=10^5$ rad/s. Neglect displacement electric currents.

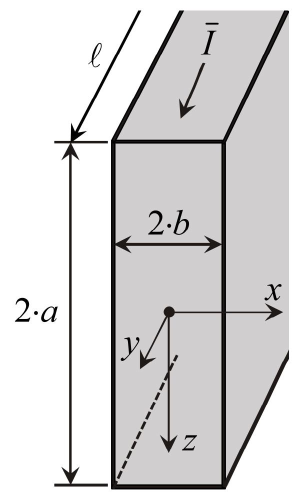


Figure 19.1. Rectangular copper conductor

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (y, z) plane and depend only on the variable x. The following holds:

$$\vec{\underline{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} \quad ; \quad \vec{\underline{H}} = \overline{H}_{\mathrm{z}} \cdot \vec{\mathrm{k}} = \overline{H} \cdot \vec{\mathrm{k}} \tag{19.61}$$

The phasor of the magnetic field intensity and the phasor of the electric field intensity are described by the 1D homogeneous Helmholtz differential equations:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.62}$$

$$\Delta \overline{E} = \frac{\partial^2 \overline{E}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{E} \tag{19.63}$$

where $\overline{\gamma}$ is the propagation constant of a good conductor, which is described by the expression:

$$\overline{\gamma} = \alpha + j \cdot \beta = (1 + j) \cdot \alpha \quad ; \quad \alpha = \beta = \sqrt{\frac{\omega \cdot \mu \cdot \kappa}{2}} \tag{19.64}$$

The particular solution of the Helmholtz differential equation (19.62) is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.65}$$

where $\overline{H}_1$ and $\overline{H}_2$ are the unknown complex constants, which must satisfy the boundary conditions. Since:

$$\overline{H}\Big|_{x=0} = 0 \tag{19.66}$$

it follows that:

$$\overline{H}_1 + \overline{H}_2 = 0 \quad ; \quad \overline{H}_2 = -\overline{H}_1 \tag{19.67}$$

and it is:

$$\overline{H} = \overline{H}_2 \cdot \left(e^{\overline{\gamma} \cdot x} - e^{-\overline{\gamma} \cdot x}\right) = 2 \cdot \overline{H}_2 \cdot \sinh(\overline{\gamma} \cdot x) \tag{19.68}$$

The remaining complex constant $\overline{H}_2$ can be determined using Ampère's law, according to Figure 19.2.

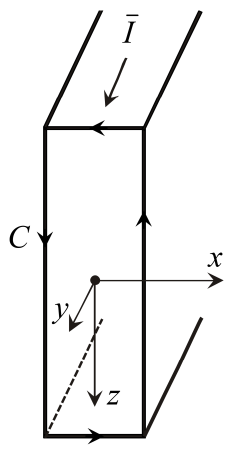


Figure 19.2. Integration curve C

According to Figure 19.2, the following holds:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \oint_C \overline{H}_z \cdot dz = \oint_C \overline{H} \cdot dz = \overline{I} \tag{19.69}$$

from which it follows that:

$$\int_a^{-a} \overline{H}\Big|_{x=b} \cdot dz + 0 + \int_{-a}^{a} \overline{H}\Big|_{x=-b} \cdot dz + 0 = \overline{I} \tag{19.70}$$

If expression (19.68) is substituted into expression (19.70), it follows that:

$$-2 \cdot a \cdot 2 \cdot \overline{H}_2 \cdot \sinh\left(\overline{\gamma} \cdot b\right) + 2 \cdot a \cdot 2 \cdot \overline{H}_2 \cdot \sinh\left(-\overline{\gamma} \cdot b\right) = \overline{I} \tag{19.71}$$

from which it follows that:

$$\overline{H}_2 = -\frac{\overline{I}}{8 \cdot a \cdot \sinh\left(\overline{\gamma} \cdot b\right)} \quad ; \quad \overline{H} = -\frac{\overline{I}}{4 \cdot a} \cdot \frac{\sinh\left(\overline{\gamma} \cdot x\right)}{\sinh\left(\overline{\gamma} \cdot b\right)} \tag{19.72}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \overline{H} \end{vmatrix} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.73}$$

from which it follows that:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.74}$$

and it is:

$$\overline{E} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{\overline{I}}{4 \cdot a} \cdot \frac{\cosh\left(\overline{\gamma} \cdot x\right)}{\sinh\left(\overline{\gamma} \cdot b\right)} = \overline{Z}_{\mathrm{m}} \cdot \frac{\overline{I}}{4 \cdot a} \cdot \frac{\cosh\left(\overline{\gamma} \cdot x\right)}{\sinh\left(\overline{\gamma} \cdot b\right)} \tag{19.75}$$

where:

$$\overline{Z}_{\mathrm{m}} = \frac{\overline{\gamma}}{\kappa} \tag{19.76}$$

is the wave impedance of a copper conductor, neglecting displacement electric currents.
In this case, the phasor of the Poynting vector is described by the expression:

$$\underline{\vec{\Gamma}} = \underline{\vec{E}} \times \underline{\vec{H}}^* = \overline{E} \cdot \overline{H}^* \cdot \left(\vec{\mathrm{j}} \times \vec{\mathrm{k}}\right) = \overline{E} \cdot \overline{H}^* \cdot \vec{\mathrm{i}} \tag{19.77}$$

whereas the complex apparent electromagnetic power entering the unit length of the conductor is described by the expression:

$$\overline{S}_{\mathrm{ap,in}} = P + j \cdot Q = I^2 \cdot \overline{Z}_{\mathrm{int}}^1 = -\oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} = -\oint_S \overline{\Gamma} \cdot \vec{\mathrm{i}} \cdot \vec{n} \cdot dS \tag{19.78}$$

which reduces to the integration over two rectangular surfaces perpendicular to the x-axis (Figure 19.3):

$$\overline{S}_{\mathrm{ap,in}} = -\left(\int_{S_1} \overline{E} \cdot \overline{H}^* \cdot \vec{\mathrm{i}} \cdot \vec{\mathrm{i}} \cdot dS + \int_{S_2} \overline{E} \cdot \overline{H}^* \cdot \vec{\mathrm{i}} \cdot \left(-\vec{\mathrm{i}}\right) \cdot dS \right) \tag{19.79}$$

from which it follows that:

$$\overline{S}_{\mathrm{ap,in}} = -\int_{S_1} \overline{E} \cdot \overline{H}^* \cdot dS + \int_{S_2} \overline{E} \cdot \overline{H}^* \cdot dS \tag{19.80}$$

$$\overline{S}_{\mathrm{ap,in}} = -\left(\overline{E} \cdot \overline{H}^*\right)_{x=b} \cdot 2 \cdot a + \left(\overline{E} \cdot \overline{H}^*\right)_{x=-b} \cdot 2 \cdot a \tag{19.81}$$

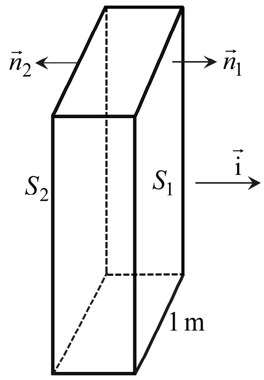


Figure 19.3. Integration surfaces

If expressions (19.72) and (19.75) are substituted into expression (19.81), the following expression is obtained:

$$\overline{S}_{\mathrm{ap,in}} = \overline{Z}_{\mathrm{m}} \cdot \frac{\overline{I} \cdot \overline{I}^*}{4 \cdot a} \cdot \frac{\cosh(\overline{\gamma} \cdot b)}{\sinh(\overline{\gamma} \cdot b)} = \overline{Z}_{\mathrm{m}} \cdot \frac{I^2}{4 \cdot a} \cdot \coth(\overline{\gamma} \cdot b) = I^2 \cdot \overline{Z}_{\mathrm{int}}^1 \tag{19.82}$$

from which it follows that the per-unit-length internal impedance of the conductor is described by the expression:

$$\overline{Z}_{\mathrm{int}}^1 = \frac{\overline{\gamma}}{\kappa} \cdot \frac{1}{4 \cdot a} \cdot \coth(\overline{\gamma} \cdot b) = \frac{\overline{Z}_{\mathrm{m}}}{4 \cdot a} \cdot \coth(\overline{\gamma} \cdot b) \tag{19.83}$$

For the conductor, it holds:

$$\overline{Z}_{\mathrm{int}}^1 = R + j \cdot \omega \cdot L_{\mathrm{int}} = R + j \cdot X_{\mathrm{Lint}} \tag{19.84}$$

where:

R - the per-unit-length resistance of the conductor,

L_{int} - the per-unit-length internal inductance of the conductor,

X_{Lint} - the per-unit-length internal inductive reactance.

For $\overline{\gamma} = (1 + j) \cdot \alpha,$ the following expression holds:

$$\coth(\overline{\gamma} \cdot x) = \frac{\sinh(2 \cdot \alpha \cdot x) - j \cdot \sin(2 \cdot \alpha \cdot x)}{\cosh(2 \cdot \alpha \cdot x) - \cos(2 \cdot \alpha \cdot x)} \tag{19.85}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\frac{\omega \cdot \mu_0 \cdot \kappa}{2}} = 1875.78884 \ \ \mathrm{m}^{-1} \quad ; \quad 2 \cdot \alpha \cdot b = 1.87578884 \tag{19.86}$$

$$\coth(\overline{\gamma} \cdot b) = 0.9137847048 \angle -16.66524733^{\mathrm{o}} \tag{19.87}$$

$$\overline{Z}_{\mathrm{m}} = 4.737082174 \times 10^{-5} \angle 45^{\mathrm{o}} \ \ \Omega \tag{19.88}$$

$$\overline{Z}_{\mathrm{int}}^1 = 1.082168309 \times 10^{-3} \angle 28.33475267^{\mathrm{o}} \ \ \Omega/\mathrm{m} \tag{19.89}$$

$$R = 9.525133293 \times 10^{-4} \ \ \Omega/\mathrm{m} \quad ; \quad X_{\mathrm{Lint}} = 5.136210755 \times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.90}$$

$$L_{\mathrm{int}} = \frac{1}{\omega} \cdot X_{\mathrm{Lint}} = 5.136210755 \times 10^{-9} \ \ \mathrm{H/m} \tag{19.91}$$

Example 19.2. Using the complex Poynting theorem, calculate the internal impedance, resistance, and internal inductance per unit length of a rectangular copper conductor. Let the conductor be placed in a slot and carry a sinusoidal electric current with a frequency of $f = 50$ Hz. Assume that the permeability of iron is infinite, all magnetic field lines in the copper conductor are parallel to the x-axis, and the magnetic field intensity along the field lines in the copper conductor is constant. The following is given: $h = 2$ cm, $b = 8$ mm, $\kappa = 57$ MS/m. Neglect displacement electric currents.

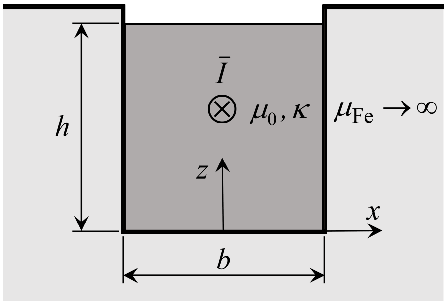


Figure 19.4. Rectangular copper conductor in a slot

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (x, y) plane and depend only on the variable z. The following holds:

$$\vec{\underline{E}} = \overline{E}_y \cdot \vec{j} = \overline{E} \cdot \vec{j} \quad ; \quad \vec{\underline{H}} = \overline{H}_x \cdot \vec{i} = \overline{H} \cdot \vec{i} \tag{19.92}$$

The phasor of the magnetic field intensity is described by the 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial z^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.93}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot z} \tag{19.94}$$

where $\overline{H}_1$ and $\overline{H}_2$ are the unknown complex constants, which must satisfy the boundary conditions:

$$\overline{H}\Big|_{z=0} = 0 \quad ; \quad \overline{H}\Big|_{z=h} = \frac{\overline{I}}{b} \tag{19.95}$$

from which it follows that:

$$\overline{H} = \frac{\overline{I}}{b} \cdot \frac{\sinh(\overline{\gamma} \cdot z)}{\sinh(\overline{\gamma} \cdot h)} \tag{19.96}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\vec{\underline{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 0 & \dfrac{\partial}{\partial z} \\ \overline{H} & 0 & 0 \end{vmatrix} = \frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial z} \cdot \vec{j} \tag{19.97}$$

from which it follows that:

$$\vec{\underline{E}} = \overline{E}_y \cdot \vec{j} = \overline{E} \cdot \vec{j} = \frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial z} \cdot \vec{j} \tag{19.98}$$

and it is:

$$\overline{E} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{\overline{I}}{b} \cdot \frac{\cosh(\overline{\gamma} \cdot z)}{\sinh(\overline{\gamma} \cdot h)} = \overline{Z}_{\mathrm{m}} \cdot \frac{\overline{I}}{b} \cdot \frac{\cosh(\overline{\gamma} \cdot z)}{\sinh(\overline{\gamma} \cdot h)} \tag{19.99}$$

where $\overline{Z}_{\mathrm{m}}$ is the wave impedance of the copper conductor, which is described by expression (19.76). In this case, the phasor of the Poynting vector is described by the expression:

$$\vec{\underline{\Gamma}} = \vec{\underline{E}} \times \vec{\underline{H}}^* = \overline{E} \cdot \overline{H}^* \cdot (\vec{\mathrm{j}} \times \vec{\mathrm{i}}) = \overline{E} \cdot \overline{H}^* \cdot (-\vec{\mathrm{k}}) \tag{19.100}$$

whereas the complex apparent electromagnetic power entering the unit length of the conductor is described by the expression:

$$\overline{S}_{\text{ap, in}} = P + j \cdot Q = I^2 \cdot \overline{Z}^1_{\text{int}} = -\oint_S \vec{\underline{\Gamma}} \cdot d\vec{S} = \oint_S \overline{\Gamma} \cdot \vec{\mathrm{k}} \cdot \vec{n} \cdot dS \tag{19.101}$$

which reduces to integration over the part of the surface $z = h$ that belongs to the unit length of the conductor:

$$\overline{S}_{\text{ap, in}} = \int_{S_z} \left(\overline{E} \cdot \overline{H}^*\right)_{z=h} (\vec{\mathrm{k}} \cdot \vec{\mathrm{k}}) \cdot dS = \left(\overline{E} \cdot \overline{H}^*\right)_{z=h} \cdot b \tag{19.102}$$

If expressions (19.96) and (19.99) are substituted into expression (19.102), It follows that:

$$\overline{S}_{\text{ap, in}} = \overline{Z}_{\mathrm{v}} \cdot \frac{\overline{I} \cdot \overline{I}^*}{b} \cdot \frac{\cosh(\overline{\gamma} \cdot h)}{\sinh(\overline{\gamma} \cdot h)} = \overline{Z}_{\mathrm{m}} \cdot \frac{I^2}{b} \cdot \coth(\overline{\gamma} \cdot h) = I^2 \cdot \overline{Z}^1_{\text{int}} \tag{19.103}$$

from which it follows that the per-unit-length internal impedance of the conductor is described by:

$$\overline{Z}^1_{\text{int}} = R + j \cdot \omega \cdot L_{\text{int}} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{1}{b} \cdot \coth(\overline{\gamma} \cdot h) = \frac{\overline{Z}_{\mathrm{m}}}{b} \cdot \coth(\overline{\gamma} \cdot h) \tag{19.104}$$

For $\overline{\gamma} = (1 + j) \cdot \alpha,$ the following expression holds:

$$\coth(\overline{\gamma} \cdot h) = \frac{\sinh(2 \cdot \alpha \cdot h) - j \cdot \sin(2 \cdot \alpha \cdot h)}{\cosh(2 \cdot \alpha \cdot h) - \cos(2 \cdot \alpha \cdot h)} \tag{19.105}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi \cdot f \cdot \mu_0 \cdot \kappa} = 106.0723763 \ \mathrm{m}^{-1} \ ; \ 2 \cdot \alpha \cdot h = 4.242895053 \tag{19.106}$$

$$\coth(\overline{\gamma} \cdot h) = 0.9870867411 \angle 1.468072449^{\circ} \tag{19.107}$$

$$\overline{Z}_{\mathrm{m}} = 2.631736722 \times 10^{-6} \angle 45^{\circ} \ \ \Omega \tag{19.108}$$

$$\overline{Z}^1_{\text{int}} = 3.247190531 \times 10^{-4} \angle 46.46807245^{\circ} \ \ \Omega/\mathrm{m} \tag{19.109}$$

$$R = 2.236530654 \times 10^{-4} \ \ \Omega/\mathrm{m} \ ; \quad X_{\text{Lint}} = 2.354182868 \times 10^{-8} \ \ \Omega/\mathrm{m} \tag{19.110}$$

$$L_{\text{int}} = \frac{1}{\omega} \cdot X_{\text{Lint}} = 7.493596808 \times 10^{-7} \ \ \mathrm{H/m} \tag{19.111}$$

Example 19.3. Let there be n identical rectangular good LIH conductors with the same characteristics (μ, κ) placed in the slot of an electric machine surrounded by a medium with infinite permeability. Let the conductors be connected so that a sinusoidal electric current of the same RMS value flows through them. Using the complex Poynting theorem, determine the increase factor of the resistance and the decrease factor of the internal inductance due to the skin effect for each conductor. Assume that all magnetic field lines in the conductors are parallel to the x-axis and that the magnetic field intensity along these lines is constant in the conductors. Neglect displacement electric currents.

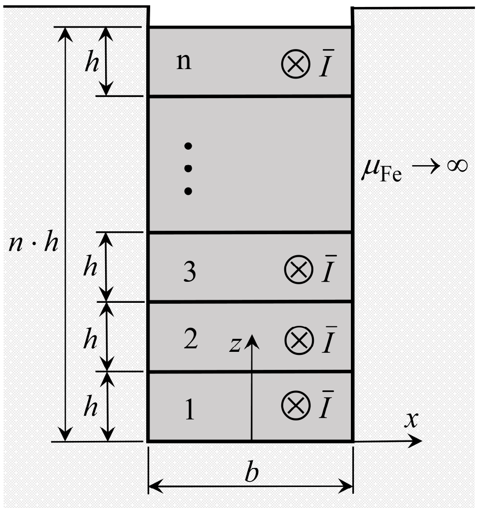


Figure 19.5. Rectangular conductors in a slot

Solution:

The per-unit-length resistance of the i-th conductor carrying a direct electric current (DC) is given by the expression:

$$R_{\mathrm{i}}^{\mathrm{DC}} = \frac{1}{\kappa \cdot b \cdot h} \tag{19.112}$$

where b is the width of the slot and h is the height of a single conductor. Since all conductors have the same dimensions and are made of the same material, the per-unit-length resistance of each conductor is identical.

It follows from the assumptions introduced that the distribution of the magnetic field intensity in all conductors carrying direct electric current can be described by the same expression given by:

$$H_{\mathrm{DC}} = \frac{I \cdot z}{b \cdot h} \tag{19.113}$$

where I is the direct electric current.

It follows that the magnetic energy stored in the magnetostatic field, per unit length of the i-th conductor, is described by the following expression:

$$W_{\mathrm{m,i}}^{\mathrm{DC}} = \frac{\mu}{2} \cdot \int_{(i-1)\cdot h}^{i \cdot h} H_{\mathrm{DC}}^2 \cdot b \cdot dz = \frac{1}{2} \cdot L_{\mathrm{int,i}}^{\mathrm{DC}} \cdot I^2 \tag{19.114}$$

from which it follows that the per-unit-length internal inductance of the i-th conductor carrying a direct electric current is given by the expression

$$L_{\mathrm{int,i}}^{\mathrm{DC}} = \frac{\mu \cdot h}{3 \cdot b} \cdot [3 \cdot i \cdot (i-1) + 1] \tag{19.115}$$

Based on the introduced assumptions, it follows that in the case of sinusoidal electric current in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (x, y) plane and depend only on the variable z. The following holds:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} \quad ; \quad \underline{\vec{H}} = \overline{H}_{\mathrm{x}} \cdot \vec{\mathrm{i}} = \overline{H} \cdot \vec{\mathrm{i}} \tag{19.116}$$

The phasor of the magnetic field intensity in the conductors is described by the 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial z^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.117}$$

whose particular solution in the i-th conductor is given by:

$$\overline{H} = \overline{A}_{\mathrm{i}} \cdot e^{-\overline{\gamma} \cdot z} + \overline{B}_{\mathrm{i}} \cdot e^{\overline{\gamma} \cdot z} \quad ; \quad (i-1) \cdot h \le z \le i \cdot h \tag{19.118}$$

where $\overline{A}_{\mathrm{i}}$ i $\overline{B}_{\mathrm{i}}$ are the unknown complex constants, which must satisfy the boundary conditions::

$$\overline{H}\Big|_{z=(i-1)\cdot h} = \frac{(i-1)\cdot \bar{I}}{b} \quad ; \quad \overline{H}\Big|_{z=i\cdot h} = \frac{i \cdot \bar{I}}{b} \tag{19.119}$$

from which it follows that:

$$\overline{H} = \frac{\bar{I}}{b} \cdot \frac{i \cdot \sinh\left[\overline{\gamma} \cdot z - \overline{\gamma}(i-1)\cdot h\right] - (i-1)\cdot \sinh\left[\overline{\gamma} \cdot z - \overline{\gamma} \cdot i \cdot h\right]}{\sinh\left(\overline{\gamma} \cdot h\right)} \tag{19.120}$$

From expression (19.97), it follows that the phasor of the electric field intensity in the i-th conductor is given by the expression:

$$\overline{E} = \frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial z} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{\bar{I}}{b} \cdot \frac{i \cdot \cosh\left[\overline{\gamma} \cdot z - \overline{\gamma}(i-1)\cdot h\right] - (i-1)\cdot \cosh\left[\overline{\gamma} \cdot z - \overline{\gamma} \cdot i \cdot h\right]}{\sinh\left(\overline{\gamma} \cdot h\right)} \tag{19.121}$$

If the complex Poynting theorem is applied to the unit length of the i-th conductor:

$$I^2 \cdot \overline{Z}^1_{\mathrm{int,i}} = -\oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} = -\oint_S \underline{\vec{\Gamma}} \cdot \vec{n} \cdot dS = \oint_S \overline{E} \cdot \overline{H}^* \cdot \vec{\mathrm{k}} \cdot \vec{n} \cdot dS \tag{19.122}$$

it follows that:

$$I^2 \cdot \overline{Z}^1_{\mathrm{int,i}} = b \cdot \left(\overline{E} \cdot \overline{H}^*\right)_{z=i\cdot h} - b \cdot \left(\overline{E} \cdot \overline{H}^*\right)_{z=(i-1)\cdot h} \tag{19.123}$$

from which it follows that the per-unit-length internal impedance of the i-th conductor is given by:

$$\overline{Z}^1_{\mathrm{int,i}} = R_{\mathrm{i}} + j \cdot \omega \cdot L_{\mathrm{int,i}} = \frac{\overline{\gamma}}{b \cdot \kappa} \cdot \left[\left(i^2 + (i-1)^2\right) \cdot \coth\left(\overline{\gamma} \cdot h\right) - \frac{2 \cdot i \cdot (i-1)}{\sinh\left(\overline{\gamma} \cdot h\right)}\right] \tag{19.124}$$

Since:

$$i^2 + (i-1)^2 = 2 \cdot i \cdot (i-1) + 1 \tag{19.125}$$

the expression (19.124) can be written in the following form:

$$\overline{Z}^1_{\mathrm{int,i}} = \frac{\overline{\gamma}}{b \cdot \kappa} \cdot \left[\coth\left(\overline{\gamma} \cdot h\right) + 2 \cdot i \cdot (i-1) \cdot \frac{\cosh\left(\overline{\gamma} \cdot h\right) - 1}{\sinh\left(\overline{\gamma} \cdot h\right)}\right] \tag{19.126}$$

If the following substitution is introduced into expression (19.126):

$$\overline{\gamma} \cdot h = (1 + j) \cdot \alpha \cdot h = (1 + j) \cdot \xi \quad ; \quad \xi = \alpha \cdot h \tag{19.127}$$

then the following expression is obtained:

$$\overline{Z}^1_{\mathrm{int,i}} = \frac{(1+j)\cdot \xi}{b \cdot \kappa \cdot h} \cdot \left[\frac{\sinh(2\cdot\xi) - j \cdot \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} + 2 \cdot i \cdot (i-1) \cdot \frac{\sinh(\xi) + j \cdot \sin(\xi)}{\cosh(\xi) + \cos(\xi)}\right] \tag{19.128}$$

From expression (19.128), it follows that the per-unit-length resistance of the i-th conductor carrying a sinusoidal electric current is given by the following expression:

$$R_{\mathrm{i}} = \frac{\xi}{b \cdot \kappa \cdot h} \cdot \left[\frac{\sinh(2\cdot\xi) + \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} + 2 \cdot i \cdot (i-1) \cdot \frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)}\right] \tag{19.129}$$

and that the per-unit-length internal inductance of the i-th conductor carrying a sinusoidal electric current is given by the expression:

$$L_{\text{int,i}} = \frac{\xi}{b \cdot \kappa \cdot h \cdot \omega} \cdot \left[\frac{\sinh(2 \cdot \xi) - \sin(2 \cdot \xi)}{\cosh(2 \cdot \xi) - \cos(2 \cdot \xi)} + 2 \cdot i \cdot (i-1) \cdot \frac{\sinh(\xi) + \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \right] \tag{19.130}$$

Since:

$$\alpha = \sqrt{\frac{\omega \cdot \mu \cdot \kappa}{2}} \quad ; \quad \frac{\xi}{\kappa \cdot h \cdot \omega} = \frac{\alpha}{\kappa \cdot \omega} = \frac{\alpha^2}{\kappa \cdot \omega \cdot \alpha} = \frac{\mu}{2 \cdot \alpha} = \frac{\mu \cdot h}{2 \cdot \xi} \tag{19.131}$$

it follows that:

$$L_{\text{int,i}} = \frac{\mu \cdot h}{b \cdot 2 \cdot \xi} \cdot \left[\frac{\sinh(2 \cdot \xi) - \sin(2 \cdot \xi)}{\cosh(2 \cdot \xi) - \cos(2 \cdot \xi)} + 2 \cdot i \cdot (i-1) \cdot \frac{\sinh(\xi) + \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \right] \tag{19.132}$$

If there are *n* identical conductors of height *h* placed in a rectangular slot (Figure 19.5), all carrying the same sinusoidal electric current, then the current crowding due to the skin effect increases from the lowest conductor (conductor 1) to the highest conductor (conductor n).

The increase factor of the resistance of the i-th conductor due to the skin effect is defined by the following expression:

$$k_{\text{Ri}} = \frac{R_{\text{i}}}{R_{\text{i}}^{\text{DC}}} \tag{19.133}$$

whereas the decrease factor of the internal inductance of the i-th conductor is defined by the expression:

$$k_{\text{Li}} = \frac{L_{\text{int, i}}}{L_{\text{int, i}}^{\text{DC}}} \tag{19.134}$$

If expressions (19.112) and (19.129) are substituted into expression (19.133), it follows that the resistance increase factor (which is also the power loss increase factor) due to the skin effect in the i-th conductor is described by the following expression:

$$k_{\text{Ri}} = \varphi_{\text{R}}(\xi) + i \cdot (i-1) \cdot \Psi_{\text{R}}(\xi) \quad ; \quad \xi = \alpha \cdot h \tag{19.135}$$

where:

$$\varphi_{\text{R}}(\xi) = \xi \cdot \frac{\sinh(2 \cdot \xi) + \sin(2 \cdot \xi)}{\cosh(2 \cdot \xi) - \cos(2 \cdot \xi)} \tag{19.136}$$

$$\Psi_{\text{R}}(\xi) = 2 \cdot \xi \cdot \frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.137}$$

If expressions (19.115) and (19.130) are substituted into expression (19.134), it follows that the decrease factor of the internal inductance due to the skin effect in the i-th conductor is described by:

$$k_{\text{Li}} = \frac{3}{3 \cdot i \cdot (i-1) + 1} \cdot [\varphi_{\text{L}}(\xi) + i \cdot (i-1) \cdot \Psi_{\text{L}}(\xi)] \quad ; \quad \xi = \alpha \cdot h \tag{19.138}$$

where:

$$\varphi_{\text{L}}(\xi) = \frac{1}{2 \cdot \xi} \cdot \frac{\sinh(2 \cdot \xi) - \sin(2 \cdot \xi)}{\cosh(2 \cdot \xi) - \cos(2 \cdot \xi)} \tag{19.139}$$

$$\Psi_{\text{L}}(\xi) = \frac{1}{\xi} \cdot \frac{\sinh(\xi) + \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.140}$$

At very high frequencies, the following approximations can be used:

$$k_{\text{Ri}} \approx [2 \cdot i \cdot (i-1) + 1] \cdot \xi \quad ; \quad k_{\text{Li}} \approx \frac{3 \cdot [2 \cdot i \cdot (i-1) + 1]}{2 \cdot \xi \cdot [3 \cdot i \cdot (i-1) + 1]} \tag{19.141}$$

Example 19.4. Calculate the internal impedance, resistance, and internal inductance per unit length of a rectangular copper conductor. Assume that the conductor is placed in a slot and carries a sinusoidal electric current with a frequency of $f = 50$ Hz. Assume that the permeability of the iron is infinite, that all magnetic field lines in the copper conductor are parallel to the x-axis, and that the magnetic field intensity along each field line in the conductor is constant. The following parameters are given: $h =$ 2 cm, $b = 8$ mm, $\kappa = 57$ MS/m. Displacement electric currents should be neglected. Solve the problem using analytical expressions for increasing the resistance and decreasing the internal inductance of the conductor in the slot. The conductor is shown in Figure 19.4.

Solution:

This example is identical to example 19.2, but should be solved using a different approach. In the case of a single conductor in the slot ($i = 1$), according to expressions (19.135) and (19.136), the increase factor of the resistance due to the skin effect in the conductor is given by the expression:

$$k_{\mathrm{R}} = k_{\mathrm{R1}} = \varphi_{\mathrm{R}}(\xi) = \xi \cdot \frac{\sinh(2\cdot\xi) + \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = \alpha \cdot h \tag{19.142}$$

whereas, according to expressions (19.138) and (19.139), the decrease factor of the internal inductance due to the skin effect in the conductor is given by the expression:

$$k_{\mathrm{L}} = k_{\mathrm{L1}} = 3 \cdot \varphi_{\mathrm{L}}(\xi) = \frac{3}{2\cdot\xi} \cdot \frac{\sinh(2\cdot\xi) - \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = \alpha \cdot h \tag{19.143}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi \cdot f \cdot \mu_0 \cdot \kappa} = 106.0723763 \ \ \mathrm{m}^{-1} \quad ; \quad \xi = \alpha \cdot h = 2.121447527 \tag{19.144}$$

$$k_{\mathrm{R}} = 2.039715957 \quad ; \quad k_{\mathrm{L}} = 0.7155857841 \tag{19.145}$$

According to the expression (19.112), the per-unit-length resistance of a conductor carrying a direct electric current is given by the expression:

$$R_{\mathrm{DC}} = R_1^{\mathrm{DC}} = \frac{1}{\kappa \cdot b \cdot h} = 1.096491228 \times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.146}$$

whereas, according to expression (19.112), the per-unit-length internal inductance of the given copper conductor carrying a direct electric current is given by the expression:

$$L_{\mathrm{int}}^{\mathrm{DC}} = L_{\mathrm{int},1}^{\mathrm{DC}} = \frac{\mu_0 \cdot h}{3 \cdot b} = 1.047197551 \times 10^{-6} \ \ \mathrm{H/m} \tag{19.147}$$

It follows that the per-unit-length resistance of the conductor carrying a sinusoidal electric current is:

$$R = k_{\mathrm{R}} \cdot R_{\mathrm{DC}} = 2.236530654 \times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.148}$$

whereas the per-unit-length internal inductance of the conductor carrying a sinusoidal current is:

$$L_{\mathrm{int}} = k_{\mathrm{L}} \cdot L_{\mathrm{int}}^{\mathrm{DC}} = 7.493596808 \times 10^{-7} \ \ \mathrm{H/m} \tag{19.149}$$

and the per-unit-length inductive reactance of the conductor carrying a sinusoidal electric current is:

$$X_{\mathrm{Lint}} = \omega \cdot L_{\mathrm{int}} = 2.354182868 \times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.150}$$

Finally, the per-unit-length internal impedance of the conductor carrying a sinusoidal electric current is:

$$\overline{Z}_{\mathrm{int}}^{1} = R + j \cdot X_{\mathrm{Lint}} = 3.247190531 \times 10^{-4} \angle 46.46807245^{\circ} \ \ \Omega/\mathrm{m} \tag{19.151}$$

Example 19.5. Calculate the internal impedance, resistance, and internal inductance per unit length of a rectangular copper conductor carrying a sinusoidal electric current, assuming that: $\ell >> a >> b$. The following is given: $a = 1$ cm, $b = 0.05$ cm, $\kappa = 56$ MS/m, $\mu = \mu_0$, $\omega = 10^5$ rad/s. Solve the problem using analytical expressions for increasing the resistance and decreasing the internal inductance of the conductor in the slot. The conductor is shown in Figure 19.1.

Solution:

This example is identical to example 19.1, but should be solved using a different approach. The given conductor can be considered as a special case of a rectangular conductor placed in a slot of an electrical machine, since each half of the conductor behaves as if it were located in a slot (Figure 19.4). In this case, the slot width is $2 \cdot a$, whereas the height of the considered half of the actual conductor in the slot is b. The magnetic field intensity is zero both in the center of the conductor and at the bottom of the slot. Therefore, half of the conductor is treated as if it were in a slot, and the resistance increase factor as well as the internal inductance decrease factor due to the skin effect are the same for both the half and the entire conductor.

In the case of a single conductor in the slot (i = 1), according to expressions (19.135) and (19.136), the increase factor of the resistance due to the skin effect in the conductor is given by the expression:

$$k_{\mathrm{R}} = k_{\mathrm{R1}} = \varphi_{\mathrm{R}}(\xi) = \xi \cdot \frac{\sinh(2\cdot\xi)+\sin(2\cdot\xi)}{\cosh(2\cdot\xi)-\cos(2\cdot\xi)} \quad ; \quad \xi = \alpha \cdot b \tag{19.152}$$

whereas, according to expressions (19.138) and (19.139), the decrease factor of the internal inductance due to the skin effect in the conductor is given by the expression:

$$k_{\mathrm{L}} = k_{\mathrm{L1}} = 3\cdot\varphi_{\mathrm{L}}(\xi) = \frac{3}{2\cdot\xi}\cdot \frac{\sinh(2\cdot\xi)-\sin(2\cdot\xi)}{\cosh(2\cdot\xi)-\cos(2\cdot\xi)} \quad ; \quad \xi = \alpha \cdot b \tag{19.153}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\frac{\omega\cdot\mu_0\cdot\kappa}{2}} = 1875.78884 \ \ \mathrm{m}^{-1} \quad ; \quad \xi = \alpha\cdot b = 0.93789442 \tag{19.154}$$

$$k_{\mathrm{R}} = 1.066814927 \quad ; \quad k_{\mathrm{L}} = 0.9809439966 \tag{19.155}$$

According to Figure 19.1, the per-unit-length resistance of the entire conductor carrying a direct electric current is given by the expression:

$$R_{\mathrm{DC}} = R_1^{\mathrm{DC}} = \frac{1}{\kappa\cdot 4\cdot a\cdot b} = 8.928571429\times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.156}$$

whereas, according to Figure 19.1, and under the introduced assumptions that reduce the problem to the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave, the magnetic field intensity in the conductor carrying a direct electric current is given by the expression:

$$H_{\mathrm{DC}} = -\frac{I_{\mathrm{enc}}}{2\cdot a} = -\frac{I\cdot x}{4\cdot a\cdot b} \quad ; \quad -b \le x \le b \tag{19.157}$$

where I is the direct electric current. It should be noted that, in the calculation of the magnetic field intensity, the integration path passes through the center of the conductor, where the magnetic field intensity is zero.

It follows that the magnetic energy stored in the magnetostatic field, per unit length of the entire copper conductor, is described by the following expression:

$$W_{\mathrm{m}}^{\mathrm{DC}} = \frac{\mu_0}{2}\cdot\int_{-b}^{b} H_{\mathrm{DC}}^2\cdot 2\cdot a\cdot dx = \frac{1}{2}\cdot L_{\mathrm{int}}^{\mathrm{DC}}\cdot I^2 \tag{19.158}$$

from which it follows that the per-unit-length internal inductance of the entire conductor carrying a direct electric current is given by the expression:

$$L_{\mathrm{int}}^{\mathrm{DC}} = \frac{\mu_0\cdot b}{12\cdot a} = 5.235987756\times 10^{-9} \ \ \mathrm{H/m} \tag{19.159}$$

It follows that the per-unit-length resistance of the entire conductor carrying a sinusoidal electric current is:

$$R = k_{\mathrm{R}}\cdot R_{\mathrm{DC}} = 9.525133279\times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.160}$$

whereas the per-unit-length internal inductance of the entire conductor carrying a sinusoidal electric current is:

$$L_{\text{int}} = k_{\text{L}} \cdot L_{\text{int}}^{\text{DC}} = 5.136210755 \times 10^{-9} \ \text{ H/m} \tag{19.161}$$

and the per-unit-length inductive reactance of the conductor carrying a sinusoidal electric current is:

$$X_{\text{Lint}} = \omega \cdot L_{\text{int}} = 5.136210755 \times 10^{-4} \ \ \Omega\text{/m} \tag{19.162}$$

Finally, the per-unit-length internal impedance of the conductor carrying a sinusoidal electric current is:

$$\overline{Z}_{\text{int}}^{1} = R + j \cdot X_{\text{Lint}} = 1.082168309 \times 10^{-3} \angle\, 28.33475267^{\circ} \ \ \Omega\text{/m} \tag{19.163}$$

Example 19.6. Using the complex Poynting theorem, calculate the per-unit-length impedance of a two-wire electric line carrying a sinusoidal electric current, under the assumption that: $\ell >> a >> b$. Given: $a = 1$ cm, $b =$ c $= 0.05$ cm, $\kappa = 56$ MS/m, $\mu = \mu_0$, $\omega = 10^5$ rad/s. Displacement electric currents should be neglected.

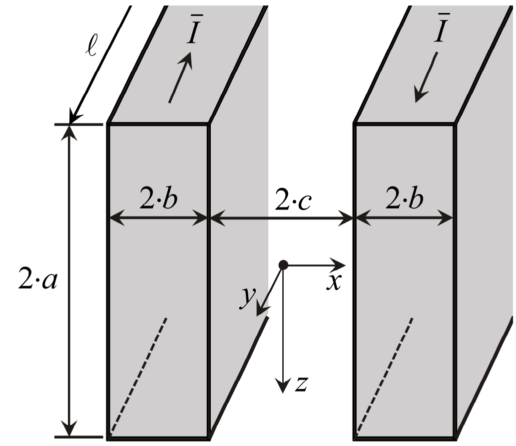


Figure 19.6. Two-wire electric line

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (y, z) plane and depend only on the variable x. The following holds:

$$\underline{\vec{E}} = \overline{E}_{\text{y}} \cdot \vec{\text{j}} = \overline{E} \cdot \vec{\text{j}} \quad ; \quad \underline{\vec{H}} = \overline{H}_{\text{z}} \cdot \vec{\text{k}} = \overline{H} \cdot \vec{\text{k}} \tag{19.164}$$

The phasor of the magnetic field intensity is described by the 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.165}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.166}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the given boundary conditions. Under the introduced assumptions, the following holds:

$$\overline{H} = 0 \quad \text{for} \quad |x| \geq c + 2 \cdot b \tag{19.167}$$

$$\overline{H} = \text{const.} \quad \text{for} \quad |x| \leq c \tag{19.168}$$

The function describing the distribution of the magnetic field intensity is an even function of x, whereas the function describing the distribution of the electric field intensity is an odd function of x.

The internal impedances of the conductors are the same. It is therefore sufficient to determine the internal impedance per unit length of only one conductor. Let this be the right-hand conductor ($x > 0$) from Figure 19.6. If Ampère's law is applied as shown in Figure 19.7:

$$\oint_C \underline{\vec{H}} \cdot d\vec{\ell} = \oint_C \overline{H}_z \cdot dz = \oint_C \overline{H} \cdot dz = \int_{-a}^{a} \overline{H}\Big|_{x=c} \cdot dz = \overline{I} \tag{19.169}$$

it follows that:

$$\overline{H}\Big|_{x=c} = \frac{\overline{I}}{2 \cdot a} \tag{19.170}$$

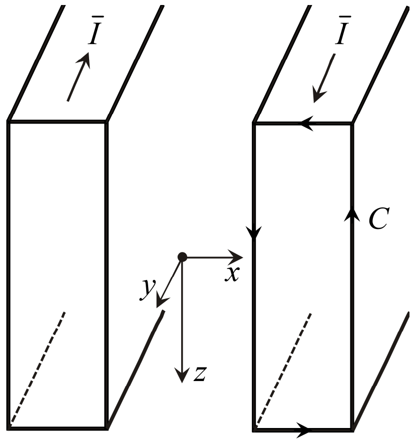


Figure 19.7. Integration curve C

Thus, the distribution of the magnetic field intensity in the right-hand conductor is described by the expression (19.166), whereas the boundary conditions are as follows:

$$\overline{H}\Big|_{x=c} = \frac{\overline{I}}{2 \cdot a} \quad ; \quad \overline{H}\Big|_{x=c+2\cdot b} = 0 \tag{19.171}$$

from which the system of linear equations follows:

$$\overline{H}_1 \cdot e^{-\overline{\gamma}\cdot c} + \overline{H}_2 \cdot e^{\overline{\gamma}\cdot c} = \frac{\overline{I}}{2 \cdot a} \tag{19.172}$$

$$\overline{H}_1 \cdot e^{-\overline{\gamma}\cdot (c + 2\cdot b)} + \overline{H}_2 \cdot e^{\overline{\gamma}\cdot (c + 2\cdot b)} = 0 \tag{19.173}$$

whose solutions are:

$$\overline{H}_1 = \frac{\overline{I}}{2 \cdot a} \cdot \frac{e^{\overline{\gamma}\cdot(c + 2\cdot b)}}{e^{2\cdot\overline{\gamma}\cdot b} - e^{-2\cdot\overline{\gamma}\cdot b}} \quad ; \quad \overline{H}_2 = -\frac{\overline{I}}{2 \cdot a} \cdot \frac{e^{-\overline{\gamma}\cdot(c + 2\cdot b)}}{e^{2\cdot\overline{\gamma}\cdot b} - e^{-2\cdot\overline{\gamma}\cdot b}} \tag{19.174}$$

It follows that the magnetic field intensity in the right-hand conductor is described by the expression:

$$\overline{H} = \frac{\overline{I}}{2 \cdot a} \cdot \frac{e^{-\overline{\gamma}\cdot(x - c - 2\cdot b)} - e^{\overline{\gamma}\cdot(x - c - 2\cdot b)}}{e^{2\cdot\overline{\gamma}\cdot b} - e^{-2\cdot\overline{\gamma}\cdot b}} \tag{19.175}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \overline{H} \end{vmatrix} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.176}$$

from which it follows that:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.177}$$

and it is:

$$\overline{E} = \overline{Z}_{\mathrm{m}} \cdot \frac{\overline{I}}{2 \cdot a} \cdot \frac{e^{-\overline{\gamma} \cdot (x - c - 2 \cdot b)} + e^{\overline{\gamma} \cdot (x - c - 2 \cdot b)}}{e^{2 \cdot \overline{\gamma} \cdot b} - e^{-2 \cdot \overline{\gamma} \cdot b}} \tag{19.178}$$

where:

$$\overline{Z}_{\mathrm{m}} = \frac{\overline{\gamma}}{\kappa} \tag{19.179}$$

is the wave impedance of the copper conductor, neglecting displacement electric currents.
In this case, the phasor of the Poynting vector is described by the expression:

$$\underline{\vec{\Gamma}} = \underline{\vec{E}} \times \underline{\vec{H}}^{*} = \overline{E} \cdot \overline{H}^{*} \cdot \left(\vec{\mathrm{j}} \times \vec{\mathrm{k}}\right) = \overline{E} \cdot \overline{H}^{*} \cdot \vec{\mathrm{i}} \tag{19.180}$$

whereas the complex apparent electromagnetic power entering the unit length of the right-hand conductor is given by the expression:

$$\overline{S}_{\mathrm{ap,in}} = P + j \cdot Q = I^{2} \cdot \overline{Z}_{\mathrm{int}}^{1} = -\oint_{S} \underline{\vec{\Gamma}} \cdot d\vec{S} = -\oint_{S} \overline{\Gamma} \cdot \vec{\mathrm{i}} \cdot \vec{n} \cdot dS \tag{19.181}$$

which reduces to an integration over the surface S_1 lying in the plane $x = c$ (Figure 19.8):

$$\overline{S}_{\mathrm{ap,in}} = -\int_{S_1} \overline{E} \cdot \overline{H}^{*} \cdot \vec{\mathrm{i}} \cdot \left(-\vec{\mathrm{i}}\right) \cdot dS = \left(\overline{E} \cdot \overline{H}^{*}\right)_{x=c} \cdot 2 \cdot a = I^{2} \cdot \overline{Z}_{\mathrm{int}}^{1} \tag{19.182}$$

because the phasor of the Poynting vector on surface S_2 is equal to zero and lies within the remaining four integration surfaces.

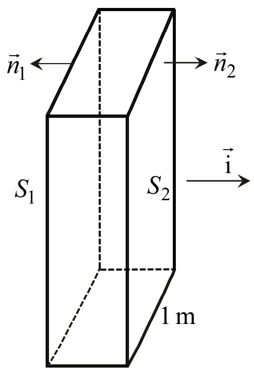


Figure 19.8. Integration surfaces

From expression (19.178), it follows that:

$$\overline{E}\Big|_{x=c} = \frac{\overline{Z}_{\mathrm{m}} \cdot \overline{I}}{2 \cdot a} \cdot \frac{e^{2 \cdot \overline{\gamma} \cdot b} + e^{-2 \cdot \overline{\gamma} \cdot b}}{e^{2 \cdot \overline{\gamma} \cdot b} - e^{-2 \cdot \overline{\gamma} \cdot b}} = \frac{\overline{Z}_{\mathrm{m}} \cdot \overline{I}}{2 \cdot a} \cdot \coth\left(2 \cdot \overline{\gamma} \cdot b\right) \tag{19.183}$$

whereas from expression (19.174), it follows that:

$$\overline{H}^{*}\Big|_{x=c} = \frac{\overline{I}^{*}}{2 \cdot a} \tag{19.184}$$

By combining the expressions (19.186), (19.184) and (19.182), the following expression is obtained:

$$\overline{S}_{\mathrm{ap,in}} = \overline{Z}_{\mathrm{m}} \cdot \frac{\overline{I} \cdot \overline{I}^{*}}{2 \cdot a} \cdot \coth\left(2 \cdot \overline{\gamma} \cdot b\right) = \overline{Z}_{\mathrm{m}} \cdot \frac{I^{2}}{2 \cdot a} \cdot \coth\left(2 \cdot \overline{\gamma} \cdot b\right) = I^{2} \cdot \overline{Z}_{\mathrm{int}}^{1} \tag{19.185}$$

from which it follows that the per-unit-length internal impedance of the conductor is given by:

$$\overline{Z}_{\mathrm{int}}^{1} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{1}{2 \cdot a} \cdot \coth\left(2 \cdot \overline{\gamma} \cdot b\right) = \frac{\overline{Z}_{\mathrm{m}}}{2 \cdot a} \cdot \coth\left(2 \cdot \overline{\gamma} \cdot b\right) \tag{19.186}$$

The per-unit-length internal impedance of the electric line is given by the expression:

$$\overline{Z}^{1}_{\text{int-linea}} = 2\cdot\overline{Z}^{1}_{\text{int}} = \frac{\overline{Z}_{\text{m}}}{a}\cdot\coth\left(2\cdot\overline{\gamma}\cdot b\right) = R + j\cdot\omega\cdot L_{\text{int}} \tag{19.187}$$

where:

R - per-unit-length resistance of the electric line,

L_{int} - per-unit-length inductance of the electric line.

For $\overline{\gamma} = (1+j)\cdot\alpha,$ the following expression holds:

$$\coth\left(2\cdot\overline{\gamma}\cdot b\right) = \frac{\sinh\left(4\cdot\alpha\cdot b\right) - j\cdot\sin\left(4\cdot\alpha\cdot b\right)}{\cosh\left(4\cdot\alpha\cdot b\right) - \cos\left(4\cdot\alpha\cdot b\right)} \tag{19.188}$$

The per-unit-length external inductance of the electric line can be easily calculated from the magnetic flux that flows between the conductors along the z-axis and is generated by a homogeneous magnetic field. The following holds:

$$L_{\text{ext}} = \frac{\overline{\Psi}}{\overline{I}} = \frac{\mu_0\cdot\overline{H}\big|_{x=c}\cdot 2\cdot c}{\overline{I}} = \mu_0\cdot\frac{c}{a} \tag{19.189}$$

The total per-unit-length impedance of the electric line is given by the expression:

$$\overline{Z}^{1}_{\text{line}} = \overline{Z}^{1}_{\text{int-line}} + j\cdot\omega\cdot L_{\text{ext}} = R + j\cdot\omega\cdot\left(L_{\text{int}} + L_{\text{ext}}\right) = R + j\cdot\omega\cdot L \tag{19.190}$$

where L is the total per-unit-length inductance of the electric line.
By substituting the given data, it follows that:

$$\alpha = \sqrt{\frac{\omega\cdot\mu_0\cdot\kappa}{2}} = 1875.78884\ \text{m}^{-1} \quad ; \quad 4\cdot\alpha\cdot b = 3.75157768 \tag{19.191}$$

$$\coth\left(2\cdot\overline{\gamma}\cdot b\right) = 0.9622413654\angle 1.54185187^{\circ} \tag{19.192}$$

$$\overline{Z}_{\text{m}} = 4.737082174\times10^{-5}\angle 45^{\circ}\ \ \Omega \tag{19.193}$$

$$\overline{Z}^{1}_{\text{int-line}} = 4.558216419\times10^{-3}\angle 46.54185187^{\circ}\ \ \Omega/\text{m} \tag{19.194}$$

$$X_{\text{ext}} = \omega\cdot L_{\text{ext}} = 6.283185307\times10^{-3}\ \ \Omega/\text{m} \tag{19.195}$$

$$\overline{Z}^{1}_{\text{line}} = \overline{Z}^{1}_{\text{int-line}} + j\cdot X_{\text{ext}} = 1.009129131\times10^{-2}\angle 71.89924283^{\circ}\ \ \Omega/\text{m} \tag{19.196}$$

$$R = 3.135253113\times10^{-3}\ \Omega/\text{m} \quad ; \quad X_{\text{L}} = 9.591889712\times10^{-3}\ \Omega/\text{m} \tag{19.197}$$

The per-unit-length resistance of a single conductor is::

$$R_{\text{cond}} = \frac{R}{2} = 1.567626557\times10^{-3}\ \ \Omega/\text{m} \tag{19.198}$$

According to expression (19.90), an isolated conductor with the same dimensions as in example 19.1, under the same assumptions, has a per-unit-length resistance of:

$$R_{\text{iso-cond}} = 9.525133293\times10^{-4}\ \ \Omega/\text{m} \tag{19.199}$$

The percentage increase in the per-unit-length resistance of the conductor due to the proximity effect of the two conductors is:

$$\Delta\text{R} = \frac{R_{\text{cond}} - R_{\text{iso-cond}}}{R_{\text{iso-cond}}}\cdot 100 = 64.5779\% \tag{19.200}$$

Note: The per-unit-length internal impedance of a single conductor can also be calculated using the resistance increase factor and the internal inductance decrease factor of an identical conductor placed in the slot of an electrical machine. In this case, the width of the imaginary slot is $2\cdot a$, whereas the height of the conductor in the slot is $2\cdot b$. According to the expression (19.152), the resistance increase factor in the conductor due to the skin effect is given by the expression:

$$k_{\mathrm{R}} = k_{\mathrm{R1}} = \varphi_{\mathrm{R}}(\xi) = \xi \cdot \frac{\sinh(2\cdot\xi) + \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = 2\cdot\alpha\cdot b \tag{19.201}$$

whereas, according to expression (19.153), the decrease factor of the internal inductance due to the skin effect in the conductor is given by the expression:

$$k_{\mathrm{L}} = k_{\mathrm{L1}} = 3\cdot\varphi_{\mathrm{L}}(\xi) = \frac{3}{2\cdot\xi} \cdot \frac{\sinh(2\cdot\xi) - \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = 2\cdot\alpha\cdot b \tag{19.202}$$

By substituting the given data, it follows that:

$$2\cdot\xi = 4\cdot\alpha\cdot b = 3.75157768 \tag{19.203}$$

$$k_{\mathrm{R}} = 1.755741743 \quad ; \quad k_{\mathrm{L}} = 0.789894992 \tag{19.204}$$

According to Figure 19.6, the per-unit-length resistance of the conductor carrying a direct electric current is given by the expression:

$$R_{\mathrm{DC}} = R_1^{\mathrm{DC}} = \frac{1}{\kappa\cdot 4\cdot a\cdot b} = 8.928571429\times 10^{-4} \ \ \Omega/\mathrm{m} \tag{19.205}$$

and it is:

$$R_{\mathrm{cond}} = \frac{R}{2} = k_{\mathrm{R}}\cdot R_{\mathrm{DC}} = 1.567626557\cdot 10^{-3} \ \ \Omega/\mathrm{m}$$

where R is the per-unit-length resistance of the two-wire electric line.

According to expression (19.115), with the substitutions $i = 1$, $h \rightarrow 2\cdot\mathrm{b}$, $b \rightarrow 2\cdot\mathrm{a}$, the per-unit-length internal inductance of a single copper conductor carrying a direct electric current is given by the expression:

$$L_{\text{int-cond}}^{\mathrm{DC}} = \frac{\mu_0\cdot b}{3\cdot a} = 2.094395102\times 10^{-8} \ \ \mathrm{H/m} \tag{19.206}$$

and it is:

$$X_{\text{int-cond}} = \omega\cdot k_{\mathrm{L}}\cdot L_{\text{int-cond}}^{\mathrm{DC}} = 1.654352202\times 10^{-3} \ \ \Omega/\mathrm{m} \tag{19.207}$$

It follows that the per-unit-length internal impedance of the conductor is:

$$\overline{Z}_{\mathrm{int}}^{1} = R_{\mathrm{cond}} + j\cdot X_{\text{int-cond}} = 2.27910821\times 10^{-3}\angle 46.541851863^{\circ} \ \ \Omega/\mathrm{m} \tag{19.208}$$

and thus, the per-unit-length internal impedance of the electric line is:

$$\overline{Z}_{\text{int-line}}^{1} = 2\cdot\overline{Z}_{\mathrm{int}}^{1} = 4.55821642\times 10^{-3}\angle 46.541851863^{\circ} \ \ \Omega/\mathrm{m} \tag{19.209}$$

The last decimal places of the results from expressions (19.194) and (19.209) differ due to rounding errors.

Example 19.7. P Using the complex Poynting theorem, calculate the total per-unit-length internal impedance of a system consisting of two identical parallel conductors carrying a sinusoidal electric current, under the assumption that: $\ell >> a >> b$. Given: $a = 1$ cm, $b = \mathrm{c} = 0.05$ cm, $\kappa = 56$ MS/m, $\mu = \mu_0$, $\omega = 10^5$ rad/s. Displacement electric currents should be neglected.

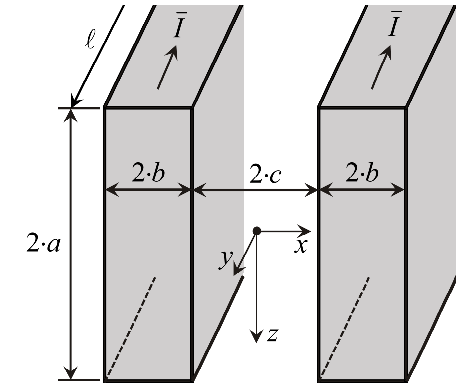


Figure 19.9. Two parallel copper conductors

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (y, z) plane and depend only on the variable x. The following holds:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} \quad ; \quad \underline{\vec{H}} = \overline{H}_{\mathrm{z}} \cdot \vec{\mathrm{k}} = \overline{H} \cdot \vec{\mathrm{k}} \tag{19.210}$$

The phasor of the magnetic field intensity is described by the 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.211}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.212}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the given boundary conditions. Under the introduced assumptions, the following holds:

$$\overline{H} = 0 \quad \text{for} \quad |x| \leq c \qquad \text{19.213)}$$

$$\overline{H} = \frac{\overline{I}}{2 \cdot a} \quad \text{for} \quad x \geq c + 2 \cdot b \quad ; \quad \overline{H} = -\frac{\overline{I}}{2 \cdot a} \quad \text{for} \quad x \leq -(c + 2 \cdot b) \tag{19.214}$$

The conductors in this and the previous example are identical, and the distributions of the magnetic field intensity and the electric field intensity within the conductors are of the same type in both cases, since the boundary conditions for the magnetic field intensity are equivalent. Therefore, the per-unit-length internal impedance of a single conductor in this problem is equal to the per-unit-length internal impedance of a single conductor obtained in the previous example. According to expression (19.208), the following holds:

$$\overline{Z}_{\mathrm{int}}^{1} = R_{\mathrm{cond}} + j \cdot X_{\mathrm{int\text{-}cond}} = 2.27910821 \times 10^{-3} \angle 46.541851863^{\mathrm{o}} \ \Omega/\mathrm{m} \tag{19.215}$$

thus, the total per-unit-length internal impedance of the two parallel conductors is:

$$\overline{Z}_{\mathrm{int\text{-}system}}^{1} = \frac{\overline{Z}_{\mathrm{int}}^{1}}{2} = 1.1395541105 \times 10^{-3} \angle 46.541851863^{\mathrm{o}} \ \Omega/\mathrm{m} \tag{19.216}$$

Note: The per-unit-length internal impedance of a single conductor can be calculated either using the complex Poynting theorem or by applying the resistance increase factor and the internal inductance decrease factor of an identical conductor placed in the slot of an electrical machine. In this case, the external inductance is considered infinite because the hypothetical return conductor is located at infinity.

Example 19.8. A copper conductor of height h and width b is placed in a rectangular slot and carries a sinusoidal electric current with a frequency of f = 50 Hz. Assume that the permeability of the iron is infinite, that all magnetic field lines within the copper conductor are parallel to the x-axis, and that the magnetic field intensity along each field line in the copper is constant. Displacement electric currents are to be neglected. If the phasor of the electric field intensity is given as $\overline{E} = \overline{E}_z = 0.0065\angle 24^{\mathrm{o}}$ V/m at the upper edge of the conductor ($y = h$), calculate the phasor of the electric field intensity at the lower edge of the conductor ($y = 0$) and the phasor of the electric current. Given: h = 1 cm, b = 5 mm, κ = 57 MS/m.

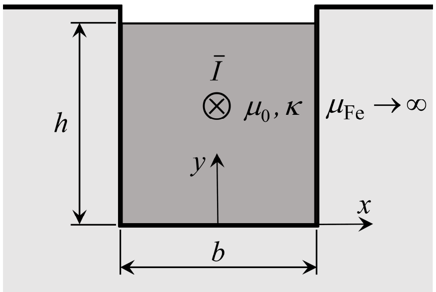


Figure 19.10. Rectangular copper conductor in a slot

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. The phasors of the electric field intensity vector and the magnetic field intensity vector lie in the (x, z) plane and depend only on the variable y. The following holds:

$$\underline{\vec{E}} = \overline{E}_z \cdot \vec{\mathrm{k}} = \overline{E} \cdot \vec{\mathrm{k}} \quad ; \quad \underline{\vec{H}} = \overline{H}_x \cdot \vec{\mathrm{i}} = \overline{H} \cdot \vec{\mathrm{i}} \tag{19.217}$$

The phasor of the magnetic field intensity is described by the 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial y^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.218}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot y} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot y} \tag{19.219}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the following boundary conditions:

$$\overline{H}\Big|_{y=0} = 0 \quad ; \quad \overline{H}\Big|_{y=h} = \frac{\overline{I}}{b} \tag{19.220}$$

from which it follows that:

$$\overline{H} = \frac{\overline{I}}{b} \cdot \frac{\sinh(\overline{\gamma} \cdot y)}{\sinh(\overline{\gamma} \cdot h)} \tag{19.221}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ 0 & \dfrac{\partial}{\partial y} & 0 \\ \overline{H} & 0 & 0 \end{vmatrix} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial y} \cdot \vec{\mathrm{k}} \tag{19.222}$$

from which it follows that:

$$\underline{\vec{E}} = \overline{E}_z \cdot \vec{k} = \overline{E} \cdot \vec{k} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial y} \cdot \vec{k} \tag{19.223}$$

and it is:

$$\overline{E} = -\frac{\overline{\gamma}}{\kappa} \cdot \frac{\overline{I}}{b} \cdot \frac{\cosh(\overline{\gamma} \cdot y)}{\sinh(\overline{\gamma} \cdot h)} \tag{19.224}$$

Given that:

$$\overline{E}_h = \overline{E}_z \Big|_{y=h} = \overline{E}\Big|_{y=h} = -\frac{\overline{\gamma}}{\kappa} \cdot \frac{\overline{I}}{b} \cdot \frac{\cosh(\overline{\gamma} \cdot h)}{\sinh(\overline{\gamma} \cdot h)} = 0.0065 \angle 24^{\circ} \text{ V/m} \tag{19.225}$$

and the following is to be calculated:

$$\overline{E}_0 = \overline{E}\Big|_{y=0} = -\frac{\overline{\gamma}}{\kappa} \cdot \frac{\overline{I}}{b} \cdot \frac{1}{\sinh(\overline{\gamma} \cdot h)} = -\overline{Z}_m \cdot \frac{\overline{I}}{b} \cdot \frac{1}{\sinh(\overline{\gamma} \cdot h)} \tag{19.226}$$

From the expressions (19.225) and (19.226), it follows that:

$$\overline{E}_0 = \frac{\overline{E}_h}{\cosh(\overline{\gamma} \cdot h)} \quad ; \quad \overline{I} = -\frac{\overline{E}_0 \cdot b}{\overline{Z}_m} \cdot \sinh(\overline{\gamma} \cdot h) \tag{19.227}$$

where:

$$\sinh(\overline{\gamma} \cdot h) = \sinh(\alpha \cdot h + j \cdot \alpha \cdot h) = \sinh(\alpha \cdot h) \cdot \cos(\alpha \cdot h) + j \cdot \cosh(\alpha \cdot h) \cdot \sin(\alpha \cdot h) \tag{19.228}$$

$$\cosh(\overline{\gamma} \cdot h) = \cosh(\alpha \cdot h + j \cdot \alpha \cdot h) = \cosh(\alpha \cdot h) \cdot \cos(\alpha \cdot h) + j \cdot \sinh(\alpha \cdot h) \cdot \sin(\alpha \cdot h) \tag{19.229}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi \cdot f \cdot \mu_0 \cdot \kappa} = 106.0723763 \text{ m}^{-1} \quad ; \quad \alpha \cdot h = 1.060723763 \tag{19.230}$$

$$\sinh(\overline{\gamma} \cdot h) = 1.541877778 \angle 66.26502004^{\circ} \tag{19.231}$$

$$\cosh(\overline{\gamma} \cdot h) = 1.361669857 \angle 54.5557039^{\circ} \tag{19.232}$$

$$\overline{Z}_m = 2.631736722 \times 10^{-6} \angle 45^{\circ} \ \Omega \tag{19.233}$$

$$\overline{E}_0 = 4.773550626 \times 10^{-3} \angle -30.5557039^{\circ} \text{ V/m} \tag{19.234}$$

$$\overline{I} = 13.98360172 \angle 170.7093161^{\circ} \text{ A} \tag{19.235}$$

Example 19.9. In a rectangular slot of width b, there is: a) a copper conductor of height $2 \cdot h$ carrying a sinusoidal electric current $2 \cdot \overline{I}$, b) two copper conductors of height h, each carrying a sinusoidal electric current $\overline{I}$. Assume that the permeability of the iron is infinite, that all magnetic field lines in the copper conductor are parallel to the x-axis, and that the magnetic field intensity along each field line in the copper is constant. Displacement electric currents are to be neglected. Determine the ratio of electric power losses in these two cases. Given: h = 1 cm, κ = 57 MS/m, f = 50 Hz.

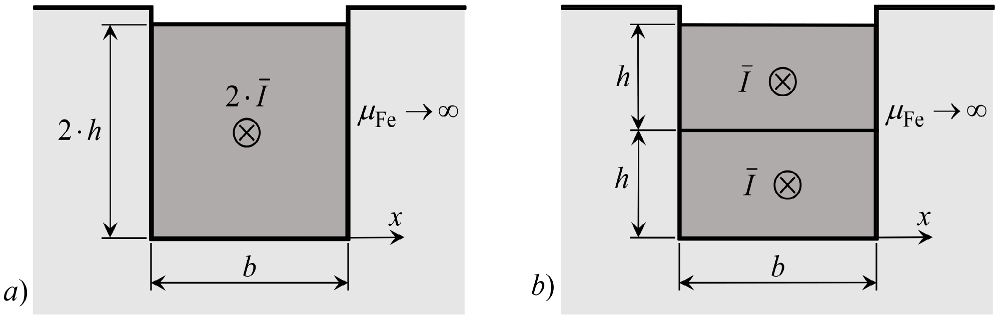


Figure 19.11. Copper conductors in a slot

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. This means that the expressions for the resistance increase factor in the conductors due to the skin effect can be used.

a) A single conductor in the slot

According to expressions (19.135) - (19.137), for a single conductor of height 2·*h*, the resistance increase factor is given by the expression:

$$k_{\mathrm{Ra}} = \varphi_{\mathrm{R}}(\xi) = \xi\cdot\frac{\sinh(2\cdot\xi)+\sin(2\cdot\xi)}{\cosh(2\cdot\xi)-\cos(2\cdot\xi)} \quad ; \quad \xi = 2\cdot\alpha\cdot h \tag{19.236}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi\cdot f\cdot\mu_0\cdot\kappa} = 106.0723763 \ \mathrm{m}^{-1} \ ; \ 2\cdot\alpha\cdot h = 2.121447526 \tag{19.237}$$

$$k_{\mathrm{Ra}} = 2.039715956 \tag{19.238}$$

b) Two conductors in the slot

According to expressions (19.135) - (19.137), for two conductors of height *h*, the resistance increase factor is given by the expressions:

$$k_{\mathrm{Rb1}} = \xi\cdot\frac{\sinh(2\cdot\xi)+\sin(2\cdot\xi)}{\cosh(2\cdot\xi)-\cos(2\cdot\xi)} \quad ; \quad \xi = \alpha\cdot h \tag{19.239}$$

$$k_{\mathrm{Rb2}} = k_{\mathrm{Rb1}} + 4\cdot\xi\cdot\frac{\sinh(\xi)-\sin(\xi)}{\cosh(\xi)+\cos(\xi)} \tag{19.240}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi\cdot f\cdot\mu_0\cdot\kappa} = 106.0723763 \ \mathrm{m}^{-1} \ ; \ \alpha\cdot h = 1.060723763 \tag{19.241}$$

$$k_{\mathrm{Rb1}} = 1.107367415 \quad ; \quad k_{\mathrm{Rb2}} = 1.910212277 \tag{19.242}$$

The required ratio of electric power losses is:

$$\frac{P_{\mathrm{a}}}{P_{\mathrm{b}}} = \frac{2\cdot k_{\mathrm{Ra}}}{k_{\mathrm{Rb1}}+k_{\mathrm{Rb2}}} = 1{,}351888709 \tag{19.243}$$

Example 19.10. In a rectangular slot of width *b*, there is: a) a copper conductor of height 5·*h* carrying a sinusoidal electric current $5\cdot\bar{I}$, b) five copper conductors of height *h*, each carrying a sinusoidal electric current $\bar{I}$. Assume that the permeability of the iron is infinite, that all magnetic field lines in the copper conductor are parallel to the *x*-axis, and that the magnetic field intensity along each field line in the copper is constant. Displacement electric currents are to be neglected. Determine the ratio of electric power losses in these two cases. Given: h = 6 mm, κ = 57 MS/m, f = 50 Hz.

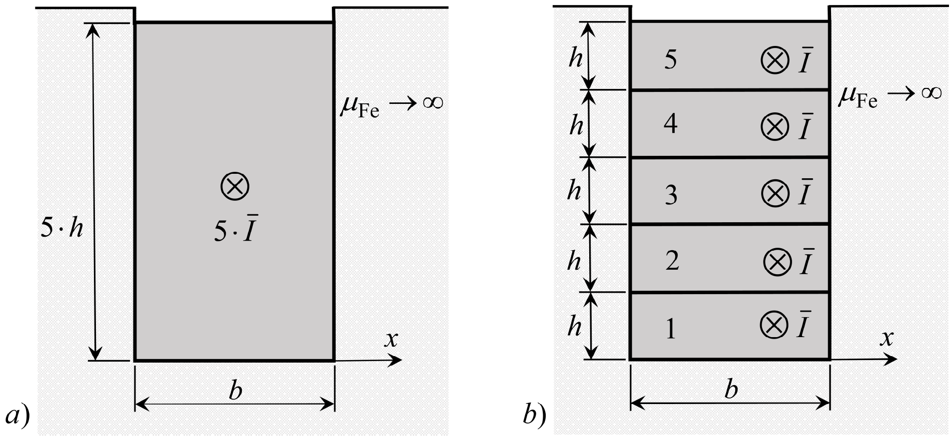


Figure 19.12. Copper conductors in a slot

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. This means that the expressions for the resistance increase factor in the conductors due to the skin effect can be used.

a) A single conductor in the slot

According to expressions (19.135) - (19.137), for a single conductor of height $5{\cdot}h$, the resistance increase factor is given by the expression:

$$k_{\mathrm{Ra}} = \varphi_{\mathrm{R}}(\xi) = \xi \cdot \frac{\sinh(2\cdot\xi) + \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = 5\cdot\alpha\cdot h \tag{19.244}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi \cdot f \cdot \mu_0 \cdot \kappa} = 106.0723763 \ \mathrm{m}^{-1} \ ; \ 5\cdot\alpha\cdot h = 3.182171289 \tag{19.245}$$

$$k_{\mathrm{Ra}} = 3.194003913 \tag{19.246}$$

b) Five conductors in the slot

According to expressions (19.135) - (19.137), for five conductors of height h, the resistance increase factor is given by the expressions:

$$k_{\mathrm{Rb1}} = \xi \cdot \frac{\sinh(2\cdot\xi) + \sin(2\cdot\xi)}{\cosh(2\cdot\xi) - \cos(2\cdot\xi)} \quad ; \quad \xi = \alpha\cdot h \tag{19.247}$$

$$k_{\mathrm{Rb2}} = k_{\mathrm{Rb1}} + 4\cdot\xi\cdot\frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.248}$$

$$k_{\mathrm{Rb3}} = k_{\mathrm{Rb1}} + 12\cdot\xi\cdot\frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.249}$$

$$k_{\mathrm{Rb4}} = k_{\mathrm{Rb1}} + 24\cdot\xi\cdot\frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.250}$$

$$k_{\mathrm{Rb5}} = k_{\mathrm{Rb1}} + 40\cdot\xi\cdot\frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.251}$$

and it is:

$$k_{\mathrm{Rb}} = \sum_{i=1}^{5} k_{\mathrm{Rb}i} = 5\cdot k_{\mathrm{Rb1}} + 80\cdot\xi\cdot\frac{\sinh(\xi) - \sin(\xi)}{\cosh(\xi) + \cos(\xi)} \tag{19.252}$$

By substituting the given data, it follows that:

$$\alpha = \sqrt{\pi \cdot f \cdot \mu_0 \cdot \kappa} = 106.0723763 \ \mathrm{m}^{-1} \ ; \ \alpha \cdot h = 0.6364342578 \tag{19.253}$$

$$k_{\mathrm{Rb1}} = 1.014492957 \quad ; \quad k_{\mathrm{Rb}} = 7.245559817 \tag{19.254}$$

The required ratio of electric power losses is:

$$\frac{P_{\mathrm{a}}}{P_{\mathrm{b}}} = \frac{5 \cdot k_{\mathrm{Ra}}}{k_{\mathrm{Rb}}} = 2.204111203 \tag{19.255}$$

Example 19.11. In a rectangular slot of width bb, there are five copper conductors of height h, each carrying a sinusoidal electric current $\bar{I}$. Assume that the permeability of the iron is infinite, that all magnetic field lines in the copper conductor are parallel to the x-axis, and that the magnetic field intensity along each field line in the copper is constant. Displacement electric currents are to be neglected. Let the resistance increase factor of the first conductor be 1.04, and let the resistance increase factor of the third conductor be 2.12. Calculate the resistance increase factors for the remaining three conductors.

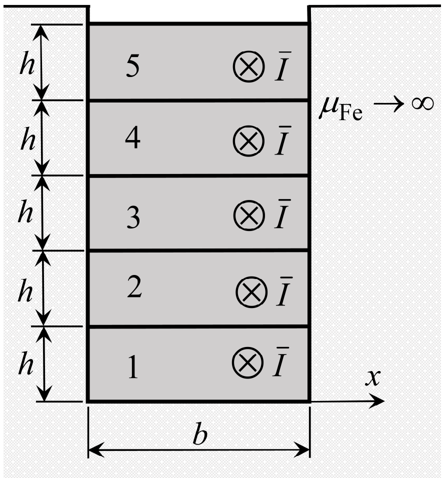


Figure 19.13. Five conductors in a rectangular slot

Solution:

From the given assumptions, it follows that in a good conductor, the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used. This means that the expressions for the resistance increase factor in the conductors due to the skin effect can be used.

According to expressions (19.135) - (19.137), for five conductors of height h, the resistance increase factor is given by the expressions:

$$\varphi_{\mathrm{R}} = \varphi_{\mathrm{R}}(\xi) \ ; \ \Psi_{\mathrm{R}} = \psi_{\mathrm{R}}(\xi) \ ; \ k_{\mathrm{R1}} = \varphi_{\mathrm{R}} \ ; \ k_{\mathrm{R2}} = \varphi_{\mathrm{R}} + 2 \cdot \Psi_{\mathrm{R}} \tag{19.256}$$

$$k_{\mathrm{R3}} = \varphi_{\mathrm{R}} + 6 \cdot \Psi_{\mathrm{R}} \ ; \ k_{\mathrm{R4}} = \varphi_{\mathrm{R}} + 12 \cdot \Psi_{\mathrm{R}} \ ; \ k_{\mathrm{R5}} = \varphi_{\mathrm{R}} + 20 \cdot \Psi_{\mathrm{R}} \tag{19.257}$$

Based on the given data, the following system of equations can be formed:

$$k_{\mathrm{R1}} = \varphi_{\mathrm{R}} = 1.04 \quad ; \quad k_{\mathrm{R3}} = \varphi_{\mathrm{R}} + 6 \cdot \Psi_{\mathrm{R}} = 2.12 \tag{19.258}$$

from which it follows that:

$$\varphi_{\mathrm{R}} = 1.04 \quad ; \quad \Psi_{\mathrm{R}} = 0.18 \tag{19.259}$$

and it is:

$$k_{\mathrm{R2}} = 1.4 \quad ; \quad k_{\mathrm{R4}} = 3.2 \quad ; \quad k_{\mathrm{R5}} = 4.64 \tag{19.260}$$

Example 19.12. Determine the functions that describe the distribution of sinusoidal magnetic flux density and surface density of the electric current in an LIH ferromagnetic sheet of thickness d, electrical conductivity κ, and magnetic permeability μ. Let the magnetic flux density vector be parallel to the z-axis, and assume that the magnetic field intensity $\overline{H} = \overline{H}_0$ is given at the edges of the sheet (for $x = \pm d/2$). Solve the problem using the magnetodynamic approximation of a linearly polarized uniform plane wave in a good conductor.

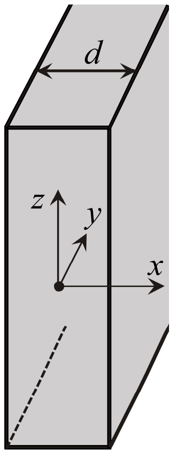


Figure 19.14. Ferromagnetic sheet

Solution:

Since the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used, the following holds:

$$\underline{\vec{E}} = \overline{E}_y \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} \quad ; \quad \underline{\vec{H}} = \overline{H}_z \cdot \vec{\mathrm{k}} = \overline{H} \cdot \vec{\mathrm{k}} \tag{19.261}$$

The phasor of the magnetic field intensity is described by the following 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.262}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.263}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the following boundary conditions:

$$\overline{H}\Big|_{x=\pm d/2} = \overline{H}_0 \tag{19.264}$$

from which it follows that:

$$\overline{H} = \overline{H}_0 \cdot \frac{\cosh(\overline{\gamma} \cdot x)}{\cosh(\overline{\gamma} \cdot d/2)} \tag{19.265}$$

and thus, the distribution of the magnetic flux density is described by the expression:

$$\overline{B} = \overline{B}_z = \mu \cdot \overline{H} = \mu \cdot \overline{H}_0 \cdot \frac{\cosh(\overline{\gamma} \cdot x)}{\cosh(\overline{\gamma} \cdot d/2)} \tag{19.266}$$

The phasor of the vector of the surface density of the electric current can be determined using Maxwell's first differential equation, which in this case is given by:

$$\underline{\vec{J}} = \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \overline{H} \end{vmatrix} = -\frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.267}$$

from which it follows that:

$$\underline{\vec{J}} = \overline{J} \cdot \vec{\mathrm{j}} = -\frac{\partial \overline{H}}{\partial y} \cdot \vec{\mathrm{j}} \tag{19.268}$$

and it is:

$$\overline{J} = \overline{J}_{\mathrm{y}} = -\overline{\gamma} \cdot \overline{H}_0 \cdot \frac{\sinh\left(\overline{\gamma} \cdot x\right)}{\cosh\left(\overline{\gamma} \cdot d/2\right)} \tag{19.269}$$

It is important to note that $\overline{J}$ is the phasor of the surface density of the eddy current, which opposes the change in the magnetic flux linkage.

Example 19.13. Let there be N conductors on each side of an LIH ferromagnetic core, each carrying a sinusoidal electric current with an RMS value I and angular frequency ω. Determine the equivalent impedance of the core. Solve the problem using the magnetodynamic approximation of a linearly polarized uniform plane wave in a good LIH conductor. Neglect the magnetic reluctance of the surrounding medium.

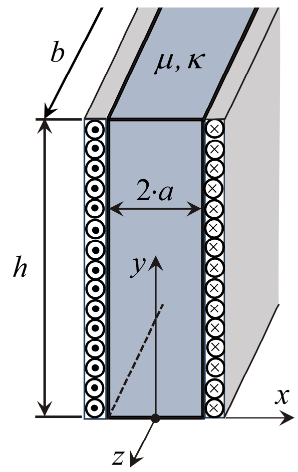


Figure 19.15. Core and conductors carrying electric current

Solution:

Since the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used, the following holds:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{z}} \cdot \vec{\mathrm{k}} = \overline{E} \cdot \vec{\mathrm{k}} \quad ; \quad \underline{\vec{H}} = \overline{H}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{H} \cdot \vec{\mathrm{j}} \tag{19.270}$$

The phasor of the magnetic field intensity is described by the following 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.271}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.272}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the following boundary conditions:

$$\underline{\overline{H}}\Big|_{x=\pm a} = \frac{N \cdot \overline{I}}{h} \tag{19.273}$$

from which it follows that:

$$\overline{H} = \frac{N \cdot \overline{I}}{h} \cdot \frac{\cosh(\overline{\gamma} \cdot x)}{\cosh(\overline{\gamma} \cdot a)} \tag{19.274}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \overline{H} & 0 \end{vmatrix} = \frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{k}} \quad ; \quad \underline{\vec{E}} = \overline{E}_z \cdot \vec{\mathrm{k}} = \overline{E} \cdot \vec{\mathrm{k}} = \frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{k}} \tag{19.275}$$

and it is:

$$\overline{E} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{N \cdot \overline{I}}{h} \cdot \frac{\sinh(\overline{\gamma} \cdot x)}{\cosh(\overline{\gamma} \cdot a)} = \overline{Z}_{\mathrm{m}} \cdot \frac{N \cdot \overline{I}}{h} \cdot \frac{\sinh(\overline{\gamma} \cdot x)}{\cosh(\overline{\gamma} \cdot a)} \quad ; \quad \overline{Z}_{\mathrm{m}} = \frac{\overline{\gamma}}{\kappa} \tag{19.276}$$

In this case, the phasor of the Poynting vector is described by the expression:

$$\underline{\vec{\Gamma}} = \underline{\vec{E}} \times \underline{\vec{H}}^* = \overline{E} \cdot \overline{H}^* \cdot \left(\vec{\mathrm{k}} \times \vec{\mathrm{j}}\right) = -\,\overline{E} \cdot \overline{H}^* \cdot \vec{\mathrm{i}} \tag{19.277}$$

whereas the complex apparent electromagnetic power entering the core is described by the expression:

$$\overline{S}_{\mathrm{ap,in}} = P + j \cdot Q = I^2 \cdot \overline{Z} = -\oint_S \underline{\vec{\Gamma}} \cdot d\vec{S} = \oint_S \overline{\Gamma} \cdot \vec{\mathrm{i}} \cdot \vec{n} \cdot dS \tag{19.278}$$

which reduces to an integration over the surface S_1, lying in the plane $x = -\,a,$ and the surface S_2, lying in the plane $x = a$ (Figure 19.16):

$$\overline{S}_{\mathrm{ap,in}} = -\left(\overline{E} \cdot \overline{H}^*\right)_{x=-a} \cdot S_1 + \left(\overline{E} \cdot \overline{H}^*\right)_{x=a} \cdot S_2 = I^2 \cdot \overline{Z} \quad ; \quad S_1 = S_2 = b \cdot h \tag{19.279}$$

since the phasor of the Poynting vector lies within the remaining four integration surfaces.

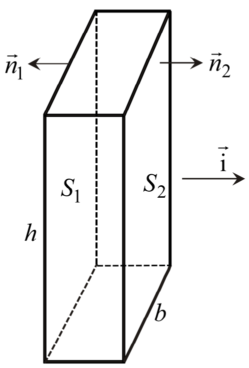


Figure 19.16. Integration surfaces

If the expressions (19.274) and (19.276) are substituted into the expression (19.279), the following expression for the equivalent impedance of the ferromagnetic core is obtained

$$\overline{Z} = \frac{2 \cdot b \cdot N^2}{h} \cdot \overline{Z}_{\mathrm{m}} \cdot \tanh(\overline{\gamma} \cdot a) = R + j \cdot \omega \cdot L \tag{19.280}$$

where R is the equivalent resistance of the core, and L is the equivalent inductance of the core.

The following holds:

$$\tanh\left(\bar{\gamma}\cdot a\right) = \tanh\left(\alpha\cdot a + j\cdot\alpha\cdot a\right) = \frac{\sinh\left(2\cdot\alpha\cdot a\right) + j\cdot\sin\left(2\cdot\alpha\cdot a\right)}{\cosh\left(2\cdot\alpha\cdot a\right) + \cos\left(2\cdot\alpha\cdot a\right)} \tag{19.281}$$

Note: According to the complex Poynting theorem in its integral form, the same solution for *R* and *L* in this example can also be obtained, in a more complicated way, using the expressions:

$$R\cdot I^2 = \kappa\cdot\int_V \underline{\vec{E}}\cdot\underline{\vec{E}}^*\cdot dV \quad ; \quad \frac{L\cdot I^2}{2} = \frac{1}{2}\cdot\mu\cdot\int_V \underline{\vec{H}}\cdot\underline{\vec{H}}^*\cdot dV \tag{19.282}$$

where it becomes necessary to use the following expressions:

$$\sinh\left(\bar{\gamma}\cdot x\right)\cdot\left(\sinh\left(\bar{\gamma}\cdot x\right)\right)^* = \left|\sinh\left(\bar{\gamma}\cdot x\right)\right|^2 = \frac{\cosh\left(2\cdot\alpha\cdot x\right) - \cos\left(2\cdot\alpha\cdot x\right)}{2} \tag{19.283}$$

$$\cosh\left(\bar{\gamma}\cdot x\right)\cdot\left(\cosh\left(\bar{\gamma}\cdot x\right)\right)^* = \left|\cosh\left(\bar{\gamma}\cdot x\right)\right|^2 = \frac{\cosh\left(2\cdot\alpha\cdot x\right) + \cos\left(2\cdot\alpha\cdot x\right)}{2} \tag{19.284}$$

and it is also useful to know the following expressions:

$$\sinh\left(\bar{\gamma}\cdot x\right)\cdot\left(\cosh\left(\bar{\gamma}\cdot x\right)\right)^* = \frac{\sinh\left(2\cdot\alpha\cdot x\right) + j\sin\left(2\cdot\alpha\cdot x\right)}{2} \tag{19.285}$$

$$\cosh\left(\bar{\gamma}\cdot x\right)\cdot\left(\sinh\left(\bar{\gamma}\cdot x\right)\right)^* = \frac{\sinh\left(2\cdot\alpha\cdot x\right) - j\sin\left(2\cdot\alpha\cdot x\right)}{2} \tag{19.286}$$

From expressions (19.283) - (19.285), the previously used expressions can be easily derived:

$$\tanh\left(\bar{\gamma}\cdot x\right) = \frac{\sinh\left(\bar{\gamma}\cdot x\right)}{\cosh\left(\bar{\gamma}\cdot x\right)} = \frac{\sinh\left(2\cdot\alpha\cdot x\right) + j\cdot\sin\left(2\cdot\alpha\cdot x\right)}{\cosh\left(2\cdot\alpha\cdot x\right) + \cos\left(2\cdot\alpha\cdot x\right)} \tag{19.287}$$

$$\coth\left(\bar{\gamma}\cdot x\right) = \frac{\cosh\left(\bar{\gamma}\cdot x\right)}{\sinh\left(\bar{\gamma}\cdot x\right)} = \frac{\sinh\left(2\cdot\alpha\cdot x\right) - j\cdot\sin\left(2\cdot\alpha\cdot x\right)}{\cosh\left(2\cdot\alpha\cdot x\right) - \cos\left(2\cdot\alpha\cdot x\right)} \tag{19.288}$$

Example 19.14. A coil of length ℓ wound tightly around a solid straight ferromagnetic core of square cross-section contains a total of *N* turns. The core is made of an LIH (linear, isotropic, and homogeneous) material with electrical conductivity κ and magnetic permeability μ. The coil carries a sinusoidal electric current with an RMS value I and angular frequency ω. Determine the equivalent impedance of the core and the electric power losses in the core, assuming that the skin depth of the electromagnetic wave is much smaller than the edge length *a* of the core. Neglect the effects at the ends of the core. Solve the problem using the magnetodynamic approximation of a linearly polarized uniform plane wave in a good conductor.

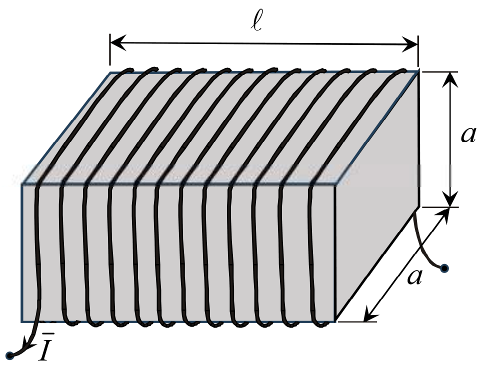


Figure 19.17. Coil wound around a solid straight ferromagnetic core

Solution:

Based on the assumption that the skin depth of the wave in the core is much smaller than the edge length ($d = 1/\alpha << a$), and by neglecting the influence of the core ends, each of the four boundary surfaces can be analyzed separately. In other words, only one quarter of the winding, positioned in front of the ferromagnetic medium that extends to infinity, is considered (Figure 19.18).

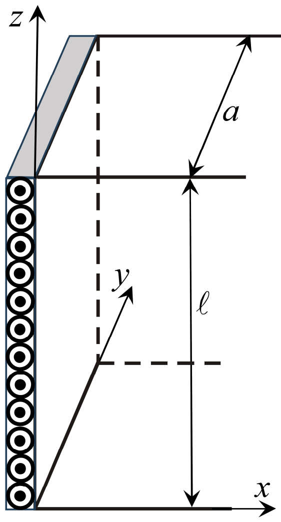


Figure 19.18. Quarter of the winding in front of the ferromagnetic medium

Since the magnetodynamic approximation of a linearly polarized uniform plane electromagnetic wave is used, the following holds:

$$\underline{\vec{E}} = \overline{E}_y \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} \quad ; \quad \underline{\vec{H}} = \overline{H}_z \cdot \vec{\mathrm{k}} = \overline{H} \cdot \vec{\mathrm{k}} \tag{19.289}$$

The phasor of the magnetic field intensity is described by the following 1D Helmholtz differential equation:

$$\Delta \overline{H} = \frac{\partial^2 \overline{H}}{\partial x^2} = \overline{\gamma}^2 \cdot \overline{H} \tag{19.290}$$

whose particular solution is:

$$\overline{H} = \overline{H}_1 \cdot e^{-\overline{\gamma} \cdot x} + \overline{H}_2 \cdot e^{\overline{\gamma} \cdot x} \tag{19.291}$$

where $\overline{H}_1$ and $\overline{H}_2$ are unknown complex constants that must satisfy the following boundary conditions:

$$\overline{H}\Big|_{x=0} = \frac{N \cdot \overline{I}}{\ell} \quad ; \quad \overline{H}\Big|_{x \to \infty} = 0 \tag{19.292}$$

from which it follows that:

$$\overline{H} = \frac{N \cdot \overline{I}}{\ell} \cdot e^{-\overline{\gamma} \cdot x} \tag{19.293}$$

The phasor of the electric field intensity vector can be determined from Maxwell's first differential equation, which in this case reads:

$$\underline{\vec{E}} = \frac{1}{\kappa} \cdot \begin{vmatrix} \vec{\mathrm{i}} & \vec{\mathrm{j}} & \vec{\mathrm{k}} \\ \frac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & \overline{H} \end{vmatrix} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.294}$$

from which it follows that:

$$\underline{\vec{E}} = \overline{E}_{\mathrm{y}} \cdot \vec{\mathrm{j}} = \overline{E} \cdot \vec{\mathrm{j}} = -\frac{1}{\kappa} \cdot \frac{\partial \overline{H}}{\partial x} \cdot \vec{\mathrm{j}} \tag{19.295}$$

and it is:

$$\overline{E} = \frac{\overline{\gamma}}{\kappa} \cdot \frac{N \cdot \overline{I}}{\ell} \cdot e^{-\overline{\gamma} \cdot x} = \overline{Z}_{\mathrm{m}} \cdot \frac{N \cdot \overline{I}}{\ell} \cdot e^{-\overline{\gamma} \cdot x} \tag{19.296}$$

where:

$$\overline{Z}_{\mathrm{m}} = \frac{\overline{\gamma}}{\kappa} \tag{19.297}$$

is the wave impedance of the ferromagnetic core, with displacement electric currents neglected.
In this case, the phasor of the Poynting vector is described by the expression:

$$\underline{\vec{\Gamma}} = \underline{\vec{E}} \times \underline{\vec{H}}^{*} = \overline{E} \cdot \overline{H}^{*} \cdot \left(\vec{\mathrm{j}} \times \vec{\mathrm{k}}\right) = \overline{E} \cdot \overline{H}^{*} \cdot \vec{\mathrm{i}} \tag{19.298}$$

whereas the complex apparent electromagnetic power entering the core is described by the expression:

$$\overline{S}_{\mathrm{ap,in}} = P + j \cdot Q = I^{2} \cdot \overline{Z} = -4 \cdot \oint_{S} \underline{\vec{\Gamma}} \cdot d\vec{S} = -4 \cdot \oint_{S} \overline{\Gamma} \cdot \vec{\mathrm{i}} \cdot \vec{n} \cdot dS \tag{19.299}$$

which reduces to an integration over surface S_1, lying in the plane $x = 0$, so that:

$$\overline{S}_{\mathrm{ap,in}} = 4 \cdot \left(\overline{E} \cdot \overline{H}^{*}\right)_{x=0} \cdot S_{1} = I^{2} \cdot \overline{Z} \quad ; \quad S_{1} = a \cdot \ell \tag{19.300}$$

If the expressions (19.293) and (19.296) are substituted into the expression (19.300), the following expression for the equivalent impedance of the ferromagnetic core is obtained:

$$\overline{Z} = 4 \cdot \frac{N^{2} \cdot a}{\ell} \cdot \frac{\overline{\gamma}}{\kappa} = R + j \cdot \omega \cdot L = R + j \cdot X_{\mathrm{L}} \tag{19.301}$$

where R is the equivalent resistance of the core, L is the equivalent inductance of the core, and X_{L} is the equivalent inductive reactance of the core.
Since $\overline{\gamma} = (1 + j) \cdot \alpha$, it follows from expression (19.301) that:

$$R = X_{\mathrm{L}} = 4 \cdot \frac{N^{2} \cdot a}{\ell} \cdot \frac{\alpha}{\kappa} = 4 \cdot \frac{N^{2} \cdot a}{\ell} \cdot \sqrt{\frac{\omega \cdot \mu}{2 \cdot \kappa}} \tag{19.302}$$

and the electric power losses in the core are:

$$P = I^{2} \cdot R = 4 \cdot \frac{N^{2} \cdot a \cdot I^{2}}{\ell} \cdot \sqrt{\frac{\omega \cdot \mu}{2 \cdot \kappa}} \tag{19.303}$$

whereas the reactive power in the core is:

$$Q = I^{2} \cdot X_{\mathrm{L}} = 4 \cdot \frac{N^{2} \cdot a \cdot I^{2}}{\ell} \cdot \sqrt{\frac{\omega \cdot \mu}{2 \cdot \kappa}} = P \tag{19.304}$$

Example 19.15. In an infinitely long slot cut into a half-space with infinite magnetic permeability, there are two thin electric conductors in the form of narrow current-carrying strips, each with a per-unit-length resistance RR. Let the strips be connected at both ends - at the beginning and at infinity - and let a total sinusoidal electric current with an RMS value I flow through them. Determine: a) the electric current in each conductor, b) analyze the effect of conductor transposition on the distribution of the total electric current between the conductors. Assume that the magnetic field between the strips is homogeneous and that the field lines are parallel to the y-axis.

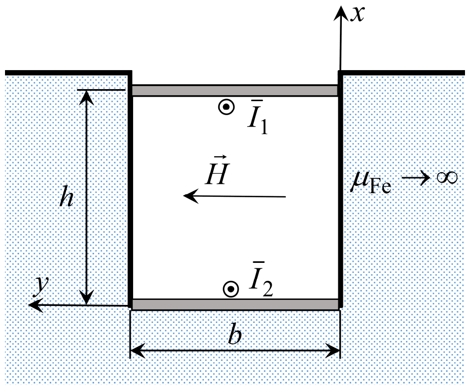


Figure 19.19. Two current-carrying strips in an infinitely long slot

Solution:

a) Distribution of the total electric current between the conductors

The phasor of the magnetic field intensity between the current-carrying strips is described by the expression:

$$\underline{\vec{H}} = \bar{H}_y \cdot \vec{j} = \bar{H} \cdot \vec{j} = \frac{\bar{I}_2}{b} \cdot \vec{j} \tag{19.305}$$

A unit length of the slot and current-carrying strips is considered, and the equivalent circuit shown in Figure 19.20 is used. For the imaginary loop, Faraday's law of electromagnetic induction applies:

$$\bar{E}_{ind} = -j \cdot \omega \cdot \bar{\Phi} = -j \cdot \omega \cdot \mu_0 \cdot \frac{\bar{I}_2}{b} \cdot h \tag{19.306}$$

where:

$$\bar{\Phi} = \mu_0 \cdot \frac{\bar{I}_2}{b} \cdot h \cdot 1 = \mu_0 \cdot \frac{\bar{I}_2}{b} \cdot h \tag{19.307}$$

is the magnetic flux per unit length of the slot, while $\bar{E}_{ind}$ is the EMF induced in the imaginary loop.

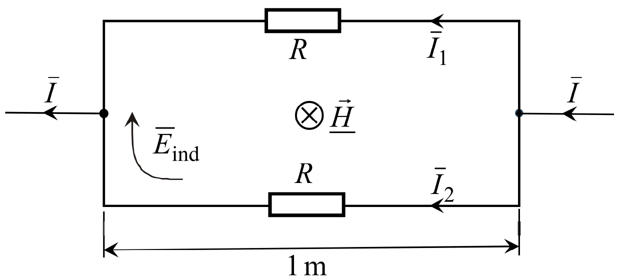


Figure 19.20. Imaginary loop associated with a unit length of the slot

The induced EMF drives an electric current through the imaginary loop, which is described by the expression:

$$\bar{I}_{ind} = \frac{\bar{E}_{ind}}{2 \cdot R} = -j \cdot \omega \cdot \mu_0 \cdot \frac{\bar{I}_2 \cdot h}{2 \cdot R \cdot b} \tag{19.308}$$

while the phasors of the electric currents in the strips are described by the expressions:

$$\bar{I}_1 = \frac{\bar{I}}{2} - \bar{I}_{ind} \quad ; \quad \bar{I}_2 = \frac{\bar{I}}{2} + \bar{I}_{ind} \tag{19.309}$$

From the system of three linear equations, described by expressions (19.308) and (19.309), the following expressions are obtained:

$$\bar{I}_1 = \frac{1 + j\cdot\omega\cdot\mu_0\cdot\dfrac{h}{R\cdot b}}{2 + j\cdot\omega\cdot\mu_0\cdot\dfrac{h}{R\cdot b}}\cdot\bar{I} \quad ; \quad \bar{I}_2 = \frac{1}{2 + j\cdot\omega\cdot\mu_0\cdot\dfrac{h}{R\cdot b}}\cdot\bar{I} \tag{19.310}$$

For $\omega = 0$ the following holds:

$$\bar{I}_1 = \bar{I}_2 = \frac{\bar{I}}{2} \tag{19.311}$$

whereas for $\omega \to \infty$ it holds that:

$$\bar{I}_1 = \bar{I} \quad ; \quad \bar{I}_2 = 0 \tag{19.312}$$

b) The effect of transposition on the distribution of the total electric current between the conductors

The purpose of conductor transposition is to ensure that equal electric currents flow through the conductors, i.e., that expression (19.311) holds.

Example 19.16. Calculate the frequency of a plane electromagnetic wave propagating through a conducting LIH medium with a phase velocity v = 2.236 m/s. The properties of the medium are given as follows: κ = 10 MS/m, μ_r = 1000, ε_r = 11.

Solution:

From expressions (17.120) and (17.121), it follows that the phase constant of the conducting medium is given by the expression:

$$\beta = \sqrt{\omega\cdot\varepsilon + \sqrt{(\omega\cdot\varepsilon)^2 + \kappa^2}}\cdot\sqrt{\frac{\omega\cdot\mu}{2}} \tag{19.313}$$

whereas the phase velocity of the wave is described by the expression:

$$v = \frac{\omega}{\beta} \tag{19.314}$$

From expressions (19.313) and (19.314), it follows that:

$$v^2\cdot\left(\omega\cdot\varepsilon + \sqrt{(\omega\cdot\varepsilon)^2 + \kappa^2}\right)\cdot\frac{\omega\cdot\mu}{2} = \omega^2 \tag{19.315}$$

and it is:

$$\sqrt{(\omega\cdot\varepsilon)^2 + \kappa^2} = \omega\cdot\left(\frac{2}{\mu\cdot v^2} - \varepsilon\right) \tag{19.316}$$

from which, after squaring expression (19.316), the following expression is easily obtained:

$$\omega = \frac{\mu\cdot\kappa\cdot v^2}{2\cdot\sqrt{1 - \mu\cdot\varepsilon\cdot v^2}} \tag{19.317}$$

Thus, the frequency of the plane wave is:

$$f = \frac{\mu\cdot\kappa\cdot v^2}{4\cdot\pi\cdot\sqrt{1 - \mu\cdot\varepsilon\cdot v^2}} = 4.999696 \text{ kHz} \tag{19.318}$$

If the displacement electric currents are neglected, then:

$$f = \frac{\mu\cdot\kappa\cdot v^2}{4\cdot\pi} = 4.999696 \text{ kHz} \tag{19.319}$$

which means that the numerical result remains the same.

Example 19.17. Calculate the phase velocity of a plane electromagnetic sinusoidal wave with a frequency of $f = 10$ MHz, propagating in a conducting LIH medium. The properties of the medium are given as follows: κ = 10 MS/m, μ_r = 1000, $\varepsilon_r = 11$.

Solution:

The phase velocity of the wave, according to the previous example, is given by the following expression:

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega \cdot \varepsilon + \sqrt{(\omega \cdot \varepsilon)^2 + \kappa^2}} \cdot \sqrt{\frac{\omega \cdot \mu}{2}}} = 99.99999997 \text{ m/s} \tag{19.320}$$

where $\omega = 2 \cdot \pi \cdot f$.

If the displacement electric currents are neglected, then:

$$v = \sqrt{\frac{2 \cdot \omega}{\mu \cdot \kappa}} = 100 \text{ m/s} \tag{19.321}$$

Example 19.18. Calculate the phase velocity of a plane electromagnetic sinusoidal wave with a frequency of f =50 MHz, propagating through an LIH soil. The properties of the soil are given as follows: κ = 0.001 S/m, μ_r = 1, $\varepsilon_r = 10$.

Solution:

The phase velocity of the wave, according to the previous examples, is given by the following expression:

$$v = \frac{\omega}{\beta} = \frac{\omega}{\sqrt{\omega \cdot \varepsilon + \sqrt{(\omega \cdot \varepsilon)^2 + \kappa^2}} \cdot \sqrt{\frac{\omega \cdot \mu}{2}}} = 7.070969467 \times 10^5 \text{ m/s} \tag{19.322}$$

where $\omega = 2 \cdot \pi \cdot f$.

If the displacement electric currents are neglected, then:

$$v = \sqrt{\frac{2 \cdot \omega}{\mu \cdot \kappa}} = 7.071067812 \times 10^5 \text{ m/s} \tag{19.323}$$

Example 19.19. Let the electric field intensity in the air be described by the following expression: $\vec{E}(\mathrm{z,t}) = 50 \cdot \cos(\omega \cdot t - \beta \cdot z) \cdot \vec{\mathrm{i}}$ V/m. Determine the expression for the instantaneous and average value of the electromagnetic power passing through an imaginary circle lying in the plane $z = a,$ with its center on the z-axis of a Cartesian coordinate system. Let the radius of the circle be r = 2.5 m. Assume that a linearly polarized plane wave is propagating in the direction of the z-axis.

Solution:

The given electric field intensity is described by the expression:

$$\vec{E}(z,t) = 50 \cdot \cos(\omega \cdot t - \beta \cdot z) \cdot \vec{\mathrm{i}} \text{ V/m} \tag{19.324}$$

and then, based on expressions (19.48) and (19.49), it follows that:

$$\vec{H}(z,t) = \frac{50}{Z_\mathrm{m}} \cdot \cos(\omega \cdot t - \beta \cdot z) \cdot \vec{\mathrm{j}} \text{ A/m} \tag{19.325}$$

where, according to expression (19.45), the wave impedance of air is:

$$Z_{\mathrm{m}} = Z_0 = \sqrt{\frac{\mu_0}{\varepsilon_0}} = 376.730313668\ \Omega \tag{19.326}$$

The instantaneous value of the electromagnetic power passing through the given circle is described by the expression:

$$p = \vec{\Gamma}\Big|_{z=a} \cdot \vec{\mathrm{k}} \cdot S = \left(\vec{E} \times \vec{H}\right)_{z=a} \cdot \vec{\mathrm{k}} \cdot S = E(a,t) \cdot H(a,t) \cdot r^2 \cdot \pi \tag{19.327}$$

where the Poynting vector $\vec{\Gamma}$, by its physical meaning, represents the surface density vector of the instantaneous electromagnetic power, and its unit of measurement is W/m².

If the expressions (19.324) and (19.325) are substituted into expression (19.327), the following expression is obtained:

$$p = \frac{50^2}{Z_{\mathrm{m}}} \cdot \cos^2(\omega \cdot t\ -\ \beta \cdot a) \cdot r^2 \cdot \pi \tag{19.328}$$

By substituting the given data, it is obtained that:

$$p = 130.2984746 \cdot \cos^2(\omega \cdot t\ -\ \beta \cdot a)\ \ \mathrm{W} \tag{19.329}$$

To facilitate the calculation of the average value of the electromagnetic power passing through the imaginary circle, expression (19.329) can be rewritten in the following form:

$$p = \frac{130.2984746}{2} \cdot \left[1 + \cos(2 \cdot \omega \cdot t - 2 \cdot \beta \cdot a)\right]\ \ \mathrm{W} \tag{19.330}$$

from which it follows that the required average value of the electromagnetic power is:

$$P_{\mathrm{av}} = \frac{130.2984746}{2} = 65.1492343\ \mathrm{W} \tag{19.331}$$

The same result is also given by the following expression:

$$P_{\mathrm{av}} = E_{\mathrm{ef}} \cdot H_{\mathrm{ef}} \cdot r^2 \cdot \pi = \ \frac{130.297}{2} = 65.1492343\ \mathrm{W} \tag{19.332}$$

where:

E_{ef} - effective, or RMS, value of the sinusoidal electric field intensity,

H_{ef} - effective, or RMS, value of the sinusoidal magnetic field intensity.

20. SINUSOIDAL PLANE WAVE AT THE BOUNDARY BETWEEN TWO MEDIA

Let the boundary separating the two half-spaces be a plane. Assume that each half-space is filled with a linear, isotropic, and homogeneous (LIH) medium.

When an electromagnetic plane wave from the first medium strikes the boundary plane, it causes polarization in the second medium and induces eddy currents if the second medium is conductive. The second medium resists the penetration of the wave through polarization and the induced electric currents in a way that corresponds to the effect of the secondary winding of a transformer. This resistance of the medium leads to reflection and refraction of the electromagnetic wave. The electromagnetic field of the polarized dipoles and the eddy currents is superimposed on the incident electromagnetic field.

20.1. Snell's Laws of Reflection and Refraction

It is important to emphasize that an electromagnetic plane wave behaves analogously to light when incident at the boundary between two LIH media.

Let a linearly polarized plane electromagnetic wave from medium 1 be incident on the boundary of medium 2. Let the incident wave propagate in the direction of the unit vector $\vec{n}_0$, the reflected wave in the direction of the unit vector $\vec{n}_1$ and the transmitted wave in the direction of the unit vector $\vec{n}_2$ (Figure 20.1). All three unit vectors lie in the same plane. Assume that the plane electromagnetic wave is sinusoidal.

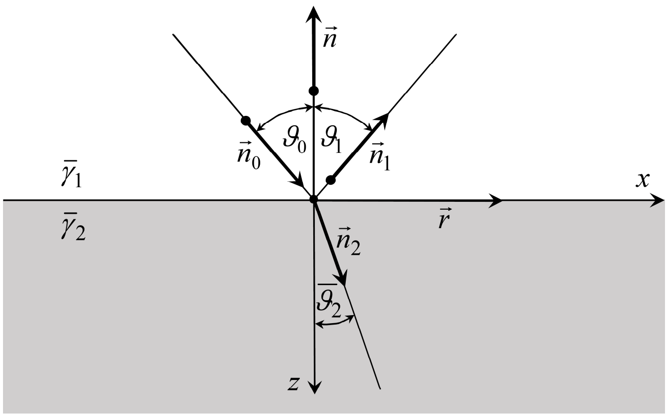


Figure 20.1. Incident, reflected and transmitted wave

The incident plane wave strikes the boundary between the two media at an angle ϑ_0 with respect to the unit normal $\vec{n}$ to the boundary surface. The reflected wave is reflected at an angle ϑ_1 with respect to the unit normal $\vec{n}$ to the boundary surface. The transmitted wave forms an angle $\underline{\vartheta}_2$ with the z-axis, which is, in the general case, a complex quantity (Figure 20.1). The properties of the two LIH media are contained in the wave constants of those media: $\underline{\gamma}_1$ and $\underline{\gamma}_2$.

For the incident sinusoidal electric field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{E}}_\mathrm{i} = \underline{\vec{E}}_0 \cdot e^{-\underline{\gamma}_1 \cdot s_\mathrm{i}} = \underline{\vec{E}}_0 \cdot e^{-\underline{\gamma}_1 \cdot \vec{n}_0 \cdot \vec{r}} \tag{20.1}$$

where the vectors $\vec{n}_0$ and $\vec{r}$ are shown in Figures 20.1 and 20.2-a.

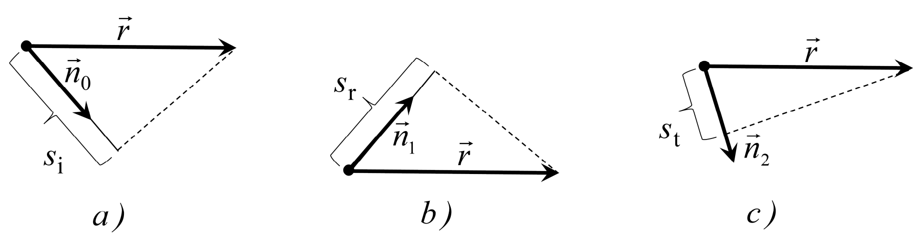


Figure 20.2. Projection of the vector $\vec{r}$ onto the unit vectors $\vec{n}_0$, $\vec{n}_1$ and $\vec{n}_2$

From expression (20.1), it follows that the incident wave has only a forward component, and its propagation is described by the vector $\vec{r}$.
In the time domain, the expression (20.1) takes the form:

$$\vec{E}_{\mathrm{i}} = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_{\mathrm{i}} \cdot e^{j\cdot\omega\cdot t}\right] = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_0 \cdot e^{-\overline{\gamma}_1\cdot\vec{n}_0\cdot\vec{r}} \cdot e^{j\cdot\omega\cdot t}\right] \qquad (20.2)$$

At the point of incidence of the electromagnetic wave, the following holds:

$$s_{\mathrm{i}} = \vec{n}_0 \cdot \vec{r} = 0 \qquad (20.3)$$

For the incident sinusoidal magnetic field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{H}}_{\mathrm{i}} = \underline{\vec{H}}_0 \cdot e^{-\overline{\gamma}_1\cdot s_{\mathrm{i}}} = \underline{\vec{H}}_0 \cdot e^{-\overline{\gamma}_1\cdot\vec{n}_0\cdot\vec{r}} \qquad (20.4)$$

It can be easily proven that the following equation holds:

$$\underline{\vec{H}}_0 = \frac{\vec{n}_0 \times \underline{\vec{E}}_0}{\overline{Z}_1} \qquad (20.5)$$

because:

$$\underline{\vec{E}}_0 \times \underline{\vec{H}}_0^* = \underline{\vec{\varGamma}}_0 = \overline{\varGamma}_0 \cdot \vec{n}_0 \quad ; \quad \overline{H}_0 = \frac{\overline{E}_0}{\overline{Z}_1} \qquad (20.6)$$

For the reflected sinusoidal electric field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{E}}_{\mathrm{r}} = \underline{\vec{E}}_1 \cdot e^{-\overline{\gamma}_1\cdot s_{\mathrm{r}}} = \underline{\vec{E}}_1 \cdot e^{-\overline{\gamma}_1\cdot\vec{n}_1\cdot\vec{r}} \qquad (20.7)$$

where the vectors $\vec{n}_1$ and $\vec{r}$ are shown in Figures 20.1 and 20.2-b.
In the time domain, the expression (20.7) takes the form:

$$\vec{E}_{\mathrm{r}} = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_{\mathrm{r}} \cdot e^{j\cdot\omega\cdot t}\right] = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_1 \cdot e^{-\overline{\gamma}_1\cdot\vec{n}_1\cdot\vec{r}} \cdot e^{j\cdot\omega\cdot t}\right] \qquad (20.8)$$

For the reflected sinusoidal magnetic field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{H}}_{\mathrm{r}} = \underline{\vec{H}}_1 \cdot e^{-\overline{\gamma}_1\cdot s_{\mathrm{r}}} = \underline{\vec{H}}_1 \cdot e^{-\overline{\gamma}_1\cdot\vec{n}_1\cdot\vec{r}} \qquad (20.9)$$

It can be easily proven that the following equation holds:

$$\underline{\vec{H}}_1 = \frac{\vec{n}_1 \times \underline{\vec{E}}_1}{\overline{Z}_1} \qquad (20.10)$$

because:

$$\underline{\vec{E}}_1 \times \underline{\vec{H}}_1^* = \underline{\vec{\varGamma}}_1 = \overline{\varGamma}_1 \cdot \vec{n}_1 \quad ; \quad \overline{H}_1 = \frac{\overline{E}_1}{\overline{Z}_1} \qquad (20.11)$$

For the transmitted sinusoidal electric field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{E}}_{\mathrm{t}} = \underline{\vec{E}}_2 \cdot e^{-\overline{\gamma}_2\cdot s_{\mathrm{t}}} = \underline{\vec{E}}_2 \cdot e^{-\overline{\gamma}_2\cdot\vec{n}_2\cdot\vec{r}} \qquad (20.12)$$

where the vectors $\vec{n}_2$ and $\vec{r}$ are shown in Figures 20.1 and 20.2-c.
In the time domain, the expression (20.12) takes the form:

$$\vec{E}_{\mathrm{t}} = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_{\mathrm{t}} \cdot e^{j\cdot\omega\cdot t}\right] = \mathrm{Re}\left[\sqrt{2} \cdot \underline{\vec{E}}_2 \cdot e^{-\overline{\gamma}_2\cdot\vec{n}_2\cdot\vec{r}} \cdot e^{j\cdot\omega\cdot t}\right] \qquad (20.13)$$

For the transmitted sinusoidal magnetic field intensity, in the phasor domain, the following equation holds:

$$\underline{\vec{H}}_{\mathrm{t}} = \underline{\vec{H}}_2 \cdot e^{-\overline{\gamma}_2\cdot s_{\mathrm{t}}} = \underline{\vec{H}}_2 \cdot e^{-\overline{\gamma}_2\cdot\vec{n}_2\cdot\vec{r}} \qquad (20.14)$$

It can be easily proven that the following equation holds:

$$\underline{\vec{H}}_2 = \frac{\vec{n}_2 \times \underline{\vec{E}}_2}{\overline{Z}_2} \tag{20.15}$$

because:

$$\underline{\vec{E}}_2 \times \underline{\vec{H}}_2^* = \underline{\vec{\Gamma}}_2 = \overline{\Gamma}_2 \cdot \vec{n}_2 \quad ; \quad \overline{H}_2 = \frac{\overline{E}_2}{\overline{Z}_2} \tag{20.16}$$

The tangential components of the electric field intensity and magnetic field intensity must be continuous, so the following equations hold:

$$\vec{n} \times \left(\underline{\vec{E}}_{\mathrm{i}} + \underline{\vec{E}}_{\mathrm{r}}\right) = \vec{n} \times \underline{\vec{E}}_{\mathrm{t}} \quad ; \quad \vec{n} \times \left(\underline{\vec{H}}_{\mathrm{i}} + \underline{\vec{H}}_{\mathrm{r}}\right) = \vec{n} \times \underline{\vec{H}}_{\mathrm{t}} \tag{20.17}$$

and it is:

$$\vec{n} \times \left(\underline{\vec{E}}_0 \cdot e^{-\overline{\gamma}_1 \cdot \vec{n}_0 \cdot \vec{r}} + \underline{\vec{E}}_1 \cdot e^{-\overline{\gamma}_1 \cdot \vec{n}_1 \cdot \vec{r}}\right) = \vec{n} \times \underline{\vec{E}}_2 \cdot e^{-\overline{\gamma}_2 \cdot \vec{n}_2 \cdot \vec{r}} \tag{20.18}$$

$$\vec{n} \times \left(\underline{\vec{H}}_0 \cdot e^{-\overline{\gamma}_1 \cdot \vec{n}_0 \cdot \vec{r}} + \underline{\vec{H}}_1 \cdot e^{-\overline{\gamma}_1 \cdot \vec{n}_1 \cdot \vec{r}}\right) = \vec{n} \times \underline{\vec{H}}_2 \cdot e^{-\overline{\gamma}_2 \cdot \vec{n}_2 \cdot \vec{r}} \tag{20.19}$$

from which it follows that the following conditions must be satisfied:

$$\vec{n} \times \left(\underline{\vec{E}}_0 + \underline{\vec{E}}_1\right) = \vec{n} \times \underline{\vec{E}}_2 \quad ; \quad \vec{n} \times \left(\underline{\vec{H}}_0 + \underline{\vec{H}}_1\right) = \vec{n} \times \underline{\vec{H}}_2 \tag{20.20}$$

$$\overline{\gamma}_1 \cdot \vec{n}_0 \cdot \vec{r} = \overline{\gamma}_1 \cdot \vec{n}_1 \cdot \vec{r} = \overline{\gamma}_2 \cdot \vec{n}_2 \cdot \vec{r} \tag{20.21}$$

From expression (20.21) and Figure 20.1, it follows that:

$$\overline{\gamma}_1 \cdot r \cdot \sin \vartheta_0 = \overline{\gamma}_1 \cdot r \cdot \sin \vartheta_1 = \overline{\gamma}_2 \cdot r \cdot \sin \overline{\vartheta}_2 \tag{20.22}$$

which leads to Snell's laws:

$$\vartheta_1 = \vartheta_0 \tag{20.23}$$

$$\sin \overline{\vartheta}_2 = \frac{\overline{\gamma}_1}{\overline{\gamma}_2} \cdot \sin \vartheta_0 \tag{20.24}$$

that is, Snell's law of reflection of the electromagnetic wave and Snell's law of refraction (transmission) of the electromagnetic wave.

According to Snell's law of refraction (20.24), in the general case, the sine of the angle of refraction is a complex quantity, and consequently, the angle of refraction itself is also complex. In the special case where both media are perfect LIH dielectrics, the following relation holds:

$$\frac{\sin \vartheta_2}{\sin \vartheta_0} = \frac{\overline{\gamma}_1}{\overline{\gamma}_2} = \frac{j \cdot \omega \cdot \sqrt{\mu_1 \cdot \varepsilon_1}}{j \cdot \omega \cdot \sqrt{\mu_2 \cdot \varepsilon_2}} = \frac{\sqrt{\mu_1 \cdot \varepsilon_1}}{\sqrt{\mu_2 \cdot \varepsilon_2}} = \frac{v_2}{v_1} \tag{20.25}$$

where v_1 and v_2 are the velocities of the electromagnetic wave in each medium, which do not depend on frequency. It is important to note that, in this case, the angle of refraction is a real quantity.

Furthermore, if for perfect LIH dielectrics it holds that $\mu_1 = \mu_2$, then:

$$\frac{\sin \vartheta_2}{\sin \vartheta_0} = \sqrt{\frac{\varepsilon_1}{\varepsilon_2}} = \sqrt{\frac{\varepsilon_{\mathrm{r}1}}{\varepsilon_{\mathrm{r}2}}} = \frac{n_1}{n_2} = n_{1,2} \quad ; \quad n_1 = \frac{c}{v_1} \quad ; \quad n_2 = \frac{c}{v_2} \tag{20.26}$$

where n_1 and n_2 are the refractive indices of the perfect LIH dielectrics, $n_{1,2}$ is the relative refractive index between the two perfect dielectrics, and c is the speed of light in a vacuum.

20.2. Fresnel Equations

At the boundary between the two media ($z = 0$), the so-called Fresnel equations for the electric field intensity apply:

$$\vec{n}\times\left(\vec{\underline{E}}_0+\vec{\underline{E}}_1\right)=\vec{n}\times\vec{\underline{E}}_2 \tag{20.27}$$

$$\vec{n}\times\left(\frac{\vec{n}_0\times\vec{\underline{E}}_0}{\overline{Z}_1}+\frac{\vec{n}_1\times\vec{\underline{E}}_1}{\overline{Z}_1}\right)=\vec{n}\times\frac{\vec{n}_2\times\vec{\underline{E}}_2}{\overline{Z}_2} \tag{20.28}$$

where:

$$\vec{\underline{H}}_0=\frac{\vec{n}_0\times\vec{\underline{E}}_0}{\overline{Z}_1} \quad ; \quad \vec{\underline{H}}_1=\frac{\vec{n}_1\times\vec{\underline{E}}_1}{\overline{Z}_1} \quad ; \quad \vec{\underline{H}}_2=\frac{\vec{n}_2\times\vec{\underline{E}}_2}{\overline{Z}_2} \tag{20.29}$$

It is particularly important to consider two specific cases when:

a) the electric field intensity vector is perpendicular to the plane of incidence,

b) the electric field intensity vector lies in the plane of incidence.

According to Figure 20.1, the plane of incidence is the plane in which the unit vectors $\vec{n}_0$, $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}$ lie.

In the general case, the electric field intensity vector can be decomposed into these two components. Fresnel equations (20.27) and (20.28) are used when the electric field intensity vector is perpendicular to the plane of incidence. In this case, the magnetic field intensity vector lies in the plane of incidence.

At the boundary between the two media ($z = 0$), the Fresnel equations for the magnetic field intensity apply:

$$\vec{n}\times\left(\vec{\underline{H}}_0+\vec{\underline{H}}_1\right)=\vec{n}\times\vec{\underline{H}}_2 \tag{20.30}$$

$$\overline{Z}_1\cdot\vec{n}\times\left(\vec{n}_0\times\vec{\underline{H}}_0+\vec{n}_1\times\vec{\underline{H}}_1\right) = \overline{Z}_2\cdot\vec{n}\times\left(\vec{n}_2\times\vec{\underline{H}}_2\right) \tag{20.31}$$

where:

$$\vec{\underline{E}}_0=-\overline{Z}_1\cdot\left(\vec{n}_0\times\vec{\underline{H}}_0\right) \quad ; \quad \vec{\underline{E}}_1=-\overline{Z}_1\cdot\left(\vec{n}_1\times\vec{\underline{H}}_1\right) \quad ; \quad \vec{\underline{E}}_2=-\overline{Z}_2\cdot\left(\vec{n}_2\times\vec{\underline{H}}_2\right) \tag{20.32}$$

Fresnel equations (20.30) and (20.31) are used when the electric field intensity vector lies in the plane of incidence. In this case, the magnetic field intensity vector is perpendicular to the plane of incidence.

20.3. Electric Field Vector Perpendicular to the Plane of Incidence

Let the electric field intensity vector be perpendicular to the plane of incidence (Figure 20.3). In this case, Fresnel equations (20.27) and (20.28) for the electric field intensity can be used to determine the expressions for the reflection and transmission coefficients, since the electric field is tangential to the boundary plane. The continuity of the tangential components of both the electric and magnetic field intensity vectors must be satisfied. For the tangential components of the electric field intensity vector, the following holds:

$$\overline{E}_{0y}=\overline{E}_0 \quad ; \quad \overline{E}_{1y}=\overline{E}_1 \quad ; \quad \overline{E}_{2y}=\overline{E}_2 \tag{20.33}$$

where, according to Figure 20.3, for the tangential components of the magnetic field intensity vector, the following holds:

$$\overline{H}_{0x}=-\overline{H}_0\cdot\cos\vartheta_0 \quad ; \quad \overline{H}_{1x}=\overline{H}_1\cdot\cos\vartheta_0 \quad ; \quad \overline{H}_{2x}=-\overline{H}_2\cdot\cos\overline{\vartheta}_2 \tag{20.34}$$

from which it follows that:

$$\overline{H}_{0x}=-\frac{\overline{E}_0}{\overline{Z}_1}\cdot\cos\vartheta_0 \quad ; \quad \overline{H}_{1x}=\frac{\overline{E}_1}{\overline{Z}_1}\cdot\cos\vartheta_0 \quad ; \quad \overline{H}_{2x}=-\frac{\overline{E}_2}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2 \tag{20.35}$$

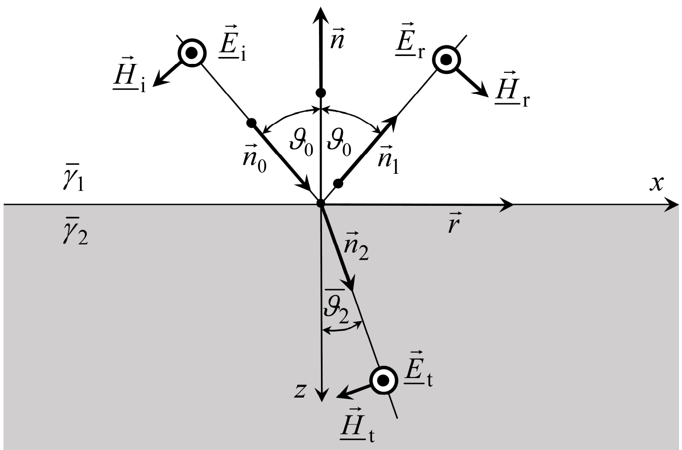


Figure 20.3. Electric field intensity vector perpendicular to the plane of incidence

From the requirement that the tangential components of the field intensities be continuous, it follows that:

$$\overline{E}_{0y} + \overline{E}_{1y} = \overline{E}_{2y} \quad ; \quad \overline{H}_{0x} + \overline{H}_{1x} = \overline{H}_{2x} \tag{20.36}$$

and, taking into account the expressions (20.33) and (20.35), the following system of linear equations is easily obtained:

$$\overline{E}_0 + \overline{E}_1 = \overline{E}_2 \tag{20.37}$$

$$\overline{E}_0 \cdot \cos\vartheta_0 - \overline{E}_1 \cdot \cos\vartheta_0 = \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \overline{E}_2 \cdot \cos\overline{\vartheta}_2 \tag{20.38}$$

To obtain the expressions for the reflection and transmission coefficients of the electric field intensity, equations (20.37) and (20.38) can be expressed as follows:

$$\overline{E}_1 - \overline{E}_2 = \overline{E}_0 \tag{20.39}$$

$$\overline{E}_1 + \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0} \cdot \overline{E}_2 = \overline{E}_0 \tag{20.40}$$

from which it follows that:

$$\left(1 + \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \cdot \overline{E}_1 = \overline{E}_0 \cdot \left(1 - \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \tag{20.41}$$

$$\left(1 + \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \cdot \overline{E}_2 = 2 \cdot \overline{E}_0 \tag{20.42}$$

From expressions (20.41) and (20.42), it is easily obtained that:

$$\overline{E}_1 = \frac{\cos\vartheta_0 - \dfrac{\overline{Z}_1}{\overline{Z}_2} \cdot \cos\overline{\vartheta}_2}{\cos\vartheta_0 + \dfrac{\overline{Z}_1}{\overline{Z}_2} \cdot \cos\overline{\vartheta}_2} \cdot \overline{E}_0 \quad ; \quad \overline{E}_2 = \frac{2 \cdot \cos\vartheta_0}{\cos\vartheta_0 + \dfrac{\overline{Z}_1}{\overline{Z}_2} \cdot \cos\overline{\vartheta}_2} \cdot \overline{E}_0 \tag{20.43}$$

The reflection coefficient for the electric field intensity is defined as the ratio of the tangential components of the electric field intensity vectors:

$$\overline{\rho}_{\perp}^{\mathrm{E}} = \frac{\overline{E}_{1\mathrm{y}}}{\overline{E}_{0\mathrm{y}}} = \frac{\overline{E}_1}{\overline{E}_0} = \frac{\cos\vartheta_0 - \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2}{\cos\vartheta_0 + \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2} \tag{20.44}$$

whereas the transmission coefficient for the electric field intensity is given by the expression:

$$\overline{\tau}_{\perp}^{\mathrm{E}} = \frac{\overline{E}_{2\mathrm{y}}}{\overline{E}_{0\mathrm{y}}} = \frac{\overline{E}_2}{\overline{E}_0} = \frac{2\cdot\cos\vartheta_0}{\cos\vartheta_0 + \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2} = 1 + \overline{\rho}_{\perp}^{\mathrm{E}} \tag{20.45}$$

From expressions (20.35) and (20.45), it follows that the reflection coefficient for the magnetic field intensity is given by the following expression:

$$\overline{\rho}_{\perp}^{\mathrm{H}} = \frac{\overline{H}_{1\mathrm{x}}}{\overline{H}_{0\mathrm{x}}} = -\frac{\overline{E}_1}{\overline{E}_0} = -\,\overline{\rho}_{\perp}^{\mathrm{E}} = \frac{\dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2 - \cos\vartheta_0}{\dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\overline{\vartheta}_2 + \cos\vartheta_0} \tag{20.46}$$

or, written differently:

$$\overline{\rho}_{\perp}^{\mathrm{H}} = \frac{\cos\overline{\vartheta}_2 - \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\vartheta_0}{\cos\overline{\vartheta}_2 + \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\vartheta_0} \tag{20.47}$$

From expressions (20.35) and (20.45), it follows that the transmission coefficient for the magnetic field intensity is given by the following expression:

$$\overline{\tau}_{\perp}^{\mathrm{H}} = \frac{\overline{H}_{2\mathrm{x}}}{\overline{H}_{0\mathrm{x}}} = \frac{\overline{E}_2}{\overline{E}_0}\cdot\frac{\overline{Z}_1\cdot\cos\overline{\vartheta}_2}{\overline{Z}_2\cdot\cos\vartheta_0} = \overline{\tau}_{\perp}^{\mathrm{E}}\cdot\frac{\overline{Z}_1\cdot\cos\overline{\vartheta}_2}{\overline{Z}_2\cdot\cos\vartheta_0} = \frac{2\cdot\cos\overline{\vartheta}_2}{\cos\overline{\vartheta}_2 + \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\vartheta_0} \tag{20.48}$$

and it holds that:

$$\overline{\tau}_{\perp}^{\mathrm{H}} = 1 + \overline{\rho}_{\perp}^{\mathrm{H}} \tag{20.49}$$

The reflection and transmission coefficients are complex quantities, which means that the reflected and transmitted waves differ from the incident wave in both magnitude and phase.

20.4. Electric Field Vector Lies in the Plane of Incidence

If the electric field intensity vector lies in the plane of incidence, then the magnetic field intensity vector is perpendicular to the plane of incidence (Figure 20.4). In this case, to determine the expressions for the reflection and transmission coefficients, Fresnel equations (20.30) and (20.31) for the magnetic field intensity can be used, since the magnetic field is tangential to the boundary plane. The continuity of the tangential components of both the electric and magnetic field intensity vectors must be satisfied. For the tangential components of the magnetic field intensity vector, the following holds:

$$\overline{H}_{0\mathrm{y}} = \overline{H}_0 \;\; ; \quad \overline{H}_{1\mathrm{y}} = \overline{H}_1 \;\; ; \quad \overline{H}_{2\mathrm{y}} = \overline{H}_2 \tag{20.50}$$

whereas, according to Figure 20.4, for the tangential components of the electric field intensity vector, the following holds:

$$\overline{E}_{0\mathrm{x}} = \overline{E}_0\cdot\cos\vartheta_0 \;\; ; \quad \overline{E}_{1\mathrm{x}} = -\,\overline{E}_1\cdot\cos\vartheta_0 \;\; ; \quad \overline{E}_{2\mathrm{x}} = \overline{E}_2\cdot\cos\overline{\vartheta}_2 \tag{20.51}$$

from which it follows that:

$$\overline{E}_{0x} = \overline{Z}_1 \cdot \overline{H}_0 \cdot \cos\vartheta_0 \quad ; \quad \overline{E}_{1x} = -\,\overline{Z}_1 \cdot \overline{H}_1 \cdot \cos\vartheta_0 \quad ; \quad \overline{E}_{2x} = \overline{Z}_2 \cdot \overline{H}_2 \cdot \cos\overline{\vartheta}_2 \tag{20.52}$$

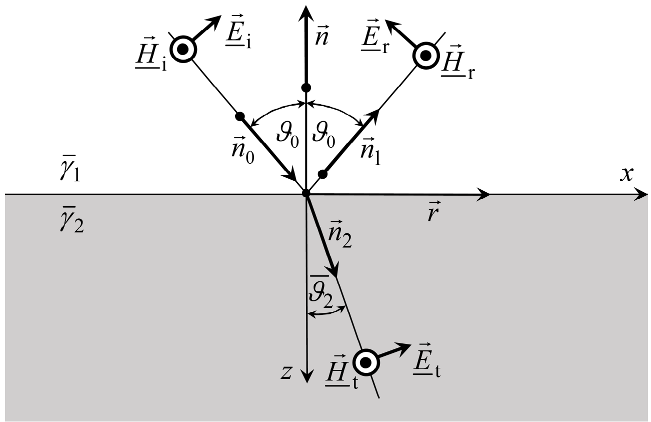


Figure 20.4. Electric field intensity vector lies in the plane of incidence

From the requirement that the tangential components of the field intensities be continuous, it follows that:

$$\overline{H}_{0y} + \overline{H}_{1y} = \overline{H}_{2y} \quad ; \quad \overline{E}_{0x} + \overline{E}_{1x} = \overline{E}_{2x} \tag{20.53}$$

and, taking into account expressions (20.50) and (20.52), the following system of linear equations is easily obtained:

$$\overline{H}_0 + \overline{H}_1 = \overline{H}_2 \tag{20.54}$$

$$\overline{H}_0 \cdot \cos\vartheta_0 - \overline{H}_1 \cdot \cos\vartheta_0 = \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \overline{H}_2 \cdot \cos\overline{\vartheta}_2 \tag{20.55}$$

To obtain the expressions for the reflection and transmission coefficients of the magnetic field intensity, equations (20.54) and (20.55) can be expressed as follows:

$$\overline{H}_1 - \overline{H}_2 = \overline{H}_0 \tag{20.56}$$

$$\overline{H}_1 + \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0} \cdot \overline{H}_2 = \overline{H}_0 \tag{20.57}$$

from which it follows that:

$$\left(1 + \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \cdot \overline{H}_1 = \overline{H}_0 \cdot \left(1 - \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \tag{20.58}$$

$$\left(1 + \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \frac{\cos\overline{\vartheta}_2}{\cos\vartheta_0}\right) \cdot \overline{H}_2 = 2 \cdot \overline{H}_0 \tag{20.59}$$

From expressions (20.58) and (20.59), it is easily obtained that:

$$\overline{H}_1 = \frac{\cos\vartheta_0 - \dfrac{\overline{Z}_2}{\overline{Z}_1} \cdot \cos\overline{\vartheta}_2}{\cos\vartheta_0 + \dfrac{\overline{Z}_2}{\overline{Z}_1} \cdot \cos\overline{\vartheta}_2} \cdot \overline{H}_0 \quad ; \quad \overline{H}_2 = \frac{2 \cdot \cos\vartheta_0}{\cos\vartheta_0 + \dfrac{\overline{Z}_2}{\overline{Z}_1} \cdot \cos\overline{\vartheta}_2} \cdot \overline{H}_0 \tag{20.60}$$

The reflection coefficient for the magnetic field intensity is defined as the ratio of the tangential components of the magnetic field intensity vectors:

$$\overline{\rho}_{\text{par}}^{\text{H}} = \frac{\overline{H}_{1\text{y}}}{\overline{H}_{0\text{y}}} = \frac{\overline{H}_1}{\overline{H}_0} = \frac{\cos\vartheta_0 - \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\overline{\vartheta}_2}{\cos\vartheta_0 + \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\overline{\vartheta}_2} \tag{20.61}$$

whereas the transmission coefficient for the magnetic field intensity is given by the expression:

$$\overline{\tau}_{\text{par}}^{\text{H}} = \frac{\overline{H}_{2\text{y}}}{\overline{H}_{0\text{y}}} = \frac{\overline{H}_2}{\overline{H}_0} = \frac{2\cdot\cos\vartheta_0}{\cos\vartheta_0 + \dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\overline{\vartheta}_2} = 1 + \overline{\rho}_{\text{par}}^{\text{H}} \tag{20.62}$$

From expressions (20.52) and (20.61), it follows that the reflection coefficient for the electric field intensity is given by the following expression:

$$\overline{\rho}_{\text{par}}^{\text{E}} = \frac{\overline{E}_{1\text{x}}}{\overline{E}_{0\text{x}}} = -\frac{\overline{H}_1}{\overline{H}_0} = -\overline{\rho}_{\text{par}}^{\text{H}} = \frac{\dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\overline{\vartheta}_2 - \cos\vartheta_0}{\dfrac{\overline{Z}_2}{\overline{Z}_1}\cdot\cos\overline{\vartheta}_2 + \cos\vartheta_0} \tag{20.63}$$

or, written differently:

$$\overline{\rho}_{\text{par}}^{\text{E}} = \frac{\cos\overline{\vartheta}_2 - \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\vartheta_0}{\cos\overline{\vartheta}_2 + \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\vartheta_0} \tag{20.64}$$

From expressions (20.52) and (20.62), it follows that the transmission coefficient for the electric field intensity is given by the following expression:

$$\overline{\tau}_{\text{par}}^{\text{E}} = \frac{\overline{E}_{2\text{x}}}{\overline{E}_{0\text{x}}} = \frac{\overline{H}_2}{\overline{H}_0}\cdot\frac{\overline{Z}_2\cdot\cos\overline{\vartheta}_2}{\overline{Z}_1\cdot\cos\vartheta_0} = \overline{\tau}_{\text{par}}^{\text{H}}\cdot\frac{\overline{Z}_2\cdot\cos\overline{\vartheta}_2}{\overline{Z}_1\cdot\cos\vartheta_0} = \frac{2\cdot\cos\overline{\vartheta}_2}{\cos\overline{\vartheta}_2 + \dfrac{\overline{Z}_1}{\overline{Z}_2}\cdot\cos\vartheta_0} \tag{20.65}$$

and it holds that:

$$\overline{\tau}_{\text{par}}^{\text{E}} = 1 + \overline{\rho}_{\text{par}}^{\text{E}} \tag{20.66}$$

In this case as well, the reflection and transmission coefficients are complex quantities, which means that the reflected and transmitted waves differ from the incident wave in both magnitude and phase.

It is important to emphasize that, according to expressions (20.46) and (20.63), the following holds:

$$\overline{\rho}_{\perp}^{\text{H}} = -\overline{\rho}_{\perp}^{\text{E}} \quad ; \quad \overline{\rho}_{\text{par}}^{\text{E}} = -\overline{\rho}_{\text{par}}^{\text{H}} \tag{20.67}$$

20.5. Normal Incidence of a Sinusoidal Plane Wave

Let a linearly polarized plane electromagnetic wave from medium 1 be normally incident on the boundary of medium 2. In this case, both the electric and magnetic field intensities have only tangential components, and there are infinitely many planes of incidence.

Normal incidence of a plane electromagnetic wave can be considered a special case of the two previously discussed cases of oblique incidence of a plane electromagnetic wave, for which the following holds:

$$\vartheta_0 = \vartheta_1 = \vartheta_2 = 0 \quad ; \quad \cos\vartheta_0 = \cos\vartheta_1 = \cos\vartheta_2 = 1 \tag{20.68}$$

In this special case, the Fresnel equations for the electric field intensity (20.37) and (20.38) take on a new form:

$$\overline{E}_0 + \overline{E}_1 = \overline{E}_2 \quad ; \quad \overline{E}_0 - \overline{E}_1 = \frac{\overline{Z}_1}{\overline{Z}_2} \cdot \overline{E}_2 \tag{20.69}$$

from which it is easily obtained that the reflection and transmission coefficients are:

$$\overline{\rho}_{\mathrm{E}} = \frac{\overline{E}_1}{\overline{E}_0} = \frac{\overline{Z}_2 - \overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2} \quad ; \quad \overline{\tau}_{\mathrm{E}} = \frac{\overline{E}_2}{\overline{E}_0} = 1 + \overline{\rho}_{\mathrm{E}} = \frac{2 \cdot \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2} \tag{20.70}$$

$$\overline{\rho}_{\mathrm{H}} = -\,\overline{\rho}_{\mathrm{E}} = \frac{\overline{Z}_1 - \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2} \quad ; \quad \overline{\tau}_{\mathrm{H}} = 1 + \overline{\rho}_{\mathrm{H}} = \frac{2 \cdot \overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2} \tag{20.71}$$

In this special case, the Fresnel equations for the magnetic field intensity (20.54) and (20.55) take on a new form:

$$\overline{H}_0 + \overline{H}_1 = \overline{H}_2 \quad ; \quad \overline{H}_0 - \overline{H}_1 = \frac{\overline{Z}_2}{\overline{Z}_1} \cdot \overline{H}_2 \tag{20.72}$$

from which it is easily obtained that the reflection and transmission coefficients are:

$$\overline{\rho}_{\mathrm{H}} = \frac{\overline{H}_1}{\overline{H}_0} = \frac{\overline{Z}_1 - \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2} \quad ; \quad \overline{\tau}_{\mathrm{H}} = \frac{\overline{H}_2}{\overline{H}_0} = 1 + \overline{\rho}_{\mathrm{H}} = \frac{2 \cdot \overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2} \tag{20.73}$$

$$\overline{\rho}_{\mathrm{E}} = -\,\overline{\rho}_{\mathrm{H}} = \frac{\overline{Z}_2 - \overline{Z}_1}{\overline{Z}_1 + \overline{Z}_2} \quad ; \quad \overline{\tau}_{\mathrm{E}} = 1 + \overline{\rho}_{\mathrm{E}} = \frac{2 \cdot \overline{Z}_2}{\overline{Z}_1 + \overline{Z}_2} \tag{20.74}$$

Conclusion: For the tangential components of the electric and magnetic field intensities of a sinusoidal plane wave, the following holds:

$$1 + \overline{\rho} = \overline{\tau} \quad ; \quad 1 + \overline{\rho}_{\perp} = \overline{\tau}_{\perp} \quad ; \quad 1 + \overline{\rho}_{\mathrm{par}} = \overline{\tau}_{\mathrm{par}} \tag{20.75}$$

However, in the application of the method of images in electrostatics, magnetostatics and stationary current fields, the reflection factor k_{R} and the transmission factor k_{T} satisfy the following:

$$k_{\mathrm{R}} + k_{\mathrm{T}} = 1 \tag{20.76}$$

20.6. Plane Wave at the Dielectric-Conductor Boundary

Let a linearly polarized plane electromagnetic wave from a perfect LIH dielectric (medium 1) strikes the boundary of a good LIH conductor (medium 2) at an angle, as shown in Figure 20.5.

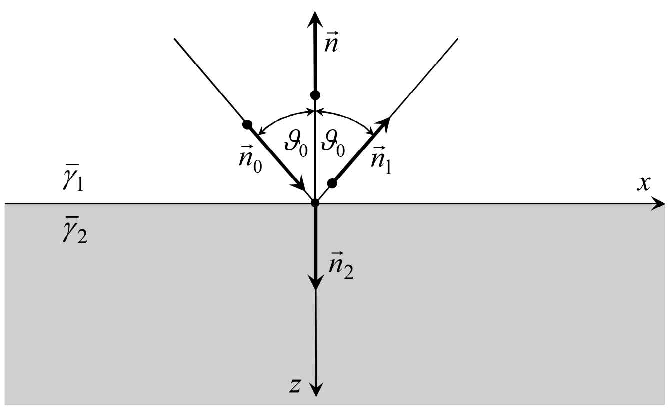


Figure 20.5. Incidence of a plane wave from a perfect dielectric onto the boundary with a good conductor

The following expressions apply to a perfect LIH dielectric:

$$\overline{\gamma}_1 = j\cdot\omega\cdot\sqrt{\mu_1\cdot\varepsilon_1} \tag{20.77}$$

whereas the following expressions apply to a good LIH conductor:

$$\overline{\gamma}_2 = \sqrt{\omega\cdot\mu_2\cdot\kappa_2}\cdot \mathrm{e}^{j\cdot\pi/4} \tag{20.78}$$

According to expressions (20.24), (20.77), and (20.78), the following holds:

$$\sin\overline{\vartheta}_2 = \frac{\overline{\gamma}_1}{\overline{\gamma}_2}\cdot\sin\vartheta_0 = \mathrm{e}^{j\cdot\pi/4}\cdot\sqrt{\frac{\omega\cdot\mu_1\cdot\varepsilon_1}{\mu_2\cdot\kappa_2}}\cdot\sin\vartheta_0 \tag{20.79}$$

Since it holds that:

$$\sqrt{\frac{\omega\cdot\mu_1\cdot\varepsilon_1}{\mu_2\cdot\kappa_2}}\approx 0 \tag{20.80}$$

even for very high frequencies on the order of GHz, it follows that:

$$\sin\overline{\vartheta}_2\approx 0 \quad ; \quad \overline{\vartheta}_2\approx 0 \tag{20.81}$$

which means that the transmitted electromagnetic wave in a good conductor propagates approximately perpendicular to the surface of the conductor, i.e., approximately perpendicular to the boundary plane between the perfect dielectric and the good conductor (Figure 20.5).
In this case, the following holds:

$$\frac{\overline{Z}_2}{\overline{Z}_1} = \frac{\mu_2\cdot\overline{\gamma}_1}{\mu_1\cdot\overline{\gamma}_2} = \mathrm{e}^{j\cdot\pi/4}\cdot\sqrt{\frac{\omega\cdot\mu_2\cdot\varepsilon_1}{\mu_1\cdot\kappa_2}}\approx 0 \tag{20.82}$$

and from the expressions (20.44) - (20.47) or alternatively from the expressions (20.61) - (20.64), it follows that the reflection and transmission coefficients of the electric and magnetic field intensities are:

$$\overline{\rho}_E\approx -1 \quad ; \quad \overline{\tau}_E\approx 0 \quad ; \quad \overline{\rho}_H\approx 1 \quad ; \quad \overline{\tau}_H\approx 2 \tag{20.83}$$

which means that the transmission coefficient of the electric field is negligible compared to that of the magnetic field. It is important to emphasize that the skin depth of the electromagnetic wave into a good conductor is very small, and that the electromagnetic wave does not penetrate a superconductor at all.

20.7. Solved Example

Example 20.1. Let the incident electric field intensity vector be described by the following expression: $\vec{E}_\mathrm{i} = 100\cdot\cos(\omega\cdot t - 6\cdot\pi\cdot z)\cdot\vec{\mathrm{i}}$ V/m. Let a linearly polarized sinusoidal plane electromagnetic wave be normally incident on the boundary plane between medium 1 ($\mu_{r1} = 1$, $\varepsilon_{r1} = 4$, $\kappa_1 = 0$) and medium 2 ($\mu_{r2} = 4$, $\varepsilon_{r2} = 9$, $\kappa_2 = 0$) at time t = 0. Write the expressions describing the time dependence of the electric field intensity vector and the magnetic field intensity vector for the incident, reflected, and transmitted waves.

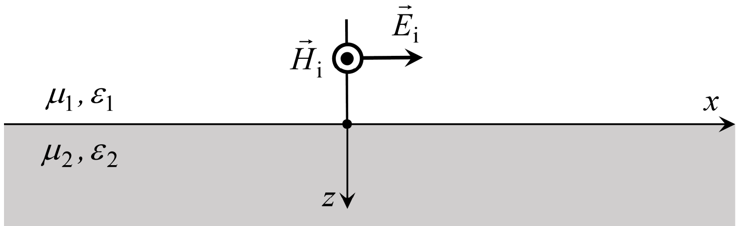


Figure 20.6. Normal incidence of a plane wave on the boundary between two perfect LIH dielectrics

Solution:

The given electric field intensity vector of the incident linearly polarized plane wave is described by the expression:

$$\vec{E}_{\mathrm{i}} = 100 \cdot \cos(\omega \cdot t - \beta_1 \cdot z) \cdot \vec{\mathrm{i}} = 100 \cdot \cos(\omega \cdot t - 6 \cdot \pi \cdot z) \cdot \vec{\mathrm{i}} \ \ \mathrm{V/m} \tag{20.84}$$

from which it follows that the phase constant of medium 1 is:

$$\beta_1 = \omega \cdot \sqrt{\mu_1 \cdot \varepsilon_1} = 6 \cdot \pi \tag{20.85}$$

and the phase constant of medium 2 is:

$$\beta_2 = \beta_1 \cdot \sqrt{\frac{\mu_{\mathrm{r}2} \cdot \varepsilon_{\mathrm{r}2}}{\mu_{\mathrm{r}1} \cdot \varepsilon_{\mathrm{r}1}}} = 3 \cdot \beta_1 = 18 \cdot \pi \tag{20.86}$$

The wave impedance of medium 1 is given by the expression:

$$Z_1 = \sqrt{\frac{\mu_1}{\varepsilon_1}} = 188.3651568 \ \Omega \tag{20.87}$$

whereas the wave impedance of medium 2 is given by the expression:

$$Z_2 = \sqrt{\frac{\mu_2}{\varepsilon_2}} = 251.1535424 \ \Omega \tag{20.88}$$

The reflection and transmission coefficients for the electric field intensity are:

$$\rho_{\mathrm{E}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = \frac{1}{7} \quad ; \quad \tau_{\mathrm{E}} = \frac{2 \cdot Z_2}{Z_1 + Z_2} = \frac{8}{7} \tag{20.89}$$

From expression (20.84), it follows that the incident magnetic field intensity vector is described by the following expression:

$$\vec{H}_{\mathrm{i}} = \frac{E_{\mathrm{i}}}{Z_1} \cdot \vec{\mathrm{j}} = 0.5308837456 \cdot \cos(\omega \cdot t - 6 \cdot \pi \cdot z) \cdot \vec{\mathrm{j}} \ \ \mathrm{A/m} \tag{20.90}$$

The reflected electric field intensity vector is described by the expression:

$$\vec{E}_{\mathrm{r}} = \rho_{\mathrm{E}} \cdot 100 \cdot \cos(\omega \cdot t + 6 \cdot \pi \cdot z) \cdot \vec{\mathrm{i}} \ \ \mathrm{V/m} = \frac{100}{7} \cdot \cos(\omega \cdot t + 6 \cdot \pi \cdot z) \cdot \vec{\mathrm{i}} \ \ \mathrm{V/m} \tag{20.91}$$

and the reflected magnetic field intensity vector is described by the following expression:

$$\vec{H}_{\mathrm{r}} = -\frac{E_{\mathrm{r}}}{Z_1} \cdot \vec{\mathrm{j}} = -0.07584053509 \cdot \cos(\omega \cdot t + 6 \cdot \pi \cdot z) \cdot \vec{\mathrm{j}} \ \ \mathrm{A/m} \tag{20.92}$$

The transmitted electric field intensity vector is described by the expression:

$$\vec{E}_{\mathrm{t}} = \tau_{\mathrm{E}} \cdot 100 \cdot \cos(\omega \cdot t - 18 \cdot \pi \cdot z) \cdot \vec{\mathrm{i}} \ \ \mathrm{V/m} = \frac{800}{7} \cdot \cos(\omega \cdot t - 18 \cdot \pi \cdot z) \cdot \vec{\mathrm{i}} \ \ \mathrm{V/m} \tag{20.93}$$

and the transmitted magnetic field intensity vector is described by the following expression:

$$\vec{H}_{\mathrm{t}} = \frac{E_{\mathrm{t}}}{Z_2} \cdot \vec{\mathrm{j}} = 0.4550432105 \cdot \cos(\omega \cdot t - 18 \cdot \pi \cdot z) \cdot \vec{\mathrm{j}} \ \ \mathrm{A/m} \tag{20.94}$$

21. ELECTRIC POWER TRANSMISSION LINES

An electric line is an assembly of one or more conductors, insulators, and various other equipment used for the transmission, distribution, and routing of electrical energy, or for telecommunication purposes. Electric power transmission lines are used for the transmission of electrical energy over long distances and for its distribution. They are generally high-voltage lines and can be classified as overhead (aerial) transmission lines and cable transmission lines. Cable transmission lines may be installed underground or underwater.

Telecommunication transmission lines are used for the transmission and distribution of analog or digital signals in public and private networks, as well as in telecommunication installations within buildings and other structures. In electronics, hollow metal tubes known as waveguides are used to transmit electromagnetic energy at high frequencies, whereas dielectric waveguides are used at very high frequencies.

In this textbook, intended for students in electric power engineering, the focus will be solely on the theory of electric power transmission lines. For numerical modeling of electric power transmission lines, electromagnetic models based on Maxwell's differential equations can be used, in which all parts of the power line are electromagnetically coupled. However, electromagnetic models of electrical networks – of which transmission lines are a part – belong to the category of advanced numerical models and will not be considered in this textbook. In this chapter, a numerical model of the electric power transmission line will be considered (*TLM – Transmission Line Model*), which is an approximation of the behavior in the electromagnetic field derived from Maxwell's differential equations.

The simplified theory of electric power transmission lines is derived from electromagnetic field theory, in which the vectors of electric and magnetic field intensity are replaced by electric currents in the conductors and electric voltages between the conductors (transverse voltages). Essentially, transmission lines are modeled as electrical networks with lumped parameters – meaning that Kirchhoff's laws are applied, but the transmission line parameters are distributed. This means that the (transverse) voltage and electric current vary along the length of the transmission line. Such a simplified model enables numerical analysis of electric power transmission lines under steady-state, sinusoidal, and transient current conditions. In other words, the TLM is used for numerical modeling of sinusoidal electrical networks and transient phenomena in electrical networks and on power transmission lines.

21.1. Equations of a Two-Wire Electric Power Transmission Line

Let R, L, C, and G be the resistance, inductance, capacitance, and conductance per unit length of the electric power transmission line. These are also known as the primary parameters of the transmission line. Both the electric voltage between the conductors and the electric current in the conductors depend on the position along the line, i.e., on the coordinate z (Figure 21.1).

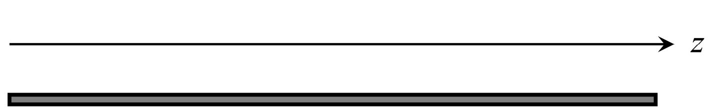


Figure 21.1. Two-wire electric power transmission line

It is important to remember that the electric voltage in a time-varying electromagnetic field depends on the integration path, so that the voltage between two points on an electric line is not uniquely defined. The transverse electric voltage is a special case of the electric voltage between two points, which is equal to the difference of the electric scalar potentials between these points. In this context, the term "electric voltage" refers specifically to the transverse electric voltage, whereas the electric voltage along the line (longitudinal electric voltage) can be referred to as the voltage drop.

A change in electric voltage (electric potential) along a conductor also occurs in electrical networks with lumped parameters, but a change in electric current along the conductor is a new concept. This occurs for two reasons: first, the insulator between the conductors is not perfect; and second, there is capacitive coupling between the conductors, allowing the capacitive component of the electric current to flow from one conductor to the other.

The differential equations of a two-wire electric power transmission line are derived from Maxwell's differential equations for an infinitesimal segment of the transmission line of length dz.

The first differential equation of a two-wire electric power transmission line can be derived from Kirchhoff's first law, which applies to an infinitesimal segment of the transmission line of length dz (Figure 21.2).

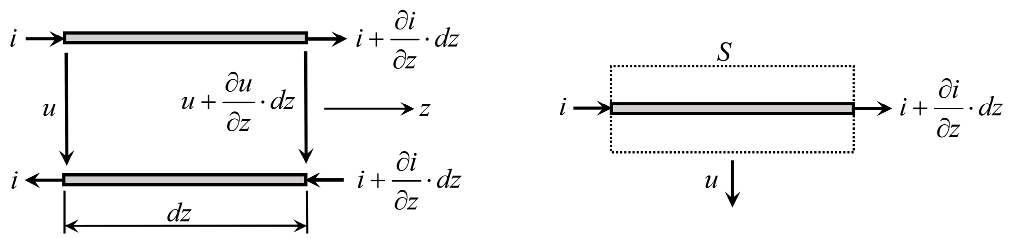


Figure 21.2. Electric voltages and currents in an infinitesimal segment of a two-wire electric power transmission line

According to Kirchhoff's first law, the sum of the electric currents entering a volume enclosed by the surface S (Figure 21.2) is equal to the sum of the electric currents leaving that volume:

$$i = \left(i + \frac{\partial i}{\partial z} \cdot dz \right) + (G \cdot dz) \cdot u + (C \cdot dz) \cdot \frac{\partial u}{\partial t} \tag{21.1}$$

where, for the conduction electric current through the infinitesimal conductance $G \cdot dz$ and the displacement electric current through the infinitesimal capacitance $C \cdot dz$, it is assumed that the electric voltage u is constant along the infinitesimal segment of the electric power transmission line.

From expression (21.1), the well-known first differential equation of the two-wire electric power transmission line is obtained, and it reads:

$$-\frac{\partial i}{\partial z} = G \cdot u + C \cdot \frac{\partial u}{\partial t} \tag{21.2}$$

and it can be represented by the equivalent circuit shown in Figure 20.3.

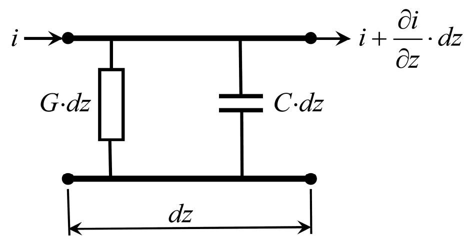


Figure 21.3. Equivalent circuit for the first differential equation of the two-wire electric power transmission line

The second differential equation of the two-wire electric power transmission line can be derived from Faraday's law of electromagnetic induction, i.e., from Maxwell's second integral equation (6.95), which reads:

$$\oint_C \vec{E} \cdot d\vec{\ell} = -\frac{\partial \Psi}{\partial t} = -\frac{\partial \Phi}{\partial t} \tag{21.3}$$

which also applies to an infinitesimal segment of the two-wire electric power transmission line (Figure 21.4).

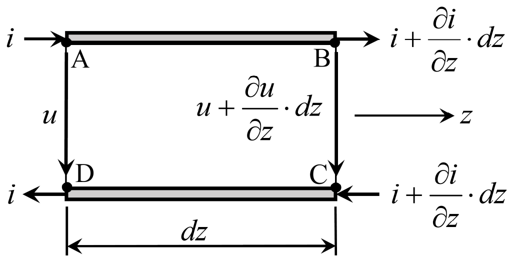


Figure 21.4. Infinitesimal segment of the two-wire electric power transmission line

Inside the conductors, the electric field intensity vector has a longitudinal component, whereas between the conductors, there is a transverse component of the electric field intensity vector. The rectangular integration path *C*, indicated in the expression (21.3), is defined by points A, B, C, and D, as shown in Figure 21.4.

According to the expression (21.3) and Figure 21.4, the following holds:

$$\int_{A}^{B}\vec{E}\cdot d\vec{\ell}+\int_{B}^{C}\vec{E}\cdot d\vec{\ell}+\int_{C}^{D}\vec{E}\cdot d\vec{\ell}+\int_{D}^{A}\vec{E}\cdot d\vec{\ell}=-L_{\text{ext}}\cdot dz\cdot\frac{\partial i}{\partial t} \tag{21.4}$$

where L_{ext} is the per-unit-length external inductance of the electric power transmission line.

Furthermore, it holds that:

$$\int_{A}^{B}\vec{E}\cdot d\vec{\ell}+\int_{C}^{D}\vec{E}\cdot d\vec{\ell}=R\cdot dz\cdot i+L_{\text{int}}\cdot dz\cdot\frac{\partial i}{\partial t} \tag{21.5}$$

$$\int_{B}^{C}\vec{E}\cdot d\vec{\ell}+\int_{D}^{A}\vec{E}\cdot d\vec{\ell}=\left(u+\frac{\partial u}{\partial z}\cdot dz\right)-u=\frac{\partial u}{\partial z}\cdot dz \tag{21.6}$$

where L_{int} is the per-unit-length external inductance of the electric power transmission line.

From expressions (21.4) - (21.6), the well-known second differential equation of the two-wire electric power transmission line is obtained, and it reads:

$$-\frac{\partial u}{\partial z}=R\cdot i+L\cdot\frac{\partial i}{\partial t} \tag{21.7}$$

where:

$$L=L_{\text{int}}+L_{\text{ext}} \tag{21.8}$$

The per-unit-length resistance *R* and the per-unit-length internal inductance L_{int} of the electric power transmission line correspond to the total electrical resistance and the total internal inductance of both conductors of the two-wire transmission line. These parameters are frequency-dependent, but in numerical models they can be approximated by their values corresponding to a direct electric current.

The second differential equation of the two-wire electric power transmission line (21.7) can be represented by the equivalent circuit shown in Figure 21.5.

By applying Kirchhoff's second law to the equivalent circuit shown in Figure 21.5, the following equation is obtained:

$$R\cdot dz\cdot i+L\cdot dz\cdot\frac{\partial i}{\partial t}+u+\frac{\partial u}{\partial z}\cdot dz-u=0 \tag{21.9}$$

from which the second differential equation of the electric power transmission line (21.7) can be derived.

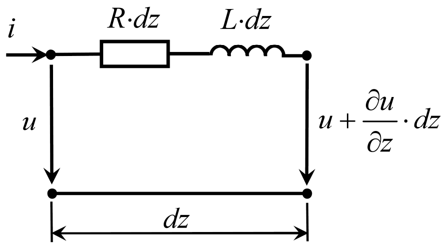


Figure 21.5. Equivalent circuit for the second differential equation of the two-wire electric power transmission line

Therefore, the differential equations of the two-wire electric power transmission line* are:

$$-\frac{\partial u}{\partial z}=R\cdot i+L\cdot\frac{\partial i}{\partial t} \tag{21.10}$$

$$-\frac{\partial i}{\partial z}=G\cdot u+C\cdot\frac{\partial u}{\partial t} \tag{21.11}$$

and they are first-order partial differential equations that do not take into account the mutual electromagnetic coupling between infinitesimal segments of the line.

The differential equations (21.10) and (21.11) can be differentiated with respect to z and t, leading to the following expressions:

$$-\frac{\partial^2 u}{\partial z^2}=R\cdot\frac{\partial i}{\partial z}+L\cdot\frac{\partial^2 i}{\partial z\partial t}\quad ;\quad -\frac{\partial^2 u}{\partial z\partial t}=R\cdot\frac{\partial i}{\partial t}+L\cdot\frac{\partial^2 i}{\partial t^2} \tag{21.12}$$

$$-\frac{\partial^2 i}{\partial z^2}=G\cdot\frac{\partial u}{\partial z}+C\cdot\frac{\partial^2 u}{\partial z\partial t}\quad ;\quad -\frac{\partial^2 i}{\partial z\partial t}=G\cdot\frac{\partial u}{\partial t}+C\cdot\frac{\partial^2 u}{\partial t^2} \tag{21.13}$$

From differential equations (21.10) - (21.13), the following second-order partial differential equations can be derived†:

$$\frac{\partial^2 u}{\partial z^2} = R\cdot G\cdot u + \left(R\cdot C+L\cdot G\right)\cdot\frac{\partial u}{\partial t} + L\cdot C\cdot\frac{\partial^2 u}{\partial t^2} \tag{21.14}$$

$$\frac{\partial^2 i}{\partial z^2} = R\cdot G\cdot i + \left(R\cdot C+L\cdot G\right)\cdot\frac{\partial i}{\partial t} + L\cdot C\cdot\frac{\partial^2 i}{\partial t^2} \tag{21.15}$$

In the phasor domain, the first-order partial differential equations of the line (21.10) and (21.11) are transformed into the following partial differential equations

$$\frac{\partial\overline{U}}{\partial z}=-\left(R+j\cdot\omega\cdot L\right)\cdot\overline{I} \tag{21.16}$$

$$\frac{\partial\overline{I}}{\partial z}=-\left(G+j\cdot\omega\cdot C\right)\cdot\overline{U} \tag{21.17}$$

whereas the second-order partial differential equations of the line (21.14) and (21.15), similarly to the case of a plane electromagnetic wave, are transformed into the homogeneous Helmholtz differential equations:

$$\frac{\partial^2\overline{U}}{\partial z^2}=\overline{\gamma}^2\cdot\overline{U} \tag{21.18}$$

* Some authors refer to partial differential equations (21.10) and (21.11) as the telegrapher's equations.

† Some authors refer to partial differential equations (21.14) and (21.15) as the telegrapher's equations.

$$\frac{\partial^2 \bar{I}}{\partial z^2} = \bar{\gamma}^2 \cdot \bar{I} \tag{21.19}$$

where the propagation constant $\bar{\gamma}$ of the transmission line is defined by the following expression:

$$\bar{\gamma} = \sqrt{(R + j \cdot \omega \cdot L) \cdot (G + j \cdot \omega \cdot C)} \tag{21.20}$$

The solutions of the homogeneous Helmholtz differential equations (21.18) and (21.19) are given by the following expressions:

$$\overline{U} = \overline{A}_1 \cdot e^{-\bar{\gamma} \cdot z} + \overline{A}_2 \cdot e^{\bar{\gamma} \cdot z} \tag{21.21}$$

$$\bar{I} = \overline{B}_1 \cdot e^{-\bar{\gamma} \cdot z} + \overline{B}_2 \cdot e^{\bar{\gamma} \cdot z} \tag{21.22}$$

where $\overline{A}_1, \overline{A}_2, \overline{B}_1,$ and $\overline{B}_2$ are unknown complex constants that are determined based on satisfying the given boundary conditions.

If the expressions (21.21) and (21.22) are substituted into expression (21.16), the following expression is obtained:

$$\bar{\gamma} \cdot \left(-\overline{A}_1 \cdot e^{-\bar{\gamma} \cdot z} + \overline{A}_2 \cdot e^{\bar{\gamma} \cdot z}\right) = -(R + j \cdot \omega \cdot L) \cdot \left(\overline{B}_1 \cdot e^{-\bar{\gamma} \cdot z} + \overline{B}_2 \cdot e^{\bar{\gamma} \cdot z}\right) \tag{21.23}$$

from which it follows that:

$$\left[-\bar{\gamma} \cdot \overline{A}_1 + (R + j \cdot \omega \cdot L) \cdot \overline{B}_1\right] \cdot e^{-\bar{\gamma} \cdot z} + \left[\bar{\gamma} \cdot \overline{A}_2 + (R + j \cdot \omega \cdot L) \cdot \overline{B}_2\right] \cdot e^{\bar{\gamma} \cdot z} = 0 \tag{21.24}$$

and it is:

$$-\bar{\gamma} \cdot \overline{A}_1 + (R + j \cdot \omega \cdot L) \cdot \overline{B}_1 = 0 \tag{21.25}$$

$$\bar{\gamma} \cdot \overline{A}_2 + (R + j \cdot \omega \cdot L) \cdot \overline{B}_2 = 0 \tag{21.26}$$

From expressions (21.25) and (21.26), it follows that:

$$\frac{\overline{A}_1}{\overline{B}_1} = -\frac{\overline{A}_2}{\overline{B}_2} = \frac{R + j \cdot \omega \cdot L}{\bar{\gamma}} = \overline{Z}_\mathrm{c} \tag{21.27}$$

where $\overline{Z}_\mathrm{c}$ is the characteristic impedance of the transmission line, also known as the wave impedance or the surge impedance of the transmission line.

From expressions (21.21) and (21.27), it follows that the characteristic impedance of the transmission line is given by the expression:

$$\overline{Z}_\mathrm{c} = \frac{R + j \cdot \omega \cdot L}{\bar{\gamma}} = \frac{\bar{\gamma}}{G + j \cdot \omega \cdot C} = \sqrt{\frac{R + j \cdot \omega \cdot L}{G + j \cdot \omega \cdot C}} = Z_\mathrm{c} \cdot e^{j \cdot \varphi_\mathrm{c}} \tag{21.28}$$

Although an analogy can be drawn between the equations of a two-wire electric power transmission line and those of a plane wave, the characteristic impedance of a transmission line should be physically distinguished from the wave impedance of a medium, which appears in the case of a sinusoidal plane wave.

The propagation constant of the transmission line can be described by the following expression:

$$\bar{\gamma} = \alpha + j \cdot \beta \tag{21.29}$$

where α is the attenuation constant of the transmission line, and β is the phase constant of the transmission line.

The propagation velocity of the electromagnetic wave along the transmission line, also known as the phase velocity, is defined by the following expression:

$$v = \frac{\omega}{\beta} \tag{21.30}$$

Let it be:

$$\overline{A}_1 = A_1 \cdot e^{j\cdot\varphi_{1u}} \quad ; \quad \overline{A}_2 = A_2 \cdot e^{j\cdot\varphi_{2u}} \tag{21.31}$$

$$\overline{B}_1 = B_1 \cdot e^{j\cdot\varphi_{1i}} \quad ; \quad \overline{B}_2 = B_2 \cdot e^{j\cdot\varphi_{2i}} \tag{21.32}$$

From expressions (21.21), (21.22), (21.31), and (21.32), it follows that:

$$\overline{U} = A_1 \cdot e^{j\cdot\varphi_{1u}} \cdot e^{-(\alpha + j\cdot\beta)\cdot z} + A_2 \cdot e^{j\cdot\varphi_{2u}} \cdot e^{(\alpha + j\cdot\beta)\cdot z} \tag{21.33}$$

$$\bar{I} = B_1 \cdot e^{j\cdot\varphi_{1i}} \cdot e^{-(\alpha + j\cdot\beta)\cdot z} + B_2 \cdot e^{j\cdot\varphi_{2i}} \cdot e^{(\alpha + j\cdot\beta)\cdot z} \tag{21.34}$$

and it is:

$$\overline{U} = A_1 \cdot e^{-\alpha\cdot z} \cdot e^{-j\cdot(\beta\cdot z - \varphi_{1u})} + A_2 \cdot e^{\alpha\cdot z} \cdot e^{j\cdot(\beta\cdot z + \varphi_{2u})} \tag{21.35}$$

$$\bar{I} = B_1 \cdot e^{-\alpha\cdot z} \cdot e^{-j\cdot(\beta\cdot z - \varphi_{1i})} + B_2 \cdot e^{\alpha\cdot z} \cdot e^{j\cdot(\beta\cdot z + \varphi_{2i})} \tag{21.36}$$

From expressions (21.35) and (21.36), it follows that the electric field intensity and the magnetic field intensity in the time domain are described by the following expressions:

$$\begin{aligned} u(z,t) = u_{\text{for}}(z,t) + u_{\text{rev}}(z,t) = \sqrt{2} \cdot A_1 \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1u}) \\ + \sqrt{2} \cdot A_2 \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2u}) \end{aligned} \tag{21.37}$$

$$\begin{aligned} i(z,t) = i_{\text{for}}(z,t) + i_{\text{rev}}(z,t) = \sqrt{2} \cdot B_1 \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1i}) \\ + \sqrt{2} \cdot B_2 \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2i}) \end{aligned} \tag{21.38}$$

The forward electromagnetic wave propagates in the direction of the z-axis, whereas the reverse electromagnetic wave propagates in the opposite direction. Each of these waves attenuates in the direction of its propagation.

The phase angles of the electric voltage and electric current are related by the phase angle of the transmission line's characteristic impedance. The following holds:

$$\frac{\overline{A}_1}{\overline{B}_1} = \frac{A_1}{B_1} \cdot e^{j\cdot(\varphi_{1u} - \varphi_{1i})} = Z_c \cdot e^{j\cdot\varphi_c} \tag{21.39}$$

$$\frac{\overline{A}_2}{\overline{B}_2} = \frac{A_2}{B_2} \cdot e^{j\cdot(\varphi_{2u} - \varphi_{2i})} = -Z_c \cdot e^{j\cdot\varphi_c} = Z_c \cdot e^{j\cdot(\varphi_c + \pi)} \tag{21.40}$$

from which it follows that:

$$B_1 = \frac{A_1}{Z_c} \quad ; \quad \varphi_{1i} = \varphi_{1u} - \varphi_c \tag{21.41}$$

$$B_2 = \frac{A_2}{Z_c} \quad ; \quad \varphi_{2i} = \varphi_{2u} - \varphi_c - \pi \tag{21.42}$$

From expressions (21.37), (21.38), (21.41), and (21.42), it follows that the electric voltage and electric current in the time domain can also be described by the following expressions:

$$u(z,t) = \sqrt{2} \cdot A_1 \cdot e^{-\alpha\cdot z} \cdot \cos(\omega\cdot t - \beta\cdot z + \varphi_{1u}) + \sqrt{2} \cdot A_2 \cdot e^{\alpha\cdot z} \cdot \cos(\omega\cdot t + \beta\cdot z + \varphi_{2u}) \tag{21.43}$$

$$i(z,t) = \sqrt{2} \cdot \frac{A_1}{Z_c} \cdot e^{-\alpha \cdot z} \cdot \cos\left(\omega \cdot t - \beta \cdot z + \varphi_{1u} - \varphi_c\right) - \sqrt{2} \cdot \frac{A_2}{Z_c} \cdot e^{\alpha \cdot z} \cdot \cos\left(\omega \cdot t + \beta \cdot z + \varphi_{2u} - \varphi_c\right) \tag{21.44}$$

21.2. Lossless Two-Wire Electric Power Transmission Line

For a lossless two-wire overhead electric power transmission line, the following holds:

$$R = G = 0 \tag{21.45}$$

$$\alpha = 0 \quad ; \quad \beta = \omega \cdot \sqrt{L \cdot C} \quad ; \quad \bar{\gamma} = j \cdot \omega \cdot \sqrt{L \cdot C} \tag{21.46}$$

$$\bar{Z}_c = Z_c = \sqrt{\frac{L}{C}} \quad ; \quad v = \frac{\omega}{\beta} = \frac{1}{\sqrt{L \cdot C}} \tag{21.47}$$

which means that the propagation of an electromagnetic wave along a lossless two-wire electric power transmission line is analogous to the propagation of an electromagnetic plane wave in a perfect LIH dielectric.

In this special case, the phase velocity of the wave does not depend on the frequency, and over a distance of one wavelength (λ), the spatial phase changes by $2 \cdot \pi$, so the following expressions hold:

$$\beta \cdot \lambda = 2 \cdot \pi \quad ; \quad \lambda = \frac{2 \cdot \pi}{\beta} = \frac{1}{f \cdot \sqrt{L \cdot C}} = \frac{v}{f} \tag{21.48}$$

Let the conductors of an isolated two-wire electric power transmission line be spaced at a distance d (Figure 21.6), and let the radius of both conductors be r_0, with $d >> r_0$.

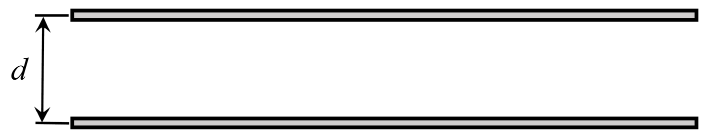


Figure 21.6. Isolated two-wire electric power transmission line

Neglecting the losses of the isolated two-wire overhead electric power transmission line, as well as its internal inductance, the per-unit-length inductance and capacitance of the line are given by the following expressions:

$$L = L_{ext} = \frac{\mu_0}{\pi} \cdot \ln\frac{d}{r_0} \quad ; \quad C = \frac{\pi \cdot \varepsilon_0}{\ln\frac{d}{r_0}} \tag{21.49}$$

and, according to the expressions (21.47), the phase velocity of the wave is equal to the speed of light:

$$v = \frac{1}{\sqrt{L \cdot C}} = \frac{1}{\sqrt{\mu_0 \cdot \varepsilon_0}} = c \tag{21.50}$$

whereas the characteristic impedance of the transmission line is given by the following expression:

$$\bar{Z}_c = Z_c = \sqrt{\frac{L}{C}} = \frac{1}{\pi} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \ln\frac{d}{r_0} = \frac{1}{\pi} \cdot Z_0 \cdot \ln\frac{d}{r_0} \tag{21.51}$$

where the wave impedance of air, Z_0, is given by the expression (19.45).

Therefore, in the case of a lossless isolated two-wire overhead electric power transmission line, electromagnetic energy is transmitted along the line at the speed of light.

Let the conductors of the coaxial two-wire cable transmission line be as shown in Figure 21.7.

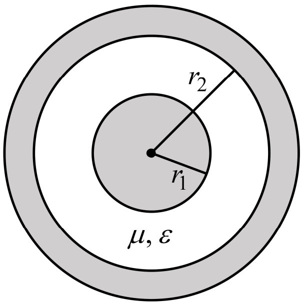


Figure 21.7. Coaxial two-wire cable transmission line

Neglecting the losses of the two-wire coaxial cable transmission line and the internal inductance of both conductors, the per-unit-length inductance and capacitance of the line are given by the following expressions:

$$L = \frac{\mu}{2 \cdot \pi} \cdot \ln \frac{r_2}{r_1} \quad ; \quad C = \frac{2 \cdot \pi \cdot \varepsilon}{\ln \frac{r_2}{r_1}} \tag{21.52}$$

and, according to expressions (21.47), the phase velocity of the wave is:

$$v = \frac{1}{\sqrt{L \cdot C}} = \frac{1}{\sqrt{\mu \cdot \varepsilon}} = \frac{c}{\sqrt{\mu_\mathrm{r} \cdot \varepsilon_\mathrm{r}}} \tag{21.53}$$

whereas the characteristic impedance of the coaxial cable transmission line is given by the following expression:

$$\overline{Z}_\mathrm{c} = Z_\mathrm{c} = \sqrt{\frac{L}{C}} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{\mu}{\varepsilon}} \cdot \ln \frac{r_2}{r_1} \tag{21.54}$$

21.3. Electric Voltage and Current Along a Two-Wire Transmission Line

Let the electric voltage and electric current be sinusoidal. In that case, they are described by expressions (21.21) and (21.22), which read:

$$\overline{U} = \overline{A}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{A}_2 \cdot e^{\overline{\gamma} \cdot z} \quad ; \quad \overline{I} = \overline{B}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{B}_2 \cdot e^{\overline{\gamma} \cdot z} \tag{21.55}$$

and according to expression (21.27):

$$\overline{U} = \overline{A}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{A}_2 \cdot e^{\overline{\gamma} \cdot z} \quad ; \quad \overline{I} = \frac{\overline{A}_1}{\overline{Z}_\mathrm{c}} \cdot e^{-\overline{\gamma} \cdot z} - \frac{\overline{A}_2}{\overline{Z}_\mathrm{c}} \cdot e^{\overline{\gamma} \cdot z} \tag{21.56}$$

The system of linear equations (21.56) in matrix notation is given by:

$$\begin{Bmatrix} \overline{U} \\ \overline{I} \end{Bmatrix} = \begin{bmatrix} e^{-\overline{\gamma} \cdot z} & e^{\overline{\gamma} \cdot z} \\ \dfrac{e^{-\overline{\gamma} \cdot z}}{\overline{Z}_\mathrm{c}} & -\dfrac{e^{\overline{\gamma} \cdot z}}{\overline{Z}_\mathrm{c}} \end{bmatrix} \cdot \begin{Bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{Bmatrix} \tag{21.57}$$

Let the length of the two-wire transmission line be $\ell,$ with the beginning of the line at $z = z_1$ and the end of the line at $z = z_2$ (Figures 21.8 and 21.9).

Figure 21.8. Single-wire representation of a two-wire transmission line

Figure 21.9. Simplified representation of electric voltages and currents in a two-wire transmission line

From expression (21.57), it follows that the electric voltage and electric current at the beginning of the line are described by the following matrix expression:

$$\begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} = \begin{bmatrix} e^{-\overline{\gamma}\cdot z_1} & e^{\overline{\gamma}\cdot z_1} \\ \dfrac{e^{-\overline{\gamma}\cdot z_1}}{\overline{Z}_c} & -\dfrac{e^{\overline{\gamma}\cdot z_1}}{\overline{Z}_c} \end{bmatrix} \cdot \begin{Bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{Bmatrix} \tag{21.58}$$

where:

$\overline{U}_1$ - the phasor of the electric voltage at the beginning of the two-wire transmission line,

$\overline{I}_1$ - the phasor of the electric current at the beginning of the two-wire transmission line.

After inverting the matrix of the system of linear equations (21.58), it follows that:

$$\begin{Bmatrix} \overline{A}_1 \\ \overline{A}_2 \end{Bmatrix} = -\frac{\overline{Z}_c}{2} \cdot \begin{bmatrix} -\dfrac{e^{\overline{\gamma}\cdot z_1}}{\overline{Z}_c} & -e^{\overline{\gamma}\cdot z_1} \\ -\dfrac{e^{-\overline{\gamma}\cdot z_1}}{\overline{Z}_c} & e^{-\overline{\gamma}\cdot z_1} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} = \begin{bmatrix} \dfrac{e^{\overline{\gamma}\cdot z_1}}{2} & \dfrac{\overline{Z}_c \cdot e^{\overline{\gamma}\cdot z_1}}{2} \\ \dfrac{e^{-\overline{\gamma}\cdot z_1}}{2} & -\dfrac{\overline{Z}_c \cdot e^{-\overline{\gamma}\cdot z_1}}{2} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} \tag{21.59}$$

From the matrix expressions (21.57) and (21.59), it follows that the distributions of the electric voltage and current along the two-wire transmission line, expressed in terms of the electric voltage and current at the beginning of the line, are given by the following matrix expression:

$$\begin{Bmatrix} \overline{U} \\ \overline{I} \end{Bmatrix} = \begin{bmatrix} e^{-\overline{\gamma}\cdot z} & e^{\overline{\gamma}\cdot z} \\ \dfrac{e^{-\overline{\gamma}\cdot z}}{\overline{Z}_c} & -\dfrac{e^{\overline{\gamma}\cdot z}}{\overline{Z}_c} \end{bmatrix} \cdot \begin{bmatrix} \dfrac{e^{\overline{\gamma}\cdot z_1}}{2} & \dfrac{\overline{Z}_c \cdot e^{\overline{\gamma}\cdot z_1}}{2} \\ \dfrac{e^{-\overline{\gamma}\cdot z_1}}{2} & -\dfrac{\overline{Z}_c \cdot e^{-\overline{\gamma}\cdot z_1}}{2} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} \tag{21.60}$$

which, after matrix multiplication, takes the following form:

$$\begin{Bmatrix} \overline{U} \\ \overline{I} \end{Bmatrix} = \begin{bmatrix} \cosh\left(\overline{\gamma}\cdot(z - z_1)\right) & -\overline{Z}_c \cdot \sinh\left(\overline{\gamma}\cdot(z - z_1)\right) \\ -\dfrac{1}{\overline{Z}_c} \cdot \sinh\left(\overline{\gamma}\cdot(z - z_1)\right) & \cosh\left(\overline{\gamma}\cdot(z - z_1)\right) \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} \tag{21.61}$$

where:

$$\cosh\left(\overline{\gamma}\cdot(z - z_1)\right) = \frac{e^{\overline{\gamma}\cdot(z - z_1)} + e^{-\overline{\gamma}\cdot(z - z_1)}}{2} \tag{21.62}$$

$$\sinh\left(\overline{\gamma}\cdot(z-z_1)\right)=\frac{e^{\overline{\gamma}\cdot(z-z_1)}-e^{-\overline{\gamma}\cdot(z-z_1)}}{2} \tag{21.63}$$

From the matrix expression (21.61), it follows that the electric voltage and current at the end of the two-wire transmission line, expressed in terms of the electric voltage and current at the beginning of the line, are given by the following matrix expression:

$$\begin{Bmatrix}\overline{U}_2\\ \overline{I}_2\end{Bmatrix}=\begin{bmatrix}\cosh\left(\overline{\gamma}\cdot\ell\right) & -\overline{Z}_{\text{c}}\cdot\sinh\left(\overline{\gamma}\cdot\ell\right)\\ -\dfrac{1}{\overline{Z}_{\text{c}}}\cdot\sinh\left(\overline{\gamma}\cdot\ell\right) & \cosh\left(\overline{\gamma}\cdot\ell\right)\end{bmatrix}\cdot\begin{Bmatrix}\overline{U}_1\\ \overline{I}_1\end{Bmatrix} \tag{21.64}$$

where:

$\overline{U}_2$ - the phasor of the electric voltage at the end of the two-wire transmission line,

$\overline{I}_2$ - the phasor of the electric current at the end of the two-wire transmission line.

After inverting the matrix of the system of linear equations (21.64), it follows that:

$$\begin{Bmatrix}\overline{U}_1\\ \overline{I}_1\end{Bmatrix}=\begin{bmatrix}\cosh\left(\overline{\gamma}\cdot\ell\right) & \overline{Z}_{\text{c}}\cdot\sinh\left(\overline{\gamma}\cdot\ell\right)\\ \dfrac{1}{\overline{Z}_{\text{c}}}\cdot\sinh\left(\overline{\gamma}\cdot\ell\right) & \cosh\left(\overline{\gamma}\cdot\ell\right)\end{bmatrix}\cdot\begin{Bmatrix}\overline{U}_2\\ \overline{I}_2\end{Bmatrix} \tag{21.65}$$

which describes the electric voltage and current at the beginning of the two-wire transmission line in terms of the electric voltage and current at the end of the line.

21.4. Endpoint Electric Currents Expressed in Terms of Electric Voltages

From the solutions of the inhomogeneous Helmholtz differential equations of the transmission line, described by expressions (21.56), it follows that:

$$\begin{Bmatrix}\overline{U}_1\\ \overline{U}_2\end{Bmatrix}=\begin{bmatrix}e^{-\overline{\gamma}\cdot z_1} & e^{\overline{\gamma}\cdot z_1}\\ e^{-\overline{\gamma}\cdot z_2} & e^{\overline{\gamma}\cdot z_2}\end{bmatrix}\cdot\begin{Bmatrix}\overline{A}_1\\ \overline{A}_2\end{Bmatrix} \tag{21.66}$$

$$\begin{Bmatrix}\overline{I}_1\\ \overline{I}_2\end{Bmatrix}=\frac{1}{\overline{Z}_{\text{c}}}\cdot\begin{bmatrix}e^{-\overline{\gamma}\cdot z_1} & -e^{\overline{\gamma}\cdot z_1}\\ e^{-\overline{\gamma}\cdot z_2} & -e^{\overline{\gamma}\cdot z_2}\end{bmatrix}\cdot\begin{Bmatrix}\overline{A}_1\\ \overline{A}_2\end{Bmatrix} \tag{21.67}$$

After inverting the matrix of the system of linear equations (21.66), it follows that:

$$\begin{Bmatrix}\overline{A}_1\\ \overline{A}_2\end{Bmatrix}=\frac{1}{2\cdot\sinh\left(\overline{\gamma}\cdot\ell\right)}\cdot\begin{bmatrix}e^{\overline{\gamma}\cdot z_2} & -e^{\overline{\gamma}\cdot z_1}\\ -e^{-\overline{\gamma}\cdot z_2} & e^{-\overline{\gamma}\cdot z_1}\end{bmatrix}\cdot\begin{Bmatrix}\overline{U}_1\\ \overline{U}_2\end{Bmatrix} \tag{21.68}$$

and after substituting expression (21.68) into expression (21.67), it is obtained that:

$$\begin{Bmatrix}\overline{I}_1\\ \overline{I}_2\end{Bmatrix}=\frac{1}{2\cdot\overline{Z}_{\text{c}}\cdot\sinh\left(\overline{\gamma}\cdot\ell\right)}\cdot\begin{bmatrix}e^{-\overline{\gamma}\cdot z_1} & -e^{\overline{\gamma}\cdot z_1}\\ e^{-\overline{\gamma}\cdot z_2} & -e^{\overline{\gamma}\cdot z_2}\end{bmatrix}\cdot\begin{bmatrix}e^{\overline{\gamma}\cdot z_2} & -e^{\overline{\gamma}\cdot z_1}\\ -e^{-\overline{\gamma}\cdot z_2} & e^{-\overline{\gamma}\cdot z_1}\end{bmatrix}\cdot\begin{Bmatrix}\overline{U}_1\\ \overline{U}_2\end{Bmatrix} \tag{21.69}$$

which, after matrix multiplication, takes the following form:

$$\begin{Bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{Bmatrix} = \frac{1}{2 \cdot \overline{Z}_c \cdot \sinh\left(\bar{\gamma} \cdot \ell\right)} \cdot \begin{bmatrix} 2 \cdot \cosh\left(\bar{\gamma} \cdot \ell\right) & -2 \\ 2 & -2 \cdot \cosh\left(\bar{\gamma} \cdot \ell\right) \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{U}_2 \end{Bmatrix} \tag{21.70}$$

from which the final expression follows:

$$\begin{Bmatrix} \bar{I}_1 \\ \bar{I}_2 \end{Bmatrix} = \begin{bmatrix} \dfrac{1}{\overline{Z}_c \cdot \tanh\left(\bar{\gamma} \cdot \ell\right)} & -\dfrac{1}{\overline{Z}_c \cdot \sinh\left(\bar{\gamma} \cdot \ell\right)} \\ \dfrac{1}{\overline{Z}_c \cdot \sinh\left(\bar{\gamma} \cdot \ell\right)} & -\dfrac{1}{\overline{Z}_c \cdot \tanh\left(\bar{\gamma} \cdot \ell\right)} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_1 \\ \overline{U}_2 \end{Bmatrix} \tag{21.71}$$

21.5. FET-Based Equations for a Segment of a Two-Wire Transmission Line

If the Finite Element Technique (FET) is used, then for the segment of the two-wire transmission line shown in Figure 21.9, new notations should be introduced as shown in Figure 21.10, where:

$\overline{U}_{n1}$ - the phasor of the electric voltage at local node 1 of the two-node finite element,

$\overline{U}_{n2}$ - the phasor of the electric voltage at local node 2 of the two-node finite element,

$\bar{I}_{n1}$ - the phasor of the electric current at local node 1 of the two-node finite element,

$\bar{I}_{n2}$ - phasor of the electric current at local node 2 of the two-node finite element.

Figure 21.10. Electric voltages and currents of the two-node finite element

Based on the comparison of the quantities in Figures 21.9 and 21.10, it is easy to conclude that the following expressions hold:

$$\overline{U}_{n1} = \overline{U}_1 \quad ; \quad \overline{U}_{n2} = \overline{U}_2 \quad ; \quad \bar{I}_{n1} = \bar{I}_1 \quad ; \quad \bar{I}_{n2} = -\bar{I}_2 \tag{21.72}$$

According to the expressions (21.71) and (21.72), in the FET, the complete local system of linear equations for the segment of the two-wire transmission line (two-node finite element) is given by:

$$\begin{bmatrix} \dfrac{1}{\overline{Z}_c \cdot \tanh\left(\bar{\gamma} \cdot \ell\right)} & -\dfrac{1}{\overline{Z}_c \cdot \sinh\left(\bar{\gamma} \cdot \ell\right)} \\ -\dfrac{1}{\overline{Z}_c \cdot \sinh\left(\bar{\gamma} \cdot \ell\right)} & \dfrac{1}{\overline{Z}_c \cdot \tanh\left(\bar{\gamma} \cdot \ell\right)} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_{n1} \\ \overline{U}_{n2} \end{Bmatrix} = \begin{Bmatrix} \bar{I}_{n1} \\ \bar{I}_{n2} \end{Bmatrix} \tag{21.73}$$

whereas the incomplete local system of linear equations for the segment of the two-wire transmission line is given by:

$$\begin{bmatrix} \dfrac{1}{\overline{Z}_\mathrm{c} \cdot \tanh\left(\overline{\gamma} \cdot \ell\right)} & -\dfrac{1}{\overline{Z}_\mathrm{c} \cdot \sinh\left(\overline{\gamma} \cdot \ell\right)} \\ -\dfrac{1}{\overline{Z}_\mathrm{c} \cdot \sinh\left(\overline{\gamma} \cdot \ell\right)} & \dfrac{1}{\overline{Z}_\mathrm{c} \cdot \tanh\left(\overline{\gamma} \cdot \ell\right)} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_\mathrm{n1} \\ \overline{U}_\mathrm{n2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{21.74}$$

In the finite element technique, the incomplete global system of linear equations is obtained by assembling the incomplete local systems of linear equations to form the incomplete global system. The vectors of the electric currents entering the local nodes of the finite elements can be set to zero vectors, since according to Kirchhoff's first law, the sum of the electric currents entering the global nodes is equal to zero. The incomplete global system of linear equations is then completed by including the prescribed global electric voltages and/or prescribed global electric currents. The finite element mesh may consist of several segments of a two-wire transmission line and, in the general case, other types of finite elements, such as lumped-parameter elements.

21.6. Loaded Two-Wire Transmission Line

Let the two-wire line be loaded with an impedance $\overline{Z}_\mathrm{L}$. Then the solutions of the homogeneous Helmholtz differential equations of the transmission line, given in (21.18) and (21.19) and described by expressions (21.56), are as follows:

$$\overline{U} = \overline{A}_1 \cdot e^{-\overline{\gamma} \cdot z} + \overline{A}_2 \cdot e^{\overline{\gamma} \cdot z} \quad ; \quad \overline{I} = \frac{\overline{A}_1}{\overline{Z}_\mathrm{c}} \cdot e^{-\overline{\gamma} \cdot z} - \frac{\overline{A}_2}{\overline{Z}_\mathrm{c}} \cdot e^{\overline{\gamma} \cdot z} \tag{21.75}$$

and can be written at node 2 (at the end of the line, as shown in Figure 21.9) in the following form:

$$\overline{U}_2 = \overline{U}_\mathrm{for} + \overline{U}_\mathrm{rev} \quad ; \quad \overline{I}_2 = \frac{\overline{U}_\mathrm{for}}{\overline{Z}_\mathrm{c}} - \frac{\overline{U}_\mathrm{rev}}{\overline{Z}_\mathrm{c}} \tag{21.76}$$

where:

$\overline{U}_\mathrm{for}$ - forward component of the electric voltage phasor at the end of the line,

$\overline{U}_\mathrm{rev}$ - reverse component of the electric voltage phasor at the end of the line.

The system of linear equations (21.76) can be written in matrix notation as follows:

$$\begin{bmatrix} 1 & 1 \\ \dfrac{1}{\overline{Z}_\mathrm{c}} & \dfrac{-1}{\overline{Z}_\mathrm{c}} \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_\mathrm{for} \\ \overline{U}_\mathrm{rev} \end{Bmatrix} = \begin{Bmatrix} \overline{U}_2 \\ \overline{I}_2 \end{Bmatrix} = \begin{Bmatrix} \overline{I}_2 \cdot \overline{Z}_\mathrm{L} \\ \overline{I}_2 \end{Bmatrix} = \overline{I}_2 \cdot \begin{Bmatrix} \overline{Z}_\mathrm{L} \\ 1 \end{Bmatrix} \tag{21.77}$$

where:

$$\overline{U}_2 = \overline{Z}_\mathrm{L} \cdot \overline{I}_2 \tag{21.78}$$

After inverting the matrix of the system of linear equations (21.77), the following is obtained:

$$\begin{Bmatrix} \overline{U}_\mathrm{for} \\ \overline{U}_\mathrm{rev} \end{Bmatrix} = \frac{\overline{I}_2}{2} \cdot \begin{bmatrix} 1 & \overline{Z}_\mathrm{c} \\ 1 & -\overline{Z}_\mathrm{c} \end{bmatrix} \cdot \begin{Bmatrix} \overline{Z}_\mathrm{L} \\ 1 \end{Bmatrix} = \frac{\overline{I}_2}{2} \cdot \begin{Bmatrix} \overline{Z}_\mathrm{L} + \overline{Z}_\mathrm{c} \\ \overline{Z}_\mathrm{L} - \overline{Z}_\mathrm{c} \end{Bmatrix} \tag{21.79}$$

from which it follows that:

$$\overline{U}_\mathrm{for} = \frac{\left(\overline{Z}_\mathrm{L} + \overline{Z}_\mathrm{c}\right) \cdot \overline{I}_2}{2} \quad ; \quad \overline{U}_\mathrm{rev} = \frac{\left(\overline{Z}_\mathrm{L} - \overline{Z}_\mathrm{c}\right) \cdot \overline{I}_2}{2} \tag{21.80}$$

It follows that the reflection coefficient and the transmission coefficient of the electric voltage are given by the following expressions:

$$\bar{\rho}_{\mathrm{u}} = \frac{\bar{U}_{\mathrm{rev}}}{\bar{U}_{\mathrm{for}}} = \frac{\bar{Z}_{\mathrm{L}} - \bar{Z}_{\mathrm{c}}}{\bar{Z}_{\mathrm{L}} + \bar{Z}_{\mathrm{c}}} \quad ; \quad \bar{\tau}_{\mathrm{u}} = 1 + \bar{\rho}_{\mathrm{u}} = \frac{2 \cdot \bar{Z}_{\mathrm{L}}}{\bar{Z}_{\mathrm{L}} + \bar{Z}_{\mathrm{c}}} \tag{21.81}$$

whereas the reflection coefficient and the transmission coefficient of the electric current are given by the following expressions:

$$\bar{\rho}_{\mathrm{i}} = -\bar{\rho}_{\mathrm{u}} = \frac{\bar{Z}_{\mathrm{c}} - \bar{Z}_{\mathrm{L}}}{\bar{Z}_{\mathrm{L}} + \bar{Z}_{\mathrm{c}}} \quad ; \quad \bar{\tau}_{\mathrm{i}} = 1 + \bar{\rho}_{\mathrm{i}} = \frac{2 \cdot \bar{Z}_{\mathrm{c}}}{\bar{Z}_{\mathrm{L}} + \bar{Z}_{\mathrm{c}}} \tag{21.82}$$

In the case where the load impedance is equal to the characteristic impedance of the transmission line:

$$\bar{Z}_{\mathrm{L}} = \bar{Z}_{\mathrm{c}} \tag{21.83}$$

such a load is called the natural load of the transmission line, or the load is said to be matched to the transmission line. In this case, there is no wave reflection, because the following holds:

$$\bar{\rho}_{\mathrm{u}} = \bar{\rho}_{\mathrm{i}} = 0 \quad ; \quad \bar{\tau}_{\mathrm{u}} = \bar{\tau}_{\mathrm{i}} = 1 \tag{21.84}$$

In the case of an open circuit, the load impedance is infinite:

$$\bar{Z}_{\mathrm{L}} \to \infty \tag{21.85}$$

and the following holds:

$$\bar{\rho}_{\mathrm{u}} = 1 \quad ; \quad \bar{\tau}_{\mathrm{u}} = 2 \quad ; \quad \bar{\rho}_{\mathrm{i}} = -1 \quad ; \quad \bar{\tau}_{\mathrm{i}} = 0 \tag{21.86}$$

In the case of a short circuit, the load impedance is equal to zero:

$$\bar{Z}_{\mathrm{L}} = 0 \tag{21.87}$$

and the following holds:

$$\bar{\rho}_{\mathrm{u}} = -1 \quad ; \quad \bar{\tau}_{\mathrm{u}} = 0 \quad ; \quad \bar{\rho}_{\mathrm{i}} = 1 \quad ; \quad \bar{\tau}_{\mathrm{i}} = 2 \tag{21.88}$$

21.7. Measurement of the Characteristic Impedance of a Two-Wire Transmission Line

The characteristic impedance of a two-wire transmission line can be determined from the data obtained through open-circuit and short-circuit tests.

From the solution of the inhomogeneous Helmholtz differential equations, the following system of linear equations (21.65) is obtained:

$$\begin{Bmatrix} \bar{U}_1 \\ \bar{I}_1 \end{Bmatrix} = \begin{bmatrix} \cosh(\bar{\gamma} \cdot \ell) & \bar{Z}_{\mathrm{c}} \cdot \sinh(\bar{\gamma} \cdot \ell) \\ \dfrac{1}{\bar{Z}_{\mathrm{c}}} \cdot \sinh(\bar{\gamma} \cdot \ell) & \cosh(\bar{\gamma} \cdot \ell) \end{bmatrix} \cdot \begin{Bmatrix} \bar{U}_2 \\ \bar{I}_2 \end{Bmatrix} \tag{21.89}$$

where the electric voltage and current at the beginning of the two-wire transmission line are expressed in terms of the electric voltage and current at the end of the line.

In the open-circuit condition $(\bar{I}_2 = 0)$, the electric voltage and current at the beginning of the two-wire transmission line are measured and, according to expression (21.89), are expressed in terms of the electric voltage at the end of the line:

$$\bar{U}_{1,\mathrm{oc}} = \bar{U}_{2,\mathrm{oc}} \cdot \cosh(\bar{\gamma} \cdot \ell) \quad ; \quad \bar{I}_{1,\mathrm{oc}} = \frac{\bar{U}_{2,\mathrm{oc}}}{\bar{Z}_{\mathrm{c}}} \cdot \sinh(\bar{\gamma} \cdot \ell) \tag{21.90}$$

from which it follows that the open-circuit input impedance of the two-wire transmission line is:

$$\bar{Z}_{\mathrm{in,oc}} = \frac{\bar{U}_{1,\mathrm{oc}}}{\bar{I}_{1,\mathrm{oc}}} = \bar{Z}_{\mathrm{c}} \cdot \coth(\bar{\gamma} \cdot \ell) \tag{21.91}$$

In the short-circuit condition $(\overline{U}_2 = 0)$, the electric voltage and current at the beginning of the two-wire transmission line are measured and, according to expression (21.89), are expressed in terms of the electric current at the end of the line:

$$\overline{U}_{1,\text{sc}} = \overline{I}_{2,\text{sc}} \cdot \overline{Z}_\text{c} \cdot \sinh(\overline{\gamma} \cdot \ell) \quad ; \quad \overline{I}_{1,\text{sc}} = \overline{I}_{2,\text{sc}} \cdot \cosh(\overline{\gamma} \cdot \ell) \tag{21.92}$$

from which it follows that the short-circuit input impedance of the two-wire transmission line is:

$$\overline{Z}_{\text{in,sc}} = \frac{\overline{U}_{1,\text{sc}}}{\overline{I}_{1,\text{sc}}} = \overline{Z}_\text{c} \cdot \tanh(\overline{\gamma} \cdot \ell) \tag{21.93}$$

It follows that:

$$\overline{Z}_{\text{in,oc}} \cdot \overline{Z}_{\text{in,sc}} = \overline{Z}_\text{c}^2 \cdot \coth(\overline{\gamma} \cdot \ell) \cdot \tanh(\overline{\gamma} \cdot \ell) = \overline{Z}_\text{c}^2 \tag{21.94}$$

and it is:

$$\overline{Z}_\text{c} = \sqrt{\overline{Z}_{\text{in,oc}} \cdot \overline{Z}_{\text{in,sc}}} = \sqrt{\frac{\overline{U}_{1,\text{oc}} \cdot \overline{U}_{1,\text{sc}}}{\overline{I}_{1,\text{oc}} \cdot \overline{I}_{1,\text{sc}}}} \tag{21.95}$$

Therefore, according to expression (21.95), the characteristic impedance of the two-wire transmission line can be calculated from the measured voltages and currents at the beginning of the line under open-circuit and short-circuit conditions.

21.8. Wave Behavior at the Boundary Between Two Lossless Transmission Lines

Let a wave of arbitrary shape propagate along a homogeneous, lossless two-wire transmission line with a characteristic impedance Z_1, approaching the boundary with another homogeneous, lossless two-wire transmission line with a characteristic impedance Z_2 (Figure 21.11).

The characteristic impedance of a lossless transmission line is independent of frequency, and at the boundary between two lossless two-wire transmission lines, the following equations hold in the time domain:

$$u_\text{i} + u_\text{r} = u_\text{t} \quad ; \quad i_\text{i} + i_\text{r} = i_\text{t} \tag{21.96}$$

where:

u_i - the incident electric voltage,

u_r - the reflected electric voltage,

u_t - the transmitted electric voltage,

i_i - the incident electric current,

i_r - the reflected electric current,

i_t - the transmitted electric current.

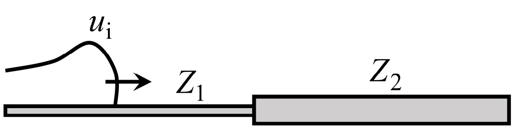


Figure 21.11. Incident wave at the boundary between two lossless transmission lines

The following holds:

$$i_\text{i} = \frac{u_\text{i}}{Z_1} \quad ; \quad i_\text{r} = -\frac{u_\text{r}}{Z_1} \quad ; \quad i_\text{t} = \frac{u_\text{t}}{Z_2} \tag{21.97}$$

and the system of linear equations (21.96) can also be written in the following form:

$$u_\mathrm{i} + u_\mathrm{r} = u_\mathrm{t} \quad ; \quad \frac{u_\mathrm{i}}{Z_1} - \frac{u_\mathrm{r}}{Z_1} = \frac{u_\mathrm{t}}{Z_2} \tag{21.98}$$

from which it follows that:

$$u_\mathrm{r} - u_\mathrm{t} = -\,u_\mathrm{i} \tag{21.99}$$

$$u_\mathrm{r} + \frac{Z_1}{Z_2} \cdot u_\mathrm{t} = u_\mathrm{i} \tag{21.100}$$

If the reflection and transmission coefficients of the electric voltage are to be determined, the reflected and transmitted electric voltages must be expressed in terms of the incident electric voltage. From the system of linear equations (21.99) and (21.100), it follows that:

$$\left(1 + \frac{Z_1}{Z_2}\right) \cdot u_\mathrm{r} = \left(1 - \frac{Z_1}{Z_2}\right) \cdot u_\mathrm{i} \tag{21.101}$$

$$\left(1 + \frac{Z_1}{Z_2}\right) \cdot u_\mathrm{t} = 2 \cdot u_\mathrm{i} \tag{21.102}$$

from which it follows that the reflection and transmission coefficients of the electric voltage are:

$$\rho_\mathrm{u} = \frac{u_\mathrm{r}}{u_\mathrm{i}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} \quad ; \quad \tau_\mathrm{u} = \frac{u_\mathrm{t}}{u_\mathrm{i}} = \frac{2 \cdot Z_2}{Z_1 + Z_2} = 1 + \rho_\mathrm{u} \tag{21.103}$$

whereas the reflection and transmission coefficients of the electric current are:

$$\rho_\mathrm{i} = -\,\rho_\mathrm{u} = \frac{Z_1 - Z_2}{Z_1 + Z_2} \quad ; \quad \tau_\mathrm{i} = 1 + \rho_\mathrm{i} = \frac{2 \cdot Z_1}{Z_1 + Z_2} \tag{21.104}$$

21.9. Petersen's Rule

The reflection and transmission of a wave at the boundary between two lossless two-wire transmission lines, as discussed in the previous subchapter, can also be analyzed using Petersen's rule, for which the equivalent circuit is shown in Figure 21.12.

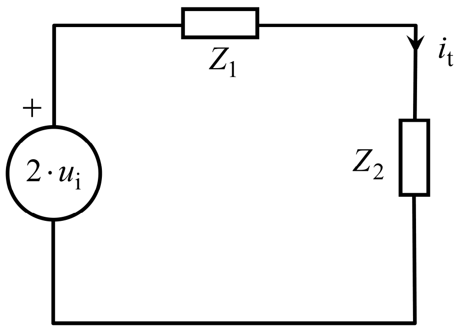


Figure 21.12. Equivalent circuit of Petersen's rule

According to Figure 21.12, the transmitted electric current is given by the following expression:

$$i_\mathrm{t} = \frac{2 \cdot u_\mathrm{i}}{Z_1 + Z_2} \tag{21.105}$$

and according to (21.97), the transmitted electric voltage is given by the following expression:

$$u_t = i_t \cdot Z_2 = \frac{2 \cdot Z_2}{Z_1 + Z_2} \cdot u_i \tag{21.106}$$

from which it follows that the transmission coefficient of the electric voltage is:

$$\tau_u = \frac{u_t}{u_i} = \frac{2 \cdot Z_2}{Z_1 + Z_2} \tag{21.106}$$

After that, the remaining reflection and transmission coefficients can be easely obtained, because:

$$\rho_u = \tau_u - 1 \quad ; \quad \rho_i = -\rho_u = 1 - \tau_u \quad ; \quad \tau_i = 1 + \rho_i = 2 - \tau_u \tag{21.107}$$

21.10. Lumped Resistor Between Two Lossless Two-Wire Transmission Lines

Let a lumped resistance R be inserted between two lossless two-wire transmission lines (Figure 21.13). Let a wave of arbitrary shape propagate along a homogeneous, lossless two-wire transmission line with a characteristic impedance Z_1. The task is to determine the expression for the voltage drop across the resistor R.

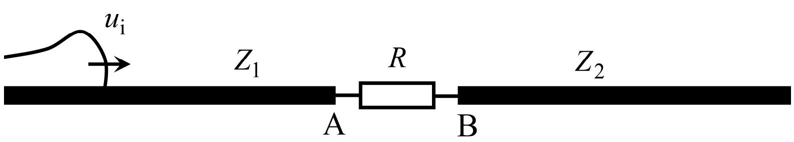


Figure 21.13. Lumped resistor R between two lossless two-wire transmission lines

From expressions (21.103) and (21.104), it follows that the reflection and transmission coefficients of the electric voltage at points A and B are given by the following expressions:

$$\rho_{uA} = \frac{u_{rA}}{u_i} = \frac{(Z_2 + R) - Z_1}{Z_1 + (Z_2 + R)} \tag{21.108}$$

$$\tau_{uA} = \frac{u_{tA}}{u_i} = \frac{2 \cdot (Z_2 + R)}{Z_1 + (Z_2 + R)} \tag{21.109}$$

$$\tau_{uB} = \frac{u_{tB}}{u_i} = \frac{2 \cdot Z_2}{(Z_1 + R) + Z_2} \tag{21.110}$$

which means that, at the point under consideration, the lumped resistance and the characteristic impedance of the transmission line to which the point does not belong are added together.

The electric voltage drop across the resistor R, i.e., the longitudinal electric voltage across resistor R, is given by the following expression:

$$u_R = (\tau_{uA} - \tau_{uB}) \cdot u_i = \frac{2 \cdot R}{Z_1 + Z_2 + R} \cdot u_i \tag{21.111}$$

The same solution can be obtained using Petersen's rule, and in this case, the equivalent circuit of Petersen's rule is shown in Figure 21.14. The following holds:

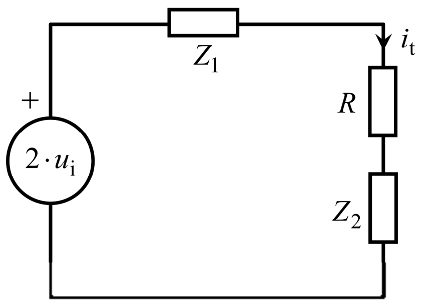
$$i_t = i_{tA} = i_{tB} = \frac{2 \cdot u_t}{Z_1 + (Z_2 + R)} \tag{21.112}$$

and it is:

$$u_R = i_t \cdot R = \frac{2 \cdot R}{Z_1 + Z_2 + R} \cdot u_i \tag{21.113}$$

whereas:

$$u_{tB} = i_t \cdot Z_2 = \frac{2 \cdot Z_2}{Z_1 + Z_2 + R} \cdot u_i \tag{21.114}$$



Slika 21.14. Petersen's equivalent circuit for a lumped resistor between lossless lines

The reflected and transmitted wave electric voltages at node A are:

$$u_{rA} = u_{tA} - u_i \tag{21.115}$$

$$u_{tA} = i_t \cdot (R + Z_2) = \frac{2 \cdot (Z_2 + R)}{Z_1 + (Z_2 + R)} \cdot u_i \tag{21.116}$$

from which it follows that:

$$u_{tA} = \frac{Z_2 - Z_1 + R}{Z_1 + Z_2 + R} \cdot u_i \tag{21.117}$$

21.11. Branching of Lossless Two-Wire Transmission Lines

Using Petersen's rule, it is also possible to analyse the incidence of a wave at a node where lossless transmission lines branch (Figure 21.15), where R_{gr} is the grounding resistance. The corresponding equivalent circuit based on Petersen's rule is shown in Figure 21.16.

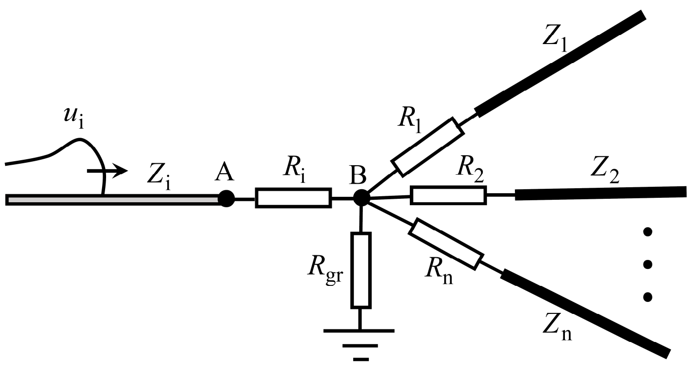


Figure 21.15. Branching of lossless two-wire transmission lines

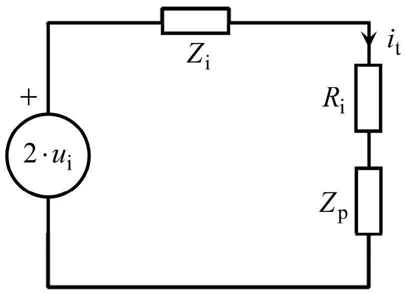


Figure 21.16. Equivalent circuit of Petersen's rule for branching of lossless transmission lines

The impedance Z_p of the parallel-connected lines at point B, including the grounding resistance, is given by the following expression:

$$Z_p = \frac{1}{\dfrac{1}{R_{gr}} + \displaystyle\sum_{k=1}^{n} \dfrac{1}{R_k + Z_k}} \tag{21.118}$$

From the equivalent circuit shown in Figure 21.16, it follows that the transmitted electric current is given by the following expression:

$$i_t = i_{tA} = i_{tB} = \frac{2 \cdot u_i}{Z_i + R_i + Z_p} \tag{21.119}$$

and the electric voltage drop across the resistor R_i is given by the following expression:

$$u_{R_i} = i_t \cdot R_i = \frac{2 \cdot R_i}{Z_i + R_i + Z_p} \cdot u_i \tag{21.120}$$

whereas the electric voltage at point B is:

$$u_B = u_{tB} = i_t \cdot Z_p = \frac{2 \cdot Z_p}{Z_i + R_i + Z_p} \cdot u_i \tag{21.121}$$

The reflected and transmitted wave electric voltages at node A are:

$$u_{rA} = u_{tA} - u_i \tag{21.122}$$

$$u_A = u_{tA} = i_t \cdot \left(R_i + Z_p\right) = \frac{2 \cdot \left(R_i + Z_p\right)}{Z_i + R_i + Z_p} \cdot u_i \tag{21.123}$$

from which it follows that:

$$u_{rA} = \frac{\left(Z_p + R_i\right) - Z_i}{Z_i + \left(R_i + Z_p\right)} \cdot u_i \tag{21.124}$$

21.12. Series Connection of Two-Wire Transmission Line Segments

In the phasor domain, segments of a two-wire transmission line can be modelled as quadripoles (also known as four-terminal or two-port networks). Before the development of advanced numerical methods, quadripole theory played an important role in the analysis of electric power transmission lines.

Two series-connected quadripoles are shown in Figure 21.17, and their simplified representation is given in Figure 21.18.

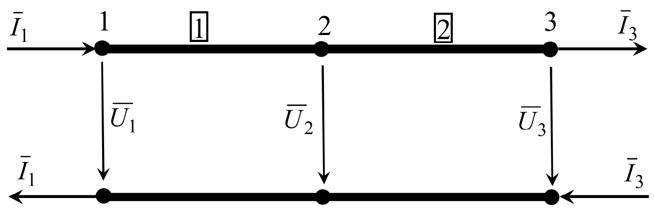


Figure 21.17. Two series-connected quadripoles

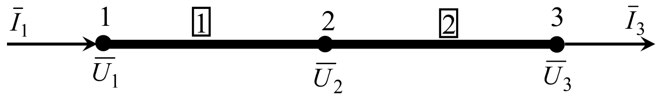


Figure 21.18. Simplified representation of two series-connected quadripoles

According to the expression (21.65), the following expression holds for the first quadripole:

$$\begin{Bmatrix} \overline{U}_1 \\ \bar{I}_1 \end{Bmatrix} = \begin{bmatrix} \cosh\left(\bar{\gamma}_1 \cdot \ell_1\right) & \overline{Z}_{c1} \cdot \sinh\left(\bar{\gamma}_1 \cdot \ell_1\right) \\ \frac{1}{\overline{Z}_{c1}} \cdot \sinh\left(\bar{\gamma}_1 \cdot \ell_1\right) & \cosh\left(\bar{\gamma}_1 \cdot \ell_1\right) \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_2 \\ \bar{I}_2 \end{Bmatrix} = \left[\overline{T}_1\right] \cdot \begin{Bmatrix} \overline{U}_2 \\ \bar{I}_2 \end{Bmatrix} \tag{21.125}$$

where:

$$\left[\overline{T}_1\right] = \begin{bmatrix} \cosh\left(\bar{\gamma}_1 \cdot \ell_1\right) & \overline{Z}_{c1} \cdot \sinh\left(\bar{\gamma}_1 \cdot \ell_1\right) \\ \frac{1}{\overline{Z}_{c1}} \cdot \sinh\left(\bar{\gamma}_1 \cdot \ell_1\right) & \cosh\left(\bar{\gamma}_1 \cdot \ell_1\right) \end{bmatrix} \tag{21.126}$$

is the transfer matrix of the first quadripole, i.e., the transfer matrix of the first segment of the two-wire transmission line.

According to the expression (21.65), the following expression holds for the second quadripole:

$$\begin{Bmatrix} \overline{U}_2 \\ \bar{I}_2 \end{Bmatrix} = \begin{bmatrix} \cosh\left(\bar{\gamma}_2 \cdot \ell_2\right) & \overline{Z}_{c2} \cdot \sinh\left(\bar{\gamma}_2 \cdot \ell_2\right) \\ \frac{1}{\overline{Z}_{c2}} \cdot \sinh\left(\bar{\gamma}_2 \cdot \ell_2\right) & \cosh\left(\bar{\gamma}_2 \cdot \ell_2\right) \end{bmatrix} \cdot \begin{Bmatrix} \overline{U}_3 \\ \bar{I}_3 \end{Bmatrix} = \left[\overline{T}_2\right] \cdot \begin{Bmatrix} \overline{U}_3 \\ \bar{I}_3 \end{Bmatrix} \tag{21.127}$$

where:

$$\left[\overline{T}_2\right] = \begin{bmatrix} \cosh\left(\bar{\gamma}_2 \cdot \ell_2\right) & \overline{Z}_{c2} \cdot \sinh\left(\bar{\gamma}_2 \cdot \ell_2\right) \\ \frac{1}{\overline{Z}_{c2}} \cdot \sinh\left(\bar{\gamma}_2 \cdot \ell_2\right) & \cosh\left(\bar{\gamma}_2 \cdot \ell_2\right) \end{bmatrix} \tag{21.128}$$

is the transfer matrix of the second quadripole.

Since electric voltage and current are continuous at the junction of the two quadripoles, the following expression holds for the two series-connected quadripoles:

$$\begin{Bmatrix} \overline{U}_1 \\ \bar{I}_1 \end{Bmatrix} = \left[\overline{T}_1\right] \cdot \left[\overline{T}_2\right] \cdot \begin{Bmatrix} \overline{U}_3 \\ \bar{I}_3 \end{Bmatrix} = \left[\overline{T}\right] \cdot \begin{Bmatrix} \overline{U}_3 \\ \bar{I}_3 \end{Bmatrix} \tag{21.129}$$

If the two-wire transmission line consists of *n* series-connected quadripoles, then the following expression holds:

$$\begin{Bmatrix} \overline{U}_{\mathrm{i}} \\ \overline{I}_{\mathrm{i}} \end{Bmatrix} = \left[\overline{T}_{\mathrm{i}}\right] \cdot \begin{Bmatrix} \overline{U}_{\mathrm{i+1}} \\ \overline{I}_{\mathrm{i+1}} \end{Bmatrix} \quad ; \quad \mathrm{i} = 1, 2, ..., \mathrm{n} \tag{21.130}$$

where the transfer matrix of the i-th quadripole is given by the following expression:

$$\left[\overline{T}_{\mathrm{i}}\right] = \begin{bmatrix} \cosh\left(\overline{\gamma}_{\mathrm{i}} \cdot \ell_{\mathrm{i}}\right) & \overline{Z}_{\mathrm{ci}} \cdot \sinh\left(\overline{\gamma}_{\mathrm{i}} \cdot \ell_{\mathrm{i}}\right) \\ \dfrac{1}{\overline{Z}_{\mathrm{ci}}} \cdot \sinh\left(\overline{\gamma}_{\mathrm{i}} \cdot \ell_{\mathrm{i}}\right) & \cosh\left(\overline{\gamma}_{\mathrm{i}} \cdot \ell_{\mathrm{i}}\right) \end{bmatrix} \tag{21.131}$$

It follows that the electric voltage and current at the beginning of the entire two-wire transmission line can be expressed in terms of the electric voltage and current at the end of the line as follows:

$$\begin{Bmatrix} \overline{U}_1 \\ \overline{I}_1 \end{Bmatrix} = \left[\overline{T}_1\right] \cdot \left[\overline{T}_2\right] \cdots \left[\overline{T}_{\mathrm{n}}\right] \cdot \begin{Bmatrix} \overline{U}_{\mathrm{n+1}} \\ \overline{I}_{\mathrm{n+1}} \end{Bmatrix} = \left[\overline{T}\right] \cdot \begin{Bmatrix} \overline{U}_{\mathrm{n+1}} \\ \overline{I}_{\mathrm{n+1}} \end{Bmatrix} \tag{21.132}$$

where, therefore, the transfer matrix of the entire line $\left[\overline{T}\right]$ is given by the following expression:

$$\left[\overline{T}\right] = \left[\overline{T}_1\right] \cdot \left[\overline{T}_2\right] \cdots \left[\overline{T}_{\mathrm{n}}\right] = \prod_{\mathrm{i}=1}^{\mathrm{n}} \left[\overline{T}_{\mathrm{i}}\right] \tag{21.133}$$

21.13. Solved Examples

Example 21.1. A lossless overhead two-wire electric power transmission line with a characteristic impedance of $Z_1 = 400\ \Omega$ transitions into a lossless cable transmission line with a characteristic impedance of $Z_2 = 50\ \Omega$. Determine the voltage and current reflection and transmission coefficients at the boundary between the overhead line and the cable transmission line.

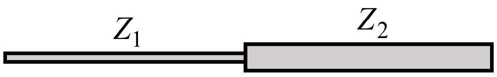


Figure 21.19. Transition from an overhead line to a cable line

Solution:

According to subchapter 21.8, starting from the system of two linear equations:

$$u_{\mathrm{i}} + u_{\mathrm{r}} = u_{\mathrm{t}} \quad ; \quad \frac{u_{\mathrm{i}}}{Z_1} - \frac{u_{\mathrm{r}}}{Z_1} = \frac{u_{\mathrm{t}}}{Z_2} \tag{21.134}$$

the expressions for the voltage reflection and transmission coefficients can be easily obtained:

$$\rho_{\mathrm{u}} = \frac{u_{\mathrm{r}}}{u_{\mathrm{i}}} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = -\frac{7}{9} \quad ; \quad \tau_{\mathrm{u}} = \frac{u_{\mathrm{t}}}{u_{\mathrm{i}}} = \frac{2 \cdot Z_2}{Z_1 + Z_2} = 1 + \rho_{\mathrm{u}} = \frac{2}{9} \tag{21.135}$$

whereas the current reflection and transmission coefficients are:

$$\rho_{\mathrm{i}} = -\rho_{\mathrm{u}} = \frac{Z_1 - Z_2}{Z_1 + Z_2} = \frac{7}{9} \quad ; \quad \tau_{\mathrm{i}} = 1 + \rho_{\mathrm{i}} = \frac{2 \cdot Z_1}{Z_1 + Z_2} = \frac{16}{9} \tag{21.136}$$

The same solution can also be obtained using Petersen's rule, as described in subchapter 21.9.

Example 21.2. A lossless two-wire electric power transmission line consists of two sections: an overhead transmission line ($L_{ov} = 3.48$ mH/km; $C_{ov} = 0.0219$ µF/km) and a power cable transmission line ($L_{ca} = 0.696$ mH/km; $C_{ca} = 0.438$ µF/km). A rectangular electric voltage surge wave with an amplitude of 17.3 kV impinges from the overhead line side. Determine the transmitted and reflected components of electric voltage and electric current at the transition point between the overhead transmission line and the cable transmission line.

Solution:

If the losses of the two-wire line are neglected, the line's characteristic impedance is described by the expression:

$$\overline{Z} = Z = \sqrt{\frac{L}{C}} \tag{21.137}$$

where L and C are the per-unit-length inductance and capacitance of the line, respectively.

It follows that the characteristic impedance of the overhead transmission line is:

$$Z_1 = Z_{ov} = \sqrt{\frac{L_{ov}}{C_{ov}}} = \sqrt{\frac{3.48\times10^{-3}}{0.0219\times10^{-6}}} = 398.6277833\ \Omega \tag{21.138}$$

whereas the characteristic impedance of the cable transmission line is:

$$Z_2 = Z_{ca} = \sqrt{\frac{L_{ca}}{C_{ca}}} = \sqrt{\frac{0.696\times10^{-3}}{0.438\times10^{-6}}} = 39.86277833\ \Omega \tag{21.139}$$

The electric voltage reflection and transmission coefficients are:

$$\rho_u = \frac{u_r}{u_i} = \frac{Z_2 - Z_1}{Z_1 + Z_2} = -\frac{9}{11} \quad ; \quad \tau_u = \frac{u_t}{u_i} = \frac{2\cdot Z_2}{Z_1 + Z_2} = 1 + \rho_u = \frac{2}{11} \tag{21.140}$$

and the reflected component of the electric voltage is:

$$u_r = \rho_u \cdot \mathrm{u_i} = -14.15454545\ \text{kV} \tag{21.141}$$

whereas the transmitted component of the electric voltage is:

$$u_t = \tau_u \cdot \mathrm{u_i} = 3.145454545\ \text{kV} \tag{21.142}$$

The incident electric current is:

$$i_i = \frac{u_i}{Z_1} = 43.39888168\ \text{A} \tag{21.143}$$

whereas the reflected component of the electric current is:

$$i_r = -\frac{u_r}{Z_1} = 35.50817592\ \text{A} \tag{21.144}$$

and the transmitted component of the electric current is:

$$i_t = \frac{u_t}{Z_2} = 78.9070576\ \text{A} \tag{21.145}$$

Example 21.3. A lossless two-wire electric power transmission line consists of two sections: an isolated overhead transmission line ($r_0 = 5$ mm; $d = 1$ m; $r_0 << d$) and a power cable transmission line ($r_1 = 1$ cm; $r_2 = 2$ cm; $\varepsilon_r = 2.1$; $\mu_r = 1$). A rectangular voltage surge with an amplitude of 17.5 kV impinges from the overhead line side. Calculate the electric voltage at the transition point between the overhead and cable transmission lines at the moment the wave reaches that point. When calculating the per-unit-length inductance and capacitance, assume that both the overhead line and the cable line are infinitely long. Neglect the internal inductance of the conductors and the influence of the ground on the characteristics of the overhead line.

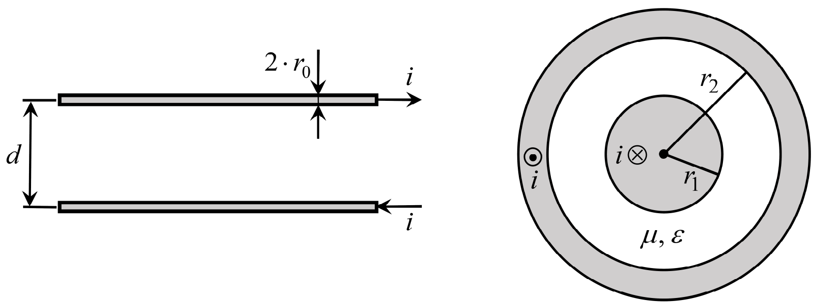


Figure 21.20. Parameters of the two-wire overhead line and the two-wire cable line

Solution:

For an isolated two-wire overhead electric power transmission line, with the aforementioned simplifications, the following expressions for the per-unit-length (external) inductance and capacitance are easily obtained:

$$L_{ov} = \frac{\mu_0}{\pi} \cdot \ln\frac{d}{r_0} \quad ; \quad C_{ov} = \frac{\pi \cdot \varepsilon_0}{\ln\frac{d}{r_0}} \tag{21.146}$$

whereas for the two-wire cable transmission line, with the aforementioned simplifications, the per-unit-length inductance and capacitance are given by:

$$L_{ca} = \frac{\mu_0}{2 \cdot \pi} \cdot \ln\frac{r_2}{r_1} \quad ; \quad C_{ca} = \frac{2 \cdot \pi \cdot \varepsilon_0 \cdot \varepsilon_r}{\ln\frac{r_2}{r_1}} \tag{21.147}$$

It follows that the characteristic impedance of the overhead transmission line is:

$$Z_1 = Z_{ov} = \sqrt{\frac{L_{ov}}{C_{ov}}} = \frac{1}{\pi} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0}} \cdot \ln\frac{d}{r_0} = 635.358235\ \Omega \tag{21.148}$$

whereas the characteristic impedance of the cable transmission line is:

$$Z_2 = Z_{ca} = \sqrt{\frac{L_{ca}}{C_{ca}}} = \frac{1}{2 \cdot \pi} \cdot \sqrt{\frac{\mu_0}{\varepsilon_0 \cdot \varepsilon_r}} \cdot \ln\frac{r_2}{r_1} = 28.67916565\ \Omega \tag{21.149}$$

The electric voltage at the transition point between the overhead and cable transmission lines, at the moment the wave reaches that point, is equal to the transmitted voltage:

$$u_t = \frac{2 \cdot Z_2}{Z_1 + Z_2} \cdot u_i = 1.511617865\ \text{kV} \tag{21.150}$$

Example 21.4. A lossless power transmission line with a characteristic impedance of Z = 400 Ω is interrupted, and a lumped resistor of R = 100 Ω is inserted at the point of discontinuity. Determine the voltage drop across the resistor, the transmitted voltage, and the reflected voltage if a rectangular voltage surge of $u_{\mathrm{i}} = U_0 = 17\ \mathrm{kV}$ impinges on the discontinuity point.

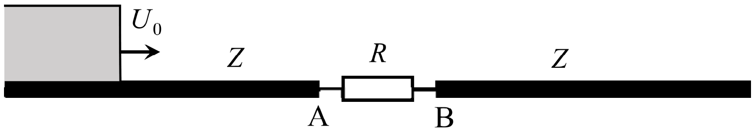


Figure 21.21. Incidence of a voltage surge at the discontinuity point

Solution:

This example can be solved using Petersen's rule (Figure 21.22).

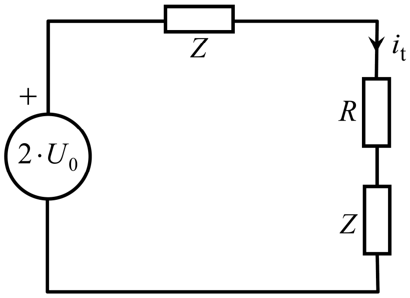


Figure 21.22. Equivalent circuit for Petersen's rule

According to Petersen's rule, the transmitted electric current is given by the following expression:

$$i_{\mathrm{t}} = i_{\mathrm{tA}} = i_{\mathrm{tB}} = \frac{2 \cdot U_0}{2 \cdot Z + R} \tag{21.151}$$

whereas the electric voltage drop across the resistor is:

$$u_{\mathrm{R}} = i_{\mathrm{t}} \cdot R = \frac{2 \cdot R}{2 \cdot Z + R} \cdot U_0 = 3.7777777778\ \mathrm{kV} \tag{21.152}$$

Furthermore, the transmitted electric voltage at point B is:

$$u_{\mathrm{t}} = u_{\mathrm{tB}} = i_{\mathrm{t}} \cdot Z = \frac{2 \cdot Z}{2 \cdot Z + R} \cdot U_0 = 15.1111111111\ \mathrm{kV} \tag{21.153}$$

whereas the reflected electric voltage at point A is:

$$u_{\mathrm{r}} = u_{\mathrm{rA}} = u_{\mathrm{R}} + u_{\mathrm{tB}} - U_0 = \frac{R}{2 \cdot Z + R} \cdot U_0 = \frac{u_{\mathrm{R}}}{2} = 1.8888888889\ \mathrm{kV} \tag{21.154}$$

Example 21.5. A perfect inductor with inductance L is inserted between two lossless power transmission lines. Determine the transmitted voltage at point B, the voltage drop across the inductor, and the reflected voltage at point A as functions of time, assuming that at time t = 0, a rectangular voltage surge $u_{\mathrm{i}} = U_0$ impinges on the inductor.

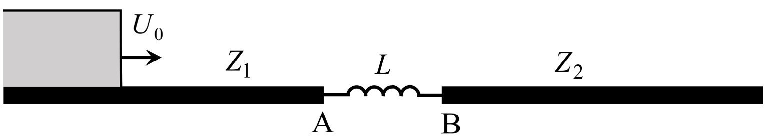


Figure 21.23. Incidence of a voltage surge at the discontinuity point

Solution:

This example can be solved using Petersen's rule (Figure 21.24).

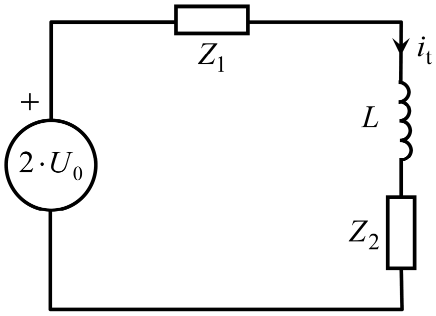


Figure 21.24. Equivalent circuit for Petersen's rule

According to Petersen's rule, the following holds:

$$2 \cdot U_0 = i_t \cdot (Z_1 + Z_2) + L \cdot \frac{di_t}{dt} \tag{21.155}$$

from which the following differential equation is obtained:

$$T \cdot \frac{di_t}{dt} + i_t = \frac{2 \cdot U_0}{Z_1 + Z_2} \quad ; \quad T = \frac{L}{Z_1 + Z_2} \tag{21.156}$$

where T is the time constant.

The differential equation (21.156) can be solved using the method of separation of variables, by first rewriting it in the following form:

$$\frac{di_t}{i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}} = -\frac{dt}{T} \tag{21.157}$$

then both sides of expression (21.157) are integrated. After integration, the result is:

$$\ln K + \ln\left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right) = -\frac{t}{T} \tag{21.158}$$

or, written differently:

$$\ln\left[K \cdot \left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right)\right] = -\frac{t}{T} \tag{21.159}$$

It follows that:

$$K \cdot \left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right) = e^{-t/T} \tag{21.160}$$

$$i_t = \frac{2 \cdot U_0}{Z_1 + Z_2} + \frac{1}{K} \cdot e^{-t/T} \tag{21.161}$$

Expression (21.161) must satisfy the given initial condition:

$$i_t\big|_{t=0} = 0 \tag{21.162}$$

from which it follows that:

$$\frac{1}{K} = -\frac{2 \cdot U_0}{Z_1 + Z_2} \tag{21.163}$$

and the final expression describing the transmitted electric current is:

$$i_t = i_{tA} = i_{tB} = \frac{2 \cdot U_0}{Z_1 + Z_2} \cdot \left(1 - e^{-t/T}\right) \tag{21.164}$$

The transmitted electric voltage at point B is described by the following expression:

$$u_t = u_{tB} = i_t \cdot Z_2 = \frac{2 \cdot U_0 \cdot Z_2}{Z_1 + Z_2} \cdot \left(1 - e^{-t/T}\right) \tag{21.165}$$

The transverse voltage drop across the inductor (lumped inductance) is described by the following expression:

$$u_L = L \cdot \frac{d i_t}{dt} = 2 \cdot U_0 \cdot e^{-t/T} \tag{21.166}$$

The reflected electric voltage at point A can be obtained from the following expression:

$$u_{rA} = u_{tA} - U_0 = u_L + u_{tB} - U_0 \tag{21.167}$$

and from the three previous expressions, it follows that:

$$u_{rA} = U_0 \cdot \frac{Z_2 - Z_1}{Z_1 + Z_2} + U_0 \cdot \frac{2 \cdot Z_1}{Z_1 + Z_2} \cdot \mathrm{e}^{-t/T} \tag{21.168}$$

In the special case when $Z_1 = Z_2 = Z,$ the following holds:

$$i_t = i_{tA} = i_{tB} = \frac{U_0}{Z} \cdot \left(1 - e^{-t/T}\right) \quad ; \quad T = \frac{L}{2 \cdot Z} \tag{21.169}$$

$$u_t = u_{tB} = i_t \cdot Z = U_0 \cdot \left(1 - e^{-t/T}\right) \tag{21.170}$$

$$u_L = 2 \cdot U_0 \cdot e^{-t/T} \tag{21.171}$$

$$u_{rA} = U_0 \cdot \mathrm{e}^{-t/T} \tag{21.172}$$

Example 21.6. A perfect capacitor with capacitance *C* is connected in parallel between two lossless power transmission lines. Determine the expressions for the capacitor voltage, the transmitted voltage, the reflected voltage, the transmitted current, the reflected current, and the capacitor current as functions of time, assuming that at time *t* =0, a rectangular voltage surge $u_i = U_0$ impinges on point A. Assume that the capacitor is initially uncharged at the beginning of the transient.

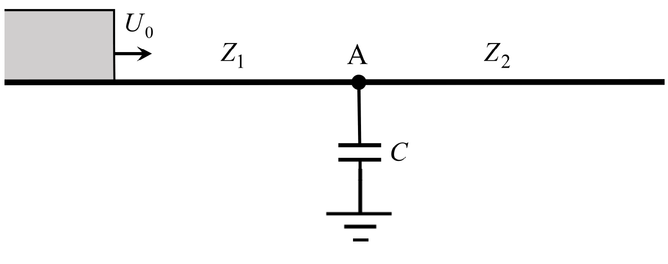


Figure 21.25. Incidence of a voltage surge at the discontinuity point

Solution:

This example can be solved using Petersen's rule (Figure 21.26).

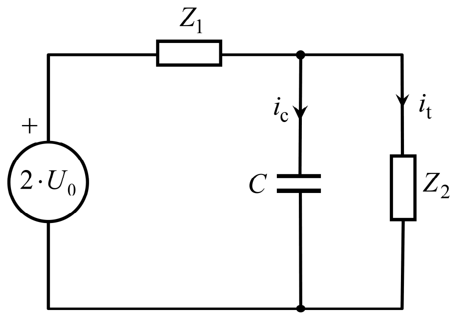


Figure 21.26. Equivalent circuit for Petersen's rule

According to Petersen's rule, the following holds:

$$2 \cdot U_0 = Z_1 \cdot (i_t + i_c) + Z_2 \cdot i_t \tag{21.173}$$

The voltage across the capacitor is equal to the transmitted voltage at point A, and it holds that:

$$u_c = u_t = \frac{1}{C} \cdot \int_0^t i_c \cdot dt = i_t \cdot Z_2 \tag{21.174}$$

from which it follows that:

$$i_c = C \cdot Z_2 \cdot \frac{di_t}{dt} \tag{21.175}$$

If expression (21.175) is substituted into expression (21.173), the following differential equation is obtained:

$$T \cdot \frac{di_t}{dt} + i_t = \frac{2 \cdot U_0}{Z_1 + Z_2} \quad ; \quad T = \frac{C \cdot Z_1 \cdot Z_2}{Z_1 + Z_2} \tag{21.176}$$

where T is the time constant.

The differential equation (21.176) can be solved using the method of separation of variables, by first rewriting it in the following form:

$$\frac{di_t}{i_t - \dfrac{2 \cdot U_0}{Z_1 + Z_2}} = -\frac{dt}{T} \tag{21.177}$$

then both sides of expression (21.177) are integrated. After integration, the result is:

$$\ln K + \ln\left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right) = -\frac{t}{T} \tag{21.178}$$

or, written differently:

$$\ln\left[K \cdot \left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right)\right] = -\frac{t}{T} \tag{21.179}$$

It follows that:

$$K \cdot \left(i_t - \frac{2 \cdot U_0}{Z_1 + Z_2}\right) = e^{-t/T} \tag{21.180}$$

$$i_t = \frac{2 \cdot U_0}{Z_1 + Z_2} + \frac{1}{K} \cdot e^{-t/T} \tag{21.181}$$

Expression (21.181) must satisfy the given initial condition:

$$i_{\mathrm{t}}\Big|_{t=0} = 0 \tag{21.182}$$

from which it follows that:

$$\frac{1}{K} = -\frac{2 \cdot U_0}{Z_1 + Z_2} \tag{21.183}$$

and the final expression describing the transmitted electric current is:

$$i_{\mathrm{t}} = \frac{2 \cdot U_0}{Z_1 + Z_2} \cdot \left(1 - e^{-t/T}\right) \tag{21.184}$$

The voltage across the capacitor and the transmitted voltage at point A are described by the following expression:

$$u_{\mathrm{c}} = u_{\mathrm{t}} = i_{\mathrm{t}} \cdot Z_2 = \frac{2 \cdot U_0 \cdot Z_2}{Z_1 + Z_2} \cdot \left(1 - e^{-t/T}\right) \tag{21.185}$$

whereas the reflected electric voltage at point A is described by the following expression:

$$u_{\mathrm{r}} = u_{\mathrm{t}} - U_0 \tag{21.186}$$

and from the previous two expressions, it follows that:

$$u_{\mathrm{r}} = U_0 \cdot \frac{Z_2 - Z_1}{Z_1 + Z_2} - U_0 \cdot \frac{2 \cdot Z_2}{Z_1 + Z_2} \cdot \mathrm{e}^{-t/T} \tag{21.187}$$

From expressions (21.175) and (21.184), it follows that the electric current through the capacitor is:

$$i_{\mathrm{c}} = C \cdot Z_2 \cdot \frac{di_{\mathrm{t}}}{dt} = U_0 \cdot \frac{2 \cdot C \cdot Z_2}{Z_1 + Z_2} \cdot \frac{1}{T} \cdot e^{-t/T} = \frac{2 \cdot U_0}{Z_1} \cdot e^{-t/T} \tag{21.188}$$

The reflected electric current is described by the following expression:

$$i_{\mathrm{r}} = -\frac{u_{\mathrm{r}}}{Z_1} = \frac{U_0}{Z_1} \cdot \frac{Z_1 - Z_2}{Z_1 + Z_2} + \frac{U_0}{Z_1} \cdot \frac{2 \cdot Z_2}{Z_1 + Z_2} \cdot \mathrm{e}^{-t/T} \tag{21.189}$$

whereas the incident electric current is described by the following expression:

$$i_{\mathrm{i}} = \frac{u_{\mathrm{i}}}{Z_1} = \frac{U_0}{Z_1} \tag{21.190}$$

In the special case when $Z_1 = Z_2 = Z,$ the following holds:

$$i_{\mathrm{t}} = \frac{U_0}{Z} \cdot \left(1 - e^{-t/T}\right) \quad ; \quad T = \frac{C \cdot Z}{2} \tag{21.191}$$

$$u_{\mathrm{c}} = u_{\mathrm{t}} = i_{\mathrm{t}} \cdot Z = U_0 \cdot \left(1 - e^{-t/T}\right) \tag{21.192}$$

$$u_{\mathrm{r}} = -U_0 \cdot \mathrm{e}^{-t/T} \tag{21.193}$$

$$i_{\mathrm{c}} = \frac{2 \cdot U_0}{Z} \cdot e^{-t/T} \quad ; \quad i_{\mathrm{r}} = \frac{U_0}{Z} \cdot \mathrm{e}^{-t/T} \quad ; \quad i_{\mathrm{i}} = \frac{U_0}{Z} \tag{21.194}$$

BIBLIOGRAPHY

[1] Assis, A. K. T.; Hernandes, J. A., *The Electric Force of a Current - Weber and the Surface Charges of Resistive Conductors Carrying Steady Currents*, Apeiron, Montreal, 2007.

[2] Barkeshli, K., *Advanced Electromagnetics and Scattering Theory*, Springer, New York, 2015.

[3] Balanis, C. A., *Advanced Engineering Electromagnetics* (2nd ed.), John Wiley & Sons, Hoboken, 2012.

[4] Bansal, R. (Ed.), *Fundamentals of Engineering Electromagnetics*, CRC Press, Boca Raton, 2006.

[5] Berberović, S., *Teorijska elektrotehnika – Odabrani primjeri* (Theoretical Electrical Engineering – Selected Examples), Graphis, Zagreb, 1998.

[6] Bosanac, T., *Teoretska elektrotehnika 1* (Theoretical Electrical Engineering 1), Tehnička knjiga, Zagreb, 1973.

[7] Bronštejn, I. N.; Semendjajev, K. A., *Matematički priručnik za inženjere i studente* (Mathematical Handbook for Engineers and Students), Tehnička knjiga, Zagreb, 1991.

[8] Edminister, P. A.; Nahvi, M., *Electromagnetics* (4th ed.), Schaum's Outline Series, McGraw-Hill, New York, 2014.

[9] Harrington, R. F., *Time-Harmonic Electromagnetic Fields*, IEEE Press / John Wiley & Sons, New York, 2001.

[10] Haus, H. A.; Melcher, J. R., *Electromagnetic Fields and Energy*, Prentice Hall, Englewood Cliffs, New Jersey, 1989.

[11] Haznadar, Z.; Štih, Ž., *Electromagnetic Fields, Waves and Numerical Methods*, IOS Press, Amsterdam, 2000.

[12] Haznadar, Z.; Štih, Ž., *Elektromagnetizam – Svezak 1 – Elektromagnetska teorija, statička i kvazistatička polja* (Electromagnetism – Volume 1 – Electromagnetic Theory, Static and Quasistatic Fields), Školska knjiga, Zagreb, 1997.

[13] Haznadar, Z.; Štih, Ž., *Elektromagnetizam – Svezak 2 – Elektromagnetski valovi i numeričke metode* (Electromagnetism – Volume 2 – Electromagnetic Waves and Numerical Methods), Školska knjiga, Zagreb, 1997.

[14] Ida, N.; Bastos, J. P. A., *Electromagnetics and Calculation of Fields,* Springer-Verlag, New York, 1992.

[15] Jackson, J. D., *Classical Electrodynamics,* John Wiley & Sons, New York, 1962.

[16] Koch, W., *Erdungen in Wechselstromanlagen über 1 kV - Berechnung und Ausführung* (Grounding in AC Installations Above 1 kV – Calculation and Implementation), Springer-Verlag, Berlin, 1955.

[17] Kurtović, M., *Predavanja iz Teorijske elektrotehnike* (Lectures in Theoretical Electrical Engineering), Unreviewed internal script, University of Split, FESB, Split, 2004.

[18] Lovrić, D.; Vujević, S., Accurate Computation of Internal Impedance of Two-Layer Cylindrical Conductors for Arguments of Arbitrary Magnitude, *IEEE Transactions on Electromagnetic Compatibility*, Vol. 60, No. 2, pp. 347-353, 2018.

[19] Lovrić, D.; Vujević, S.; Krolo, I., Computation of Per-Unit-Length Internal Impedance of a Multilayer Cylindrical Conductor with Possible Dielectric Layers, *Facta Universitatis, Series: Electronics and Energetics*, Vol. 33, No. 4, pp. 605-616, 2020.

[20] Magnuson, P. C.; Alexander, G. C.; Tripathi, V. K.; Weisshaar, A., *Transmission Lines and Wave Propagation* (4th ed.), CRC Press, Boca Raton, 2001.

[21] Maksimović, N. G.; Batranin, F. E., *Zbirka zadataka iz teoretske elektrotehnike* (Problem Collection in Theoretical Electrical Engineering), University of Zagreb, ETF, Zagreb, 1974.

[22] Matjan, J., *Auditorne vježbe iz teoretske elektrotehnike* (Classroom Exercises in Theoretical Electrical Engineering), University of Zagreb, ETF, Zagreb, 1975.

[23] Moore, J.; Piezer, R., *Moment Methods in Electromagnetics – Techniques and Applications*, Research Studies Press Ltd., Letchworth, 1984.

[24] Nojman, L. R.; Kalantarov, P. L., *Teoriski osnovi elektrotehnike – Knjiga prva – Fizički osnovi elektrotehnike* (Theoretical Foundations of Electrical Engineering – Volume 1 – Physical Fundamentals), Naučna knjiga, Belgrade, 1951.

[25] Nojman, L. R.; Kalantarov, P. L., *Teoriski osnovi elektrotehnike – Knjiga druga – Teorija naizmeničnih struja* (Theoretical Foundations of Electrical Engineering – Volume 2 – AC Theory), Naučna knjiga, Belgrade, 1958.

[26] Nojman, L. R.; Kalantarov, P. L., *Teoriski osnovi elektrotehnike – Knjiga treća – Teorija elektromagnetskog polja* (Theoretical Foundations of Electrical Engineering – Volume 3 – Electromagnetic Field Theory), Naučna knjiga, Belgrade, 1952.

[27] Ollendorf, F., *Erdströme – Zweite, neubearbeitete Auflage* (Earth Currents – Second, Revised Edition), Verlag, Basel, 1969.

[28] Panofsky, W. K. H.; Phillips, M., *Classical Electricity and Magnetism* (2nd ed.), Addison-Wesley Publishing Company, London, 1962.

[29] Patra, H. P.; Mallick, K., *Geosounding Principles, 2 – Time-Varying Geoelectric Sounding*, Elsevier, Amsterdam, 1980.

[30] Poljak, D., *Advanced Modeling in Computational Electromagnetic Compatibility*, JohnWiley & Sons, New York, 2007.

[31] Popović, B. D., *Elektromagnetika* (Electromagnetics), Građevinska knjiga, Belgrade, 1980.

[32] Popović, Z.; Popović, B. D., *Introductory Electromagnetics*, Prentice Hall, Upper Saddle River, New Jersey, 1999.

[33] Spiegel, M. R., *Vector Analysis and An Introduction to Tensor Analysis,* Schaum's Outline Series, McGraw-Hill, New York, 1959.

[34] Straton, J. A., *Electromagnetic Theory,* McGraw-Hill, New York, 1941.

[35] Sunde, E. D., *Earth Conduction Effects in Transmission Systems,* D. Van Nostrand Company, Toronto, 1949.

[36] Surutka, J., *Elektromagnetika* – 7th edition (Electromagnetics), Grafička knjiga, Belgrade, 1989.

[37] Van Bladel, J. G., *Electromagnetic Fields* (2nd ed.), IEEE Press / John Wiley & Sons, Hoboken, New Jersey, 2007.

[38] Veličković, D. M., *Elektromagnetika – Prva sveska* – 3rd edition (Electromagnetics – Volume 1), Edicija: Osnovni udžbenici, University of Niš, Faculty of Electronic Engineering, Niš, 2004.

[39] Vujević, S.; Boras, V.; Sarajčev, P., A Novel Algorithm for Internal Impedance Computation of Solid and Tubular Cylindrical Conductors, *International Review of Electrical Engineering (IREE) – Part B*, Vol. 4, No. 6, pp. 1418-1425, 2009.

[40] Vujević, S.; Modrić, T.; Lovrić, D., The Difference between Voltage and Potential Difference, *Proceedings of the Joint INDS'11 & ISTET'11*, pp. 137-143, Klagenfurt, Austria, 2011.

[41] Vujević, S.; Balaž, Z.; Lovrić, D., Ground Impedance of Cylindrical Metal Plate Buried in Homogeneous Earth, *Proceedings of the Joint INDS'11 & ISTET'11*, pp. 197-203, Klagenfurt, Austria, 2011.

[42] Vujević, S.; Lovrić, D.; Modrić, T., Scalar Electric Potential in the Air Due To a Direct Current Point Source Buried in the Earth, *Proceedings of the XVII-th International Symposium on Electrical Apparatus and Technologies SIELA 2012*, *Volume I*, pp. 344-351, Burgas, Bulgaria, 2012.

[43] Vujević, S.; Lovrić, D.; Boras, V., High-Accurate Numerical Computation of Internal Impedance of Cylindrical Conductors for Complex Arguments of Arbitrary Magnitude, *IEEE Transactions on Electromagnetic Compatibility*, Vol. 56, pp. 1431-1438, 2014.

[44] Vujević, S.; Modrić, T.; Vukić, B., Internal Impedance of Two-Layer Cylindrical Conductors, *International Review of Electrical Engineering (IREE)*, Vol. 9, No. 1, pp. 235-243, 2014.

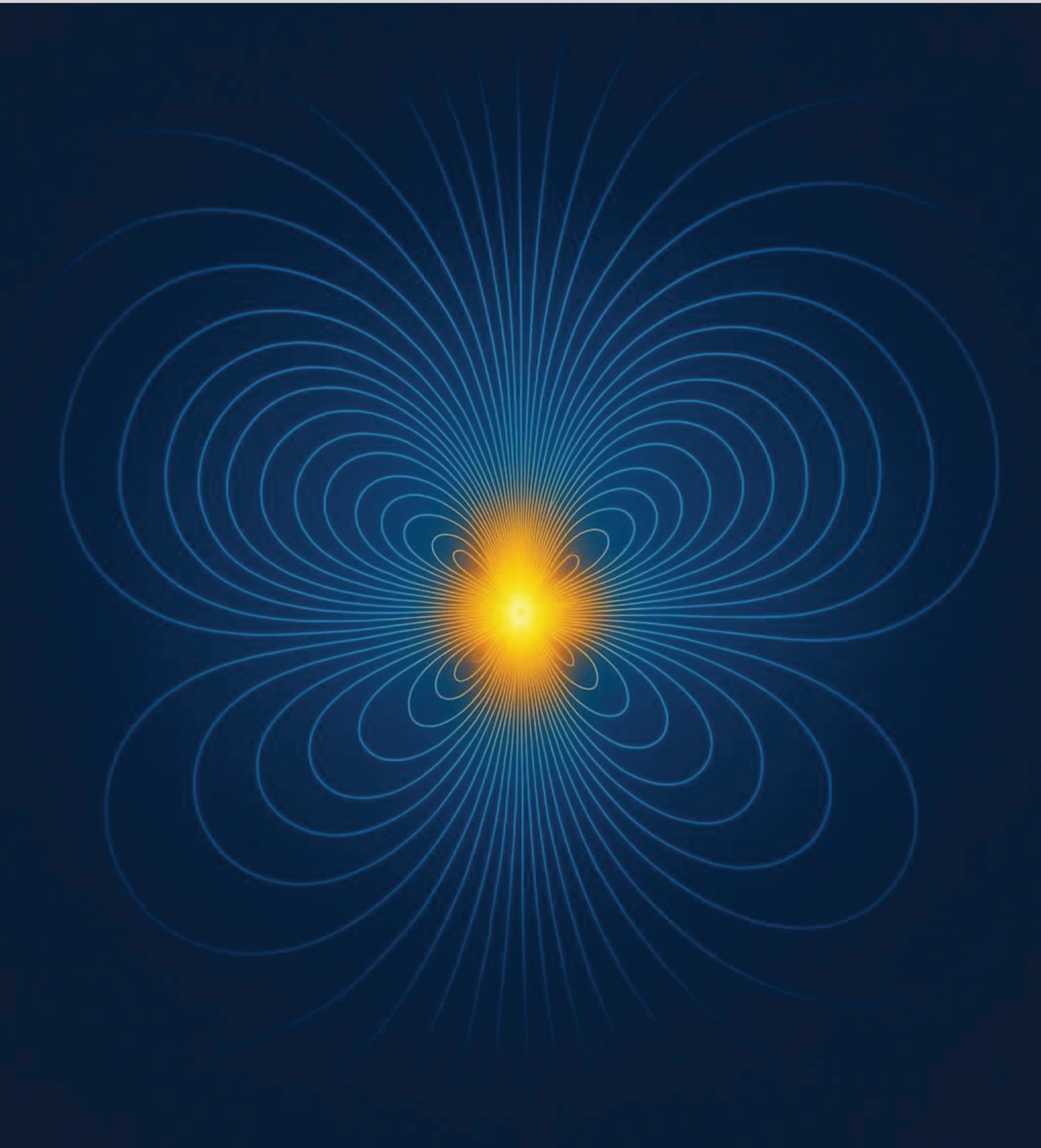
[45] Vujević, S.; Lovrić, D., On Continuous Numerical Fourier Transform for Transient Analysis of Lightning Current Related Phenomena, *Electric Power Systems Research*, Vol. 119, pp. 364-369, 2015.

[46] Vujević, S.; Lovrić, D., Inverse Continuous Numerical Fourier Transform for Transient Analysis of Electromagnetic Phenomena, *IEEE Transactions on Electromagnetic Compatibility,* Vol. 57, No. 5, pp. 1149-1154, 2015.

[47] Vujević, S.; Lovrić, D.; Balaž, Z., Self and Mutual Ground Impedances of Cylindrical Metal Plates Buried in Homogeneous Earth, *International Journal of Numerical Modelling - Electronic Networks Devices and Fields*, Vol. 28, No. 1, pp. 33-49, 2015.

[48] Vujević, S.; Lovrić, D.; Krolo, I.; Duvnjak, I., Computation of Electric and Magnetic Field Distribution inside a Multilayer Cylindrical Conductor, *Progress In Electromagnetics Research M (PIER M)*, Vol. 88, pp. 53-63, 2020.

[49] Vujević, S., *Teorijska elektrotehnika* (Theoretical Electrical Engineering), University of Split, FESB, Split, 2024.



UNIVERSITY OF SPLIT
FACULTY OF ELECTRICAL ENGINEERING, MECHANICAL ENGINEERING AND NAVAL ARCHITECTURE

e-ISBN: 978-953-290-150-4